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Binary Relation for Tripled Fixed Point Theorem in Metric Spaces

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abstract: In this paper we present a new extension of tripled fixed point theorems in metric spaces endowed with a reflexive binary relation that is not necessarily neither transitive nor antisymmetric. The key feature in this tripled fixed point theorems is that the contractivity condition on the nonlinear map is only assumed to hold on elements that are comparable in the binary relation. Next on the basis of the tripled fixed point theorems, we prove the existence and uniqueness of positive definite solutions of a nonlinear matrix equation of type

$$X = Q + \Sigma_{i=1}^{m} A_i^* G(X) A_i - \Sigma_{j=1}^{n} B_j^* K(X) B_j - \Sigma_{r=1}^{t} C_r^* L(X) C_r$$

Key Words: Tripled fixed point, Reflexive relation, Matrix equations, Positive define solution.

Contents

1. Introduction

The Banach contraction principle [10] is a classical and powerful tool in non linear analysis and has been generalized by many authors. Bhaskar and Lakshmikantham [14] introduced the concept of a coupled fixed point of mapping $F : X \times X \to X$ and investigated some coupled fixed point theorems in partially ordered metric spaces. During the last few decades, many authors discussed on coupled fixed point results in various spaces and considered this concept to study nonlinear differential equations, nonlinear integral equations and matrix equations ([1,4,5,6,7,8,22,23,24,25,26]). Recently Berinde and Borcut [11] introduced the notion of tripled fixed points in partially ordered metric spaces, which refer to the operator as $F : X \times X \times X \to X$, motivated by the fact that through the coupled fixed point technique we cannot solve a system with the following form:

$$x^2 + 2yz - 6x + 3 = 0,$$

$$y^2 + 2xz - 6y + 3 = 0,$$

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$$z^2 + 2yx - 6z + 3 = 0.$$

In a subsequent series, Berinde and Borcut [11], introduced the concept of tripled coincidence point and obtained the tripled coincidence point theorems.

Definition 1.1. *Let $(X, \preceq)$ be a partially ordered set, $F : X^3 \to X$ mapping. The mapping F is said to have the mixed monotone property if for any $x, y, z \in X$,*

(i) $x_1, x_2 \in X, x_1 \preceq x_2 \Longrightarrow F(x_1, y, z) \preceq F(x_2, y, z)$,

(ii) $y_1, y_2 \in X, y_1 \preceq y_2 \Longrightarrow F(x, y_1, z) \preceq F(x, y_2, z)$,

(iii) $z_1, z_2 \in X, z_1 \preceq z_2 \Longrightarrow F(x, y, z_1) \preceq F(x, y, z_2)$.

Definition 1.2. *An element $(x, y, z) \in X^3$ is called a tripled fixed point of $F : X^3 \to X$ if $F(x, y, z) = x, F(y, x, y) = y$, and $F(z, y, x) = z$.*

Definition 1.3. *Let $(X, \preceq)$ be a partially ordered set, $F : X^3 \to X$ and $g : X \to X$ two mappings. The mapping F is said to have the mixed $g-$monotone property if for any $x, y, z \in X$.*

(i) $x_1, x_2 \in X, g(x_1) \preceq g(x_2) \Longrightarrow F(x_1, y, z) \preceq F(x_2, y, z)$,

(ii) $y_1, y_2 \in X, g(y_1) \preceq g(y_2) \Longrightarrow F(x, y_1, z) \preceq F(x, y_2, z)$,

(iii) $z_1, z_2 \in X, g(z_1) \preceq g(z_2) \Longrightarrow F(x, y, z_1) \preceq F(x, y, z_2)$.

