



New Results on the Blow-up of Solutions in L^2 at Finite Time $\ln T_1^*$ for a Damped Emden-Fowler Type Degenerate Wave Equation

Khaled Zennir and Svetlin G. Georgiev

ABSTRACT: In this article we consider a new class of an Emden-Fowler type semilinear degenerate wave equation with memory. The main contributions here is to exhibit that the memory lets the global solutions of the degenerate problem still non-exist in L^2 at finite time

$$\ln T_1^*, \quad \text{s.t.} \quad T_1^* = \frac{2}{p-1} \left(\int_{r_1}^{r_2} |u_0| dx \right) \left(\int_{r_1}^{r_2} u_0 u_1 dx \right)^{-1}.$$

This is to extend recent result by [19] for the dissipative case.

Key Words: Local solution, Emden-Fowler wave equation, Blow-up.

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1. Introduction

The study of the Emden-Fowler equation originated from earlier theories concerning gaseous dynamics in astrophysics around the turn of the 20-th century. The fundamental problem in the study of stellar structure at that time was to study the equilibrium configuration of the mass of spherical clouds of gas. In this article we consider the nonexistence of global solutions in time of the Emden-Fowler type semilinear wave equation with viscoelasticity in the next new form

$$t^2 u'' - (a(x)u_x)_x + \int_1^t \frac{1}{s} \mu(\ln s) \left(a(x)u_x \left(\frac{t}{s} \right) \right)_x ds = u^p \quad \text{in } [1, T) \times (r_1, r_2) \quad (1.1)$$

with boundary value null and initial values

$$u(1, x) = u_0(x) \in H^2(r_1, r_2) \cap H_0^1(r_1, r_2),$$

$$u'(1, x) = u_1(x) \in H_0^1(r_1, r_2)$$

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where $p > 1$, r_1 and r_2 are real numbers and the scalar function μ (so-called relaxation kernel) is assumed to satisfy:

1. The function $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a C^1 nonincreasing function satisfying

$$\mu(0) > 0, 1 - \int_0^\infty e^{s/2} \mu(s) ds = l > 0. \quad (1.2)$$

2. The self-adjoint non-positive operator of the form

$$\mathbb{A}u = (a(x)u_x(t))_x$$

represent the class of degenerate term considered in Eq.(1.2), where $a \in C^1[r_1, r_2]$ may vanish in a subset Ω_0 of $[r_1, r_2]$ and $a(x) \geq A^2 > 0$ for some constant A .

The relationship between mathematics and physics succession of philosophical and epistemological order problem. Since long times, mathematicians and physicists have different positions on this report. The idea claiming that mathematics is the language of physics, is one of the explanations of the problem of relations between the two disciplines. The present research aims to extend the study of Emden-Fowler type simple wave equation to the case when the degenerate viscoelastic term is injected in $[r_1, r_2]$ where it seems that there is no result about this topic. Thus, a wider class of phenomena can be modeled.

Under the assumption that the gaseous cloud is under convective equilibrium (first proposed in 1862 by Lord Kelvin [27]), Lane studied the equation

$$\frac{d}{dt} \left(t^2 \frac{du}{dt} \right) + t^2 u^p = 0, \quad (1.3)$$

for the cases $p = 1.5$ and 2.5 . The equation (1.3) is commonly referred to as the Lane-Emden equation [4]. Astrophysicists were interested in the behavior of the solutions of (1.3) which satisfy the initial condition: $u(0) = 1$, $u'(0) = 0$. Special cases of (1.3), namely, when $p = 3$ the explicit solution to

$$\frac{d}{dt} \left(t^2 \frac{du}{dt} \right) + t^2 u = 0, \quad u(0) = 1, \quad u'(0) = 0$$

is $u = \sin(t)/t$, and when $p = 5$, the explicit solution to

$$\frac{d}{dt} \left(t^2 \frac{du}{dt} \right) + t^2 u^5 = 0, \quad u(0) = 1, \quad u'(0) = 0$$

is $u = 1/\sqrt{1+t^2/3}$.

Many properties of solutions to the Lane-Emden equation were studied by Ritter [24] in a series of 18 papers published during 1878-1889. The publication of Emden's treatise *Gaskugeln* [9] marks the end of first epoch in the study of stellar configurations governed by (1.3). The mathematical foundation for the study of such an equation and also of the more general equation

$$\frac{d}{dt} \left(t^\rho \frac{du}{dt} \right) + t^\sigma u^\gamma = 0, \quad t \geq 0, \quad (1.4)$$

was made by Fowler [10,11,12,13] in a series of four papers during 1914-1931. The first serious study on the generalized Emden-Fowler equation

$$\frac{d^2u}{dt^2} + a(t)|u|^\gamma \operatorname{sgn} u = 0, \quad t \geq 0$$

was made by Atkinson et al. Recently, M.-R. Li in [19] considered and studied the blow-up phenomena of solutions to the Emden-Fowler type semilinear wave equation

$$t^2 u_{tt} - u_{xx} = u^p \quad \text{in } [1, T) \times (a, b).$$

The main subject of this paper is to exhibit the role of the degenerate viscoelasticity, which makes our problem (1.1) dissipative, in the Blowing up of solutions in L^2 at finite time

$$\ln T_1^*, s.t. T_1^* = \frac{2}{p-1} T_1^* = \frac{2}{p-1} \left(\int_{r_1}^{r_2} |u_0| dx \right) \left(\int_{r_1}^{r_2} u_0 u_1 dx \right)^{-1},$$

for Emden-Fowler type wave equation when the energy is null which will be the main results of subsection 3.1. In the subsection 3.2, we will discuss the blow up in finite time $\ln T_2^* < \ln T_1^*$ of problem (1.1) for large class of solution in the case when the associated energy is negative. The questions of local existence and uniqueness will be also considered in Section 2.

2. Preliminaries, local Existence of unique solution

We omit the space variable x of $u(x, t), u'(x, t)$ and for simplicity reason denote $u(x, t) = u$ and $u'(x, t) = u'$, when there is no confusion, where $u' = \frac{d}{dt}u, u'' = \frac{d^2}{dt^2}u$. The constants c used throughout this paper are positive generic constants which may be different in various settings. Define the space

$$V_a = \left\{ w \in L^2(r_1, r_2) : \int_{r_1}^{r_2} a(x)|u_x|^2 dx < \infty, w(r_1) = w(r_2) = 0 \right\} \quad (2.1)$$

which is a Hilbert space endowed with the norm

$$\|u\|_{V_a}^2 = \int_{r_1}^{r_2} a|u_x|^2 dx$$

Note that

$$\|u_x\|_2 \leq \frac{1}{|A|} \|u\|_{V_a}$$

for any $u \in V_a$. Indeed, we have

$$\begin{aligned} \|u_x\|_2^2 &= \int_{r_1}^{r_2} |u_x|^2 dx \\ &= \int_{r_1}^{r_2} \frac{a(x)}{a(x)} |u_x|^2 dx \\ &\leq \frac{1}{A^2} \int_{r_1}^{r_2} a(x) |u_x|^2 dx \\ &= \frac{1}{A^2} \|u\|_{V_a}^2, \end{aligned}$$

whereupon we get the desired estimate.

Under some suitable transformations, we can get the local existence of solutions to Eq. (1.1). Consider the integral

$$\int_1^t \frac{1}{s} \mu(\ln s) \left(a(x) u_1 \left(\frac{t}{x} \right) \right)_x ds.$$

In it we make the change $s = e^y$ and we get

$$\int_1^t \frac{1}{s} \mu(\ln s) \left(a(x) u_1 \left(\frac{t}{x} \right) \right)_x ds = \int_0^{\ln t} \mu(y) (a(x) u_x (te^{-y}))_x dy.$$

Therefore we get the equation

$$t^2 u'' - (a(x) u_x)_x + \int_0^{\ln t} \mu(y) (a(x) u_x (te^{-y}))_x dy = u^p, \text{ in } [1, T] \times (r_1, r_2). \quad (2.2)$$

Taking the transform

$$\tau = \ln t, \quad v = u, \quad u_{xx} = v_{xx},$$

then

$$u' = t^{-1} v_\tau, \quad t^2 u'' = -v_\tau + v_{\tau\tau},$$

equation (1.1) takes the form

$$\begin{aligned} v_{\tau\tau} - (a(x) v_x)_x + \int_0^\tau \mu(s) (a(x) v_x(\tau - s))_x ds &= v_\tau + v^p \quad \text{in } [0, \ln T] \times (r_1, r_2), \\ v(x, 0) &= u_0(x), \quad u_\tau(x, 0) = u_1(x) \\ v(r_1, \tau) &= v(r_2, \tau) = 0 \end{aligned} \quad (2.3)$$

Let

$$v(\tau, x) = e^{\tau/2}w(\tau, x),$$

$$v_\tau(\tau, x) = e^{\tau/2}w_\tau(\tau, x) + \frac{1}{2}v(\tau, x),$$

$$v_{\tau\tau}(\tau, x) = e^{\tau/2}w_{\tau\tau}(\tau, x) + e^{\tau/2}w_\tau(\tau, x) + \frac{1}{4}e^{\tau/2}w(\tau, x),$$

then (2.3) can be rewritten as

$$\begin{aligned} & e^{\tau/2}w_{\tau\tau} - e^{\tau/2}(a(x)w_x)_x + \int_0^\tau e^{s/2}\mu(s)(a(x)w_x(\tau-s))_x ds \\ &= \frac{1}{4}e^{\tau/2}w + e^{p\tau/2}w^p, \end{aligned}$$

then

$$w_{\tau\tau} - (a(x)w_x)_x + e^{-\tau/2} \int_0^\tau e^{s/2}\mu(s)(a(x)w_x(\tau-s))_x ds = \frac{1}{4}w + e^{(p-1)\tau/2}w^p. \quad (2.4)$$

The following technical Lemma will play an important role.