Definition 1.4. *An element $(x, y, z) \in X^3$ is called a tripled coincidence point of the mappings $F : X^3 \to X$ and $g : X \to X$ if*

$$F(x, y, z) = gx, F(y, x, y) = gy \ \text{ and } \ F(z, y, x) = gz.$$

Definition 1.5. *An element $(x, y, z) \in X^3$ is called a tripled common fixed point of the mappings $F : X^3 \to X$ and $g : X \to X$ if*

$$F(x, y, z) = gx = x, F(y, x, y) = gy = y \ \text{ and } \ F(z, y, x) = gz = z.$$

Definition 1.6. *An element $x \in X$ is called a common fixed point of the mappings $F : X^3 \to X$ and $g : X \to X$ if $F(x, x, x) = gx = x$.*

Definition 1.7. *Let X be a non empty set. The mappings $F : X^3 \to X$ and $g : X \to X$ are commuting if for all $x, y, z \in X$,*

$$g(F(x, y, z)) = F(g(x), g(y), g(z)).$$

Definition 1.8. *Let (X, d) be a metric space. The mappings F and g where $F : X^3 \to X$ and $g : X \to X$ are said to be compatible if*

$$\lim_{n\to\infty} d(g(F(x_n, y_n, z_n)), F(g(x_n), g(y_n), g(z_n))) = 0,$$

$$\lim_{n\to\infty} d(g(F(y_n, x_n, y_n)), F(g(y_n), g(x_n), g(y_n))) = 0$$

and

$$\lim_{n\to\infty} d(g(F(z_n, y_n, x_n)), F(g(z_n), g(y_n), g(x_n))) = 0$$

whenever $\{x_n\}$, $\{y_n\}$ *and* $\{z_n\}$ *are sequences in* X *such that*

$$\lim_{n\to\infty} F(x_n, y_n, z_n) = \lim_{n\to\infty} g(x_n) = x,$$

$$\lim_{n\to\infty} F(y_n, x_n, y_n) = \lim_{n\to\infty} g(y_n) = y$$

and

$$\lim_{n\to\infty} F(z_n, y_n, x_n) = \lim_{n\to\infty} g(z_n) = z$$

for some $x, y, z \in X$.

In [11] Berinde and Borcut proved the following theorem.

Theorem 1.9. *Let* $(X, \preceq)$ *be a partially ordered set and* (X, d) *be a complete metric space. Let* $F : X^3 \to X$ *be a continuous mapping having the mixed monotone property on* X. *Assume that there exist constants* $a, b, c \in [0, 1)$ *such that* $a+b+c \prec 1$ *for which,*

$$d(F(x, y, z), F(u, v, w)) \preceq ad(x, u) + bd(y, v) + cd(z, w) \tag{1.1}$$

for all $x \preceq u, y \preceq v, z \preceq w$. *Assume either,*

1. *(1)* F *is continuous,*
2. *(2)* X *has the following properties:*
   - *(a) if non decreasing sequence* $x_n \to x$ *and* $z_n \to z$, *then* $x_n \preceq x$ *and* $z_n \preceq z$ *for all* n,
   - *(b) if non increasing sequence* $y_n \to y$, *then* $y_n \preceq x$ *for all* n.

If there exist $x_0, y_0, z_0 \in X$ *such that* $x_0 \preceq F(x_0, y_0, z_0)$, $y_0 \preceq F(y_0, x_0, y_0)$ *and* $z_0 \preceq F(z_0, y_0, x_0)$. *Then there exist* $x, y, z \in X$ *such that,*

$$F(x, y, z) = x, F(y, x, y) = y, \quad \text{and} \quad F(z, y, x) = z.$$

For more generalized results of Theorem 1.9 in different spaces are refer to [19,16,20,21].

Motivated by the interesting works [11], we first introduce the notions of binary relation for tripled fixed point and later establish the existence and convergence theorems of tripled fixed point in metric spaces. Moreover, we apply these results to prove the existence and uniqueness of positive definite solutions of a nonlinear matrix equation of type

$$X = Q + \Sigma_{i=1}^{m} A_i^* G(X) A_i - \Sigma_{j=1}^{n} B_j^* K(X) B_j - \Sigma_{r=1}^{t} C_r^* L(X) C_r$$

and give illustrative examples of our theorems.