Lemma 2.1. *For any $w \in C^1(0, T, H^1(r_1, r_2))$ we have for any nonincreasing differentiable function α satisfying $\alpha(\tau) > 0$*

$$\begin{aligned} & \int_{r_1}^{r_2} \alpha(\tau) \int_0^\tau e^{s/2}\mu(\tau-s)(a(x)w_x(s))_x w'(\tau) ds dx \\ &= \frac{1}{2} \frac{d}{d\tau} \alpha(\tau) \int_0^\tau e^{s/2}\mu(\tau-s) \int_{r_1}^{r_2} a(x)|w_x(\tau) - w_x(s)|^2 dx ds \\ &- \frac{1}{2} \frac{d}{d\tau} \alpha(\tau) \int_0^\tau e^{s/2}\mu(s) ds \int_{r_1}^{r_2} a(x)|w_x(\tau)|^2 dx \\ &- \frac{1}{2} \alpha \int_0^\tau \left(e^{s/2}\mu(\tau-s) \right)' \int_{r_1}^{r_2} a(x)|w_x(\tau) - w_x(s)|^2 dx ds \\ &+ \frac{1}{2} \alpha(\tau) e^{\tau/2} \mu(\tau) \int_{r_1}^{r_2} a(x)|w_x(\tau)|^2 dx \\ &- \frac{1}{2} \alpha'(\tau) \int_0^\tau e^{s/2}\mu(\tau-s) \int_{r_1}^{r_2} a(x)|w_x(\tau) - w_x(s)|^2 dx ds \\ &+ \frac{1}{2} \alpha'(\tau) \int_0^s e^{s/2}\mu(s) ds \int_{r_1}^{r_2} a(x)|w_x(\tau)|^2 dx. \end{aligned}$$

Proof. We have

$$\begin{aligned}
& \int_{r_1}^{r_2} \alpha(\tau) \int_0^\tau e^{s/2} \mu(\tau-s) (a(x)w_x(s))_x w'(\tau) ds dx \\
&= -\alpha(\tau) \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x)w'_x(\tau)w_x(s) dx ds \\
&= -\alpha(\tau) \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} w'_x(v) [a(x)w_x(s) - a(x)w_x(\tau)] dx ds \\
&\quad -\alpha(\tau) \int_0^\tau e^{s/2} \mu(s) \int_{r_1}^{r_2} w'_x(\tau)a(x)w_x(\tau) dx ds.
\end{aligned}$$

Consequently,

$$\begin{aligned}
& \int_{r_1}^{r_2} \alpha(\tau) \int_0^\tau e^{s/2} \mu(\tau-s) (a(x)w_x(s))_x w'(\tau) ds dx \\
&= \frac{1}{2} \alpha(\tau) \int_0^\tau e^{s/2} \mu(\tau-s) \frac{d}{d\tau} \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
&\quad -\alpha(\tau) \int_0^\tau e^{s/2} \mu(s) \left(\frac{d}{d\tau} \frac{1}{2} \int_{r_1}^{r_2} a(x) |w_x(\tau)|^2 dx \right) ds
\end{aligned}$$

which implies,

$$\begin{aligned}
& \int_{r_1}^{r_2} \alpha(\tau) \int_0^\tau e^{s/2} \mu(\tau-s) (a(x)w_x(s))_x w'(\tau) ds dx \\
&= \frac{1}{2} \frac{d}{d\tau} \left[\alpha(\tau) \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \right] \\
&\quad - \frac{1}{2} \frac{d}{d\tau} \left[\alpha(\tau) \int_0^\tau e^{s/2} \mu(s) \int_{r_1}^{r_2} a(x) |w_x(v)|^2 dx ds \right] \\
&\quad - \frac{1}{2} \alpha(\tau) \int_0^\tau \left(e^{s/2} \mu(\tau-s) \right)' \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
&\quad + \frac{1}{2} \alpha(\tau) e^{\tau/2} \mu(\tau) \int_{r_1}^{r_2} a(x) |w_x(\tau)|^2 dx. \\
&\quad - \frac{1}{2} \alpha'(\tau) e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
&\quad + \frac{1}{2} \alpha'(\tau) \int_0^\tau e^{s/2} \mu(s) ds \int_{r_1}^{r_2} a(x) |w_x(\tau)|^2 dx ds.
\end{aligned}$$