2. Preliminaries

Throughout the paper X will be a topological space and R is a reflexive relation on X. We start our consideration by introducing the following definitions and basic properties of tripled fixed point in metric spaces. In whole work we denote $X \times X \times X = X^3$.

Remark 2.1. *Let X be a nonempty set and let $f : X^3 \to X$ be a mapping. Then*

(i) We will denote

$$f^0(x, y, z) = x, \quad f^0(y, x, y) = y \ \text{ and } \ f^0(z, y, x) = z$$

and

$$f^n(x, y, z) = f(f^{n-1}(x, y, z), f^{n-1}(y, x, y), f^{n-1}(z, y, x)).$$

(ii) The Cartesian product of f, g and h is denoted by $f \times g \times h$, and defined by

$$f \times g \times h(x, y, z) = (f(x, y, z), g(y, x, y), h(z, y, x)).$$

Definition 2.2. *Let X be a nonempty set and $f : X \times X \times X \to X$ be a mapping. Then an element $(x, y, z) \in X \times X \times X$ is called tripled fixed point of f , if*

$$f(x, y, z) = x, f(y, x, y) = y \ \text{ and } \ f(z, y, x) = z$$

and an element $x \in X$ is called a fixed point of f , if $f(x, x, x) = x$. We will denote the set of all the tripled fixed point of f by F_f^T and the set of all the fixed points of f by F_f.

3. Main results

In this section we will prove the tripled fixed point theorems with respect to a reflexive relation.

Definition 3.1. *Let X be a topological space and let $f, g, h : X^3 \to X$ be three map. Then*

(i) An element $(x, y, z) \in X^3$ is called a tripled attractor basin element of f with respect to $(x^*, y^*, z^*) \in X^3$, if

$$f^n(x, y, z) \to x^*,$$

$$f^n(y, x, y) \to y^*,$$

$$f^n(z, y, x) \to z^*,$$

as $n \to \infty$ and an element $x \in X$ is called a attractor basin element of f with respect to (x^*, y^*, z^*) by $A_f^T(x^*, y^*, z^*)$ and the set of all the attractor basin of f with respect to $x^* \in X$ by $A_f(x^*)$.

(ii) *The mapping f is called orbitally continuous if $(x, y, z), (a, b, c) \in X^3$ and*

$$f^{n_k}(x, y, z) \to a,$$
$$f^{n_k}(y, x, y) \to b,$$
$$f^{n_k}(z, y, x) \to c$$

as $k \to \infty$ imply

as $k \to \infty$.

$$f^{n_{k-1}}(x, y, z) \to f(a, b, c),$$
$$f^{n_{k-1}}(y, x, y) \to f(b, a, b),$$
$$f^{n_{k-1}}(z, y, x) \to f(c, b, a)$$

(iii) *The mapping f is called a Picard operator, if there exits $x^* \in X$ such that*

(1) $F_f = \{x^*\}$.

(2) $A_f(x^*) = X$.

Also f is called a weakly Picard operator, if the sequences $\{f^n(x, x, x)\}_{n \in \mathbb{N}}$ convergent for all $x \in X$ and the limits (which may depend on x) are a fixed point of f.

Definition 3.2. *Let X be nonempty set and let R be a reflexive relation on X, for every $(p, q, r) \in X^3$ we define*

$$X_R(p, q, r) = \{(x, y, z) \in X^3 : xRp \bigwedge qRy \bigwedge zRr\}.$$

Note that $(x, y, z) \in X_R(p, q, r)$ if and only if $(q, p, q) \in X_R(y, x, y)$, $(r, q, p) \in X_R(z, y, x)$ and $(x, y, z) \in X_R(y, x, y) \cap X_R(z, y, x)$.