This completes the proof. \square

We introduce the modified energy associated to problem (2.4)

$$\begin{aligned}
 2E_w(\tau) &= \int_{r_1}^{r_2} |w_\tau|^2 dx + (1 - \int_0^\tau e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \\
 &+ \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
 &- \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx - \frac{2}{p+1} e^{\frac{(p-1)\tau}{2}} \int_{r_1}^{r_2} |w|^{p+1} dx.
 \end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
 2E_w(0) &= \int_{r_1}^{r_2} (u_1 - \frac{1}{2}u_0)^2 dx + \int_{r_1}^{r_2} a(x) |u_{0x}|^2 dx \\
 &+ \int_{r_1}^{r_2} u_0 u_1 dx - \frac{2}{p+1} \int_{r_1}^{r_2} |u_0|^{p+1} dx.
 \end{aligned}$$

Direct differentiation, using (1.2), (2.4), leads to

$$E'_w(\tau) \leq 0.$$

We now can obtain Lemma 2.2.

Lemma 2.2. *Suppose that $v \in C^1(0, T, H_0^1(r_1, r_2)) \cap C^2(0, T, L^2(r_1, r_2))$ is a solution of the semi-linear wave equation (2.4). Then for $\tau \geq 0$,*

$$E_w(\tau) \leq E_w(0) - \frac{p-1}{p+1} \int_0^\tau e^{\frac{(p-1)s}{2}} \int_{r_1}^{r_2} |w|^{p+1} dx ds, \tag{2.6}$$

Proof. Taking the L^2 product of (2.4) with w_τ yields

$$\begin{aligned}
 &\int_{r_1}^{r_2} w_{\tau\tau} w_\tau dx - \int_{r_1}^{r_2} \left((a(x)w_x)_x - e^{-\tau/2} \int_0^\tau e^{s/2} \mu(s) (a(x)w_x(t-s))_x ds \right) w_\tau dx \\
 &= \frac{1}{4} \int_{r_1}^{r_2} w w_\tau dx + \int_{r_1}^{r_2} e^{(p-1)\tau/2} w^p w_\tau dx.
 \end{aligned}$$

Thus, by Lemma 2.1 with $\alpha(\tau) = e^{-\tau/2}$, we have

$$\begin{aligned}
 &\frac{1}{2} \frac{d}{d\tau} \left[\int_{r_1}^{r_2} |w_\tau|^2 dx + (1 - \int_0^\tau e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \right] \\
 &+ \frac{1}{2} \frac{d}{d\tau} \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
 &= \frac{1}{8} \frac{d}{d\tau} \int_{r_1}^{r_2} |w|^2 dx + \frac{1}{p+1} \frac{d}{d\tau} \int_{r_1}^{r_2} e^{(p-1)\tau/2} w^{p+1} w_\tau dx \\
 &+ \frac{2(p-1)}{p+1} \int_{r_1}^{r_2} e^{(p-1)\tau/2} w^{p+1} dx
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}\alpha(\tau) \int_0^\tau \left(e^{s/2}\mu(\tau-s) \right)' \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
& - \frac{1}{2}\mu(\tau) \int_{r_1}^{r_2} a(x) |w_x(t)|^2 dx \\
& + \frac{1}{2}\alpha'(\tau) \int_0^\tau e^{s/2}\mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
& - \frac{1}{2}\alpha'(\tau) \int_0^s e^{s/2}\mu(s) ds \int_{r_1}^{r_2} a(x) |w_x(\tau)|^2 dx.
\end{aligned}$$

Then, by conditions on μ, α and (2.5), the assertion (2.6) is proved. \square

3. Blow up results

3.1. First result for $E_u(0) = 0$

Under small amplitude initial data, we prove that w blows up in L^2 at finite time $\ln T^*$ in the following Theorem 3.1.

Theorem 3.1. *Suppose that $w \in C^1(0, T, H_0^1(r_1, r_2)) \cap C^2(0, T, L^2(r_1, r_2))$ is a weak solution of equation (2.4) with*

$$e(0) := \int_{r_1}^{r_2} u_0 u_1(x) dx > 0, \quad E_u(0) = 0$$

and $0 < r_2 - r_1 \leq 1$. Then there exists T_1^* such that

$$\int_{r_1}^{r_2} |u(t, x)|^2 dx \rightarrow +\infty \quad \text{as } t \rightarrow T_1^*,$$

where

$$T_1^* = \frac{2}{p-1} \frac{\int_{r_1}^{r_2} |u_0| dx}{\int_{r_1}^{r_2} u_0 u_1 dx}.$$

Before we beginning to prove Theorem 3.1, we need to state and prove the next intermediate Lemma

Lemma 3.2. *Suppose that w is a weak solution of equation (2.4). Then*

$$\begin{aligned}
& \int_{r_1}^{r_2} e^{\frac{p-1}{2}s} w^{p+1}(s, x) dx \\
& \geq \frac{p+1}{2} \left[\int_{r_1}^{r_2} |w_s|^2 dx + \left(1 - \int_0^t e^{s/2}\mu(s) ds \right) \int_{r_1}^{r_2} a(x) |w_x|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right] \\
& + \int_0^\tau e^{s/2}\mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds - (p+1)E_w(0)e^{\frac{p-1}{2}s}
\end{aligned}$$

$$\begin{aligned}
 &+ \frac{p^2 - 1}{2} \int_0^s e^{\frac{p-1}{2}(s-r)} \left[\int_{r_1}^{r_2} |w_s|^2 dx + (1 - \int_0^t e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \right] dr \\
 &+ \frac{p^2 - 1}{2} \int_0^s -\frac{1}{4} e^{\frac{p-1}{2}(s-r)} \int_{r_1}^{r_2} |w|^2 dx dr \\
 &+ \frac{p^2 - 1}{2} \int_0^s e^{\frac{p-1}{2}(s-r)} \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds.
 \end{aligned}$$

Proof. Set

$$\begin{aligned}
 L(s) &= \frac{1}{p+1} \int_0^s e^{\frac{p-1}{2}r} \int_{r_1}^{r_2} |w|^{p+1} dx dr, \\
 F(s) &= \int_{r_1}^{r_2} |w_s|^2 dx + (1 - \int_0^s e^{\tau/2} \mu(\tau) d\tau) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \\
 &\quad - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx + \int_0^\tau e^{s/2} \mu(\tau - s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds,
 \end{aligned}$$

By Lemma 2.1 and Lemma 2.2, Eq.(2.6) can be rewritten as

$$E_w(0) \geq F - 2L' + (p - 1)L, \tag{3.1}$$

therefore,

$$\begin{aligned}
 (e^{\frac{p-1}{2}s} L)' &= e^{\frac{p-1}{2}s} \left(L' - \frac{p-1}{2} L \right) \\
 &\geq \frac{1}{2} e^{\frac{p-1}{2}s} (F - E_w(0)),
 \end{aligned}$$

and

$$\begin{aligned}
 e^{\frac{p-1}{2}s} L &\geq \frac{1}{2} \int_0^s e^{\frac{p-1}{2}r} (F(r) - E_w(0)) dr \\
 &\geq \frac{1}{2} \int_0^s e^{\frac{p-1}{2}r} F(r) dr - \frac{E_w(0)}{p-1} \left(1 - e^{\frac{p-1}{2}s} \right),
 \end{aligned}$$

and

$$L \geq \frac{1}{2} \int_0^s e^{\frac{p-1}{2}(s-r)} F(r) dr - \frac{E_w(0)}{p-1} \left(e^{\frac{p-1}{2}s} - 1 \right);$$

this implies

$$\begin{aligned}
 &\frac{1}{p+1} \int_0^s e^{\frac{p-1}{2}r} \int_{r_1}^{r_2} |w|^{p+1} dx dr \\
 &\geq \frac{1}{2} \int_0^s e^{\frac{p-1}{2}(s-r)} \left[\int_{r_1}^{r_2} |w_s|^2 dx + (1 - \int_0^t e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \right] dr \\
 &+ \frac{1}{2} \int_0^s -\frac{1}{4} e^{\frac{p-1}{2}(s-r)} \int_{r_1}^{r_2} |w|^2 dx dr
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \int_0^s e^{\frac{p-1}{2}(s-r)} \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
& - \frac{E_w(0)}{p-1} (e^{\frac{p-1}{2}s} - 1),
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^s \int_{r_1}^{r_2} e^{\frac{p-1}{2}r} w^{p+1}(r, x) dx dr \\
& \geq \frac{p+1}{2} \int_0^s e^{\frac{p-1}{2}(s-r)} \left[\int_{r_1}^{r_2} |w_s|^2 dx + (1 - \int_0^t e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \right] dr \\
& + \frac{p+1}{2} \int_0^s -\frac{1}{4} e^{\frac{p-1}{2}(s-r)} \int_{r_1}^{r_2} |w|^2 dx dr \\
& + \frac{p+1}{2} \int_0^s e^{\frac{p-1}{2}(s-r)} \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
& - \frac{p+1}{p-1} E_w(0) (e^{\frac{p-1}{2}s} - 1),
\end{aligned}$$