Definition 3.3. *Let X be nonempty set and let R be a reflexive relation on X, $f : X^3 \to X$. Then*

(i) *We say that f has the mixed $R-$monotone property on X, if*

$$f \times f \times f(R_X(x, y, z)) \subseteq X_R(f \times f \times f(x, y, z))$$

for all $(x, y, z) \in X_3$.

(ii) *An element $(x, y, z) \in X^3$ is called a $R-$tripled fixed point of f , if $f \times f \times f(x, y, z) \in X_R(x, y, z)$.*

(iii) *A sequence $\{x_n, y_n, z_n\}_{n \in \mathbb{N}} \subseteq X$ is called a $R-$monotone sequence, if*

$$(x_n, y_n, z_n) \in X_R(x_{n-1}, y_{n-1}, z_{n-1})$$

for all $n \in N$.

We begin the following theorem that establishes the existence of a tripled fixed point for a orbitally continuous function $F : X^3 \to X$ with respect to a reflexive relation R in topological space X.

Theorem 3.4. *Let X be a topological space and R be a reflexive relation on X. Assume that $F : X^3 \to X$ is a mapping having the following properties :*

(i) For each $(x, y, z), (p, q, r) \in X^3$ there exists a $(u, v, w) \in X^3$ such that (x, y, z), $(p, q, r) \in X_R(u, v, w)$.

(ii) There exists $(x_0, y_0, z_0), (x^*, y^*, z^*) \in X^3$ such that $(x_0, y_0 z_0 \in A_f(x^*, y^*, z^*)$.

(iii) For each $(x, y, z), (p, q, r) \in X^3$ if

$$(x, y, z) \in X_R(p, q, r)$$

and

$$(p, q, r) \in A_f(x^*, y^*, z^*)$$

then $(x, y, z) \in A_f(x^*, y^*, z^*)$. Then $A_f(x^*, y^*, z^*) = X^3$. Moreover, if f is orbitally continuous then, it is also a Picard operator and $F_f = \{x^*\}$.

Proof: Let $(x, y, z) \in X^3$ be arbitrary, then from (i) there exists $(p, q, r) \in X^3$ such that $(x, y, z), (x_0, y_0, z_0) \in X_R(p, q, r)$. From $(x_0, y_0, z_0) \in X_R(p, q, r)$, we have $(q, p, q) \in X_R(y_0, x_0, y_0)$, $(r, q, p) \in X_R(z_0, y_0, x_0)$ and from (ii) and (iii) we get that $(p, q, r) \in A_f(x^*, y^*, z^*)$, also from $(x, y, z) \in X_R(p, q, r)$, $(x, y, z) \in A_f(p, q, r)$ and (iii) we obtain $(x, y, z) \in A_f(x^*, y^*, z^*) = X^3$. Now, let f be an orbitally continuous mapping, then (ii) follows that $f(x^*, y^*, z^*) = x^*$, $f(y^*, x^*, y^*) = y^*$ and $f(z^*, y^*, x^*) = z^*$. Also, from $(y^*, x^*, y^*), (z^*, y^*, x^*) \in A_f(x^*, y^*, z^*)$, we get $x^* = y^* = z^*$. Therefore $A_f(x^*) = x^* = X$ which this shows that the operator f is Picard.

Remark 3.5. *Note that the assumption (iii) in Theorem 15 is essential. To see this, let $X = \mathbb{N}$ with discrete topology τ . Suppose that R is the division relation on X and $f : X^3 \to X$ be defined by $f(x, y, z) = x$. Also $A_f(x, y, z) = \{x, y, z\}$ for all $x, y, z \in X$ and there exists Then for every $(x, y, z) \in X_R(p, q, r)$ and $(p, q, r) \in A_f(a, b, c)$ such that $(x, y, z) \in A_f(a, b, c)$. Moreover, f is continuous and $F_f = \mathbb{N}$, thus f is not a Picard operator.*

In the following theorem we prove a generalization of [11] for a orbitally continuous mapping with respect to a reflexive relation on the metric space X.