and

$$\begin{aligned}
& \int_{r_1}^{r_2} e^{\frac{p-1}{2}s} w^{p+1}(s, x) dx \\
& \geq \frac{p+1}{2} \left[\int_{r_1}^{r_2} |w_s|^2 dx + (1 - \int_0^t e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} a(x) |w_x|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right] \\
& + \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds - (p+1) E_w(0) e^{\frac{p-1}{2}s} \\
& + \frac{p^2-1}{2} \int_0^s e^{\frac{p-1}{2}(s-r)} \left[\int_{r_1}^{r_2} |w_s|^2 dx + (1 - \int_0^t e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \right] dr \\
& + \frac{p^2-1}{2} \int_0^s -\frac{1}{4} e^{\frac{p-1}{2}(s-r)} \int_{r_1}^{r_2} |w|^2 dx dr \\
& + \frac{p^2-1}{2} \int_0^s e^{\frac{p-1}{2}(s-r)} \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds. \quad (3.2)
\end{aligned}$$

This completes the proof. \square

We are now ready to prove Theorem 3.1

Proof. (Theorem 3.1)

Let

$$A(s) := \int_{r_1}^{r_2} |w(s, x)|^2 dx,$$

then we have

$$A'(s) = 2 \int_{r_1}^{r_2} w w_s(s, x) dx.$$

and

$$\begin{aligned}
 A''(s) &= 2 \int_{r_1}^{r_2} w w_{ss}(s, x) dx + 2 \int_{r_1}^{r_2} w_s^2(s, x) dx \\
 &= 2 \int_{r_1}^{r_2} (w(a(x)w_x)_x - w e^{-\tau/2} \int_0^t e^{s/2} \mu(s) (a(x)w_x)_x (t-s) ds) dx \\
 &= 2 \int_{r_1}^{r_2} \frac{1}{4} w^2 + w_s^2 + e^{\frac{p-1}{2}s} w^{p+1} dx \\
 &= 2 \int_{r_1}^{r_2} (-a(x)w_x^2 + w_x e^{-\tau/2} \int_0^t e^{s/2} \mu(s) a(x)w_x (t-s) ds) dx \\
 &= 2 \int_{r_1}^{r_2} \frac{1}{4} w^2 + w_s^2 + e^{\frac{p-1}{2}s} w^{p+1} dx.
 \end{aligned}$$

By Lemma 2.1, Lemma 3.2 and (3.2), we obtain

$$\begin{aligned}
 A''(s) &\geq 2 \left(\left(\int_0^t e^{s/2} \mu(s) ds - 1 \right) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \right) & (3.3) \\
 &+ 2 \left(\frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx + \int_{r_1}^{r_2} |w_s|^2 dx \right) \\
 &- 2 \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
 &+ 2(p+1) \left(\left(\int_0^t e^{s/2} \mu(s) ds - 1 \right) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \right) \\
 &+ 2(p+1) \left(\frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx + \int_{r_1}^{r_2} |w_s|^2 dx \right) \\
 &- (p+1) \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
 &+ (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \left(\left(\int_0^t e^{s/2} \mu(s) ds - 1 \right) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \right) \\
 &+ (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \left(\frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx + \int_{r_1}^{r_2} |w_s|^2 dx \right) \\
 &- (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
 &- 2(p+1) E_w(0) e^{\frac{p-1}{2}s} \\
 &\geq \left[(p+3) \int_{r_1}^{r_2} |w_s|^2 dx - \frac{p-1}{4} \int_{r_1}^{r_2} |w|^2 dx \right] \\
 &+ (p-1) \left(1 - \int_0^t e^{s/2} \mu(s) ds \right) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \\
 &+ (p-1) \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds
 \end{aligned}$$

$$\begin{aligned}
& -2(p+1)E_w(0)e^{\frac{p-1}{2}s} \\
& + (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \left(\int_{r_1}^{r_2} |w_s|^2 dx + \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right) dr \\
& + (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \left(\int_{r_1}^{r_2} \left(\int_0^t e^{s/2} \mu(s) ds - 1 \right) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \right) dr \\
& - (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds dr.
\end{aligned}$$

As in [19], let us set

$$J(s) := A(s)^{-k}, \quad k = \frac{p-1}{4} > 0.$$

Then

$$J'(s) = -kA(s)^{-k-1}A'(s),$$

and

$$\begin{aligned}
J''(s) &= -kA(s)^{-k-2}[A(s)A''(s) - (k+1)A'(s)^2] \\
&\leq -kA(s)^{-k-1}[A''(s) - 4(k+1) \int_{r_1}^{r_2} w_s^2 dx]. \tag{3.4}
\end{aligned}$$

Since $E_u(0) = 0$, we have

$$\begin{aligned}
& A''(s) - 4(k+1) \int_{r_1}^{r_2} |w_s|^2 dx \\
& \geq \left[(p+3) \int_{r_1}^{r_2} |w_s|^2 dx - \frac{p-1}{4} \int_{r_1}^{r_2} |w|^2 dx \right] \\
& + (p-1) \left(1 - \int_0^t e^{s/2} \mu(s) ds \right) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \\
& + (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \left(\int_{r_1}^{r_2} |w_s|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right) dr \\
& + (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \left(\left(1 - \int_0^t e^{s/2} \mu(s) ds \right) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \right) dr \\
& + (p-1) \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
& + (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds dr \\
& - 4(k+1) \int_{r_1}^{r_2} |w_s|^2 dx,
\end{aligned}$$

whereupon,

$$\begin{aligned}
 & A''(s) - 4(k+1) \int_{r_1}^{r_2} |w_s|^2 dx \\
 & \geq (p-1) \left[\left(1 - \int_0^t e^{s/2} \mu(s) ds\right) \int_{r_1}^{r_2} a(x) |w_x|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right] \\
 & + (p-1) \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\
 & + (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \left(\int_{r_1}^{r_2} |w_s|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right) dr \\
 & + (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \left(\left(1 - \int_0^t e^{s/2} \mu(s) ds\right) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \right) dr \\
 & + (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds dr \\
 & \geq (p-1)(1 - (r_2 - r_1)^2) \left(\int_{r_1}^{r_2} a(x) |w_x|^2 dx \right) \\
 & \geq (p-1)(1 - (r_2 - r_1)^2) \left(\int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \right) \\
 & + (p+1) \int_0^s e^{\frac{p-1}{2}(s-r)} \left(\int_{r_1}^{r_2} a(x) |w_s|^2 dx \right) dr \\
 & + (p+1) \int_0^s e^{\frac{p-1}{2}(s-r)} \left(\left(1 - \int_0^t e^{s/2} \mu(s) ds\right) \int_{r_1}^{r_2} a(x) |w_x|^2 dx \right) dr \\
 & + (p+1) \int_0^s e^{\frac{p-1}{2}(s-r)} \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds dr > 0,
 \end{aligned}$$

where $r_2 \leq 1 + r_1$.

Therefore, by (3.4) we obtain that for, $r_2 - r_1 \leq 1$, $J''(s) < 0$ for all $s \geq 0$,

$$\begin{aligned}
 J'(s) \leq J'(0) &= -\frac{p-1}{4} A(0)^{-\frac{p+3}{4}} A'(0) \\
 &= -\frac{p-1}{2} e(0) \int_{r_1}^{r_2} |u_0|^{-(p+3)} dx,
 \end{aligned}$$

and

$$\begin{aligned}
 J(s) &\leq J(0) - \frac{p-1}{2} e(0) \int_{r_1}^{r_2} |u_0|^{-(p+3)} dx s \\
 &= \int_{r_1}^{r_2} \|u_0\|^{-(p-1)} dx - \frac{p-1}{2} e(0) \int_{r_1}^{r_2} |u_0|^{-(p+3)} dx s \\
 &= \int_{r_1}^{r_2} |u_0|^{-(p+3)} dx \left(\int_{r_1}^{r_2} |u_0| dx - \frac{p-1}{2} e(0) s \right).
 \end{aligned}$$

Then

$$J(s) \rightarrow 0 \quad \text{as } s \rightarrow T^* = \frac{2}{p-1} \frac{\int_{r_1}^{r_2} |u_0| dx}{e(0)}. \tag{3.5}$$

Thus w solution of (2.4) blows up in L^2 at finite time T^* . □

3.2. Second result for $E_u(0) < 0$

In the following theorem we shall state and prove our second blowing up result

Theorem 3.3. *Suppose that $w \in C^1(0, T, H_0^1(r_1, r_2)) \cap C^2(0, T, L^2(r_1, r_2))$ is a weak solution of equation (1.1) with*

$$e(0) = \int_{r_1}^{r_2} u_0 u_1(x) dx > 0, \quad E_u(0) < 0$$

and $0 < r_2 - r_1 \leq 1$. Then, there exists T_2^* such that

$$\frac{1}{\int_{r_1}^{r_2} |u(t, x)|^2 dx} \rightarrow 0 \quad \text{as } t \rightarrow \ln T_2^*.$$