Theorem 3.6. *Let (X, d) be a metric space and R be a reflexive relation on X. If $f : X^3 \to X$ is a mapping such that:*

(i) f having the mixed $R-$monotone property on X.

(ii) (X, d) be a complete metric space.

(iii) f having a $R-$tripled fixed point. i.e., there exists $(x_0, y_0, z_0) \in X^3$ such that $f \times f \times f(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$.

(iv) There exists $k \in [0,1)$ *such that*

$$d(f(x,y,z), f(p,q,r)) \leq \frac{k}{3}[d(x,u) + d(y,v) + d(z,r)].$$

(v) f *is an orbitally continuous mapping. Then:*

(a) There exists $x^*, y^*, z^* \in X$ *such that* $f(x^*, y^*, z^*) = x^*$, $f(y^*, x^*, y^*) = y^*$ *and* $f(z^*, y^*, x^*) = z^*$.

(b) The sequences $x_n n \in \mathbb{N}, y_n n \in \mathbb{N}$ *and* $z_n n \in \mathbb{N}$ *defined by*

$$x_{n+1} = f(x_n, y_n, z_n),$$

$$y_{n+1} = f(y_n, x_n, y_n)$$

and

$$z_{n+1} = f(z_n, y_n, x_n)$$

converges respectively to x^*, y^* *and* z^*.

(c) The error estimation is given by

$$\max_{n\in\mathbb{N}}\{d(x_n, x^*), d(y_n, y^*), d(z_n, z^*)\}$$
$$\leq \frac{k^n}{3(1-k)}[d(f(x_0,y_0,z_0),x_0) + d(f(y_0,x_0,y_0),y_0) + d(f(z_0,y_0,x_0),z_0)].$$

Proof. Since $f \times f \times f(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$, so from (i) it follows that

$$(f^2(x_0,y_0,z_0), f^2(y_0,x_0,y_0), f^2(z_0,y_0,x_0))$$

in

$$X_R(f(x_0,y_0,z_0), f(y_0,x_0,y_0), f(z_0,y_0,x_0)).$$

Further, we can easily verify that for any $n \in \mathbb{N}$,

$$(f^n(x_0,y_0,z_0), f^n(y_0,x_0,y_0), f^n(z_0,y_0,x_0))$$
$$\in X_R(f^{n-1}(x_0,y_0,z_0), f^{n-1}(y_0,x_0,y_0), f^{n-1}(z_0,y_0,x_0)). \tag{3.1}$$

Now, we claim that, for $n \in \mathbb{N}$

$$\begin{aligned}
d(f^{n+1}(x_0,y_0,z_0), f^n(x_0,y_0,z_0)) &\leq \frac{k^n}{3}[d(f(x_0,y_0,z_0),x_0) \\
&\quad + d(f(y_0,x_0,y_0),y_0) + d(f(z_0,y_0,x_0),z_0)] \\
d(f^{n+1}(y_0,x_0,y_0), f^n(y_0,x_0,y_0)) &\leq \frac{k^n}{3}[d(f(x_0,y_0,z_0),x_0) \\
&\quad + d(f(y_0,x_0,y_0),y_0) + d(f(y_0,x_0,y_0),y_0)] \\
d(f^{n+1}(z_0,y_0,x_0), f^n(z_0,y_0,x_0)) &\leq \frac{k^n}{3}[d(f(x_0,y_0,z_0),x_0) \\
&\quad + d(f(y_0,x_0,y_0),y_0) \\
&\quad + d(f(z_0,y_0,x_0),z_0)]. \qquad (3.2)
\end{aligned}$$