Further, we have $\ln T_2^* < \ln T_1^*$, and the estimate

$$\int_{r_1}^{r_2} w^2 dx \geq \int_{r_1}^{r_2} u_0^2 dx - 2E_u(0) \frac{p+1}{p-1} \left[se^{\frac{p-1}{2}s} - \frac{2}{p-1} (e^{\frac{p-1}{2}s} - 1) \right].$$

Proof. By (3.3), Lemma 2.1, $E_u(0) < 0$, $e(0) > 0$ and $0 < r_2 - r_1 \leq 1$, we have

$$\begin{aligned} J''(s) &\leq -k \left(\int_{r_1}^{r_2} w^2 dx \right)^{-k-1} \left[A''(s) - (p+3) \int_{r_1}^{r_2} w_s^2(s, x) dx \right] \\ &= -k \left(\int_{r_1}^{r_2} w^2 dx \right)^{-k-1} \left[-2(p+1)E_w(0)e^{\frac{p-1}{2}s} \right. \\ &\quad + (p-1) \left((1 - \int_0^t e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} a(x) |w_x|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right) \\ &\quad + (p-1) \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds \\ &\quad + (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \left(\int_{r_1}^{r_2} |w_s|^2 dx \right. \\ &\quad \left. + (1 - \int_0^t e^{s/2} \mu(s) ds) \int_{r_1}^{r_2} a(x) |w_x|^2 dx - \frac{1}{4} \int_{r_1}^{r_2} |w|^2 dx \right) dr \\ &\quad \left. + (p^2-1) \int_0^s e^{\frac{p-1}{2}(s-r)} \int_0^\tau e^{s/2} \mu(\tau-s) \int_{r_1}^{r_2} a(x) |w_x(s) - w_x(\tau)|^2 dx ds dr \right] \\ &\leq 2k(p+1)E_u(0)e^{\frac{p-1}{2}s} J(s)^{1+\frac{1}{k}} < 0, \end{aligned} \tag{3.6}$$

where $k = (p - 1)/4$, we can obtain the same conclusions as in Theorem 3.1. By the inequality (3.6) and $J' < 0$ we can estimate J further,

$$\begin{aligned} J''(s) &\leq 2k(p + 1)E_u(0)e^{\frac{p-1}{2}s}J(s)^{1+\frac{1}{k}} \\ &= \frac{1}{2}(p^2 - 1)E_u(0)e^{\frac{p-1}{2}s}J(s)^{1+\frac{1}{k}} < 0, \end{aligned}$$

and

$$\begin{aligned} J'(s) &\leq J'(0) + \frac{s}{2}(p^2 - 1)E_u(0)e^{\frac{p-1}{2}s}J(s)^{1+\frac{1}{k}} \\ &\leq \frac{s}{2}(p^2 - 1)E_w(0)e^{\frac{p-1}{2}s}J(s)^{1+\frac{1}{k}}, \end{aligned}$$

and

$$\begin{aligned} -k(J(s)^{-\frac{1}{k}})' &= J(s)^{-1-\frac{1}{k}}J'(s) \\ &\leq \frac{E_u(0)}{2}(p^2 - 1)se^{\frac{p-1}{2}s}, \end{aligned}$$

and

$$\begin{aligned} -k(J(s)^{-\frac{1}{k}} - J(0)^{-\frac{1}{k}}) &\leq \frac{E_u(0)}{2}(p^2 - 1)\left(\frac{2}{p-1}se^{\frac{p-1}{2}s} - \left(\frac{2}{p-1}\right)^2(e^{\frac{p-1}{2}s} - 1)\right) \\ &= E_w(0)(p + 1)\left[se^{\frac{p-1}{2}s} - \frac{2}{p-1}(e^{\frac{p-1}{2}s} - 1)\right], \end{aligned}$$

which implies

$$\int_{r_1}^{r_2} w^2 dx \geq \int_{r_1}^{r_2} u_0^2 dx - 2\frac{p+1}{p-1}E_u(0)\left[se^{\frac{p-1}{2}s} - \frac{2}{p-1}(e^{\frac{p-1}{2}s} - 1)\right]$$

Then u solution of our initial problem (1.1) blows up in L^2 at finite time $\ln T_2^*$. This completes the proof. \square

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Khaled Zennir,

First address:

*Department of Mathematics,
College of Sciences and Arts, Al-Ras,
Qassim University,
Kingdom of Saudi Arabia.*

Second address:

*Laboratory LAMAHIS,
Department of mathematics,
University 20 Août 1955- Skikda, 21000, Algeria.
E-mail address: k.zennir@qu.edu.sa*

and

Svetlin G. Georgiev,

University of Sofia,

*Faculty of Mathematics and Informatics,
Department of Differential Equations*

E-mail address: svetlingeorgiev1@gmail.com