Indeed, for $n = 1$, using (iii) and (iv), we get

$$
\begin{aligned}
d(f^2(x_0, y_0, z_0), f(x_0, y_0, z_0)) &= d(f(f(x_0, y_0, z_0)), f(x_0, y_0, z_0)) \\
&\leq \frac{k}{3}[d(f(x_0, y_0, z_0), x_0) \\
&\quad +d(f(y_0, x_0, y_0), y_0) + d(f(z_0, y_0, x_0), z_0)] \\
d(f^2(y_0, x_0, y_0), f(y_0, x_0, y_0)) &= d(f(f(y_0, x_0, y_0)), f(y_0, x_0, y_0)) \\
&\quad \leq \frac{k}{3}[d(f(x_0, y_0, z_0), x_0) \\
&\quad +d(f(y_0, x_0, y_0), y_0) + d(f(y_0, x_0, y_0), y_0)] \\
d(f^2(z_0, y_0, x_0), f(z_0, y_0, x_0)) &= d(f(f((z_0, y_0, x_0)), f(z_0, y_0, x_0)) \\
&\leq \frac{k}{3}[d(f(x_0, y_0, z_0), x_0) \\
&\quad +d(f(y_0, x_0, y_0), y_0) + d(f(z_0, y_0, x_0), z_0)].
\end{aligned}
$$

Now assume that 3.2 holds, Using (iv) we get

$$
\begin{aligned}
d(f^{n+2}(x_0, y_0, z_0), f^{n+1}(x_0, y_0, z_0)) &= d(f(f^{n+1}(x_0, y_0, z_0)), f(f^n(x_0, y_0, z_0))) \\
&\leq \frac{k}{3}[d(f^{n+1}(x_0, y_0, z_0), f^n(x_0, y_0, z_0)) \\
&\quad +d(f^{n+1}(y_0, x_0, y_0), f^n(y_0, x_0, y_0)) \\
&\quad +d(f^{n+1}(z_0, y_0, x_0), f^n(z_0, y_0, x_0))] \\
&\leq \frac{k^{n+1}}{3}[d(f(x_0, y_0, z_0), x_0) \\
&\quad +d(f(y_0, x_0, y_0), y_0) + d(f(z_0, y_0, x_0), z_0)]
\end{aligned}
$$

$$
\begin{aligned}
d(f^{n+2}(y_0, x_0, y_0), f^{n+1}(y_0, x_0, y_0)) &= d(f(f^{n+1}(y_0, x_0, y_0)), f(f^n(y_0, x_0, y_0))) \\
&\leq \frac{k}{3}[d(f^{n+1}(x_0, y_0, z_0), f^n(x_0, y_0, z_0)) \\
&\quad +d(fn + 1(y_0, x_0, y_0), fn(y_0, x_0, y_0)) \\
&\quad +d(fn + 1(y_0, x_0, y_0), fn(y_0, x_0, y_0))] \\
&\leq \frac{k^{n+1}}{3}(f(x_0, y_0, z_0), x_0) \\
&\quad +d(f(y_0, x_0, y_0), y_0) + d(f(y_0, x_0, y_0), y_0)]
\end{aligned}
$$

$$\begin{aligned} d(f^{n+2}(z_0,y_0,x_0),f^{n+1}(z_0,y_0,x_0)) &= d(f(f^{n+1}(z_0,y_0,x_0)),f(f^n(z_0,y_0,x_0))) \\ &\leq \frac{k}{3}[d(f^{n+1}(z_0,y_0,x_0),f^n(z_0,y_0,x_0)) \\ &\quad +d(fn+1(y_0,x_0,y_0),fn(y_0,x_0,y_0)) \\ &\quad +d(fn+1(x_0,y_0,z_0),fn(x_0,y_0,z_0))] \\ &\leq \frac{k^{n+1}}{3}[d(f(x_0,y_0,z_0),x_0) \\ &\quad +d(f(y_0,x_0,y_0),y_0)+d(f(z_0,y_0,x_0),z_0)]. \end{aligned}$$

This implies that $\{f^n(x_0,y_0,z_0)\}_{n\in\mathbb{N}}$, $fn(y,x,y)_{n\in\mathbb{N}}$ and $\{fn(z,y,x)\}_{n\in\mathbb{N}}$ are Cauchy sequences in X. Because, if $m>n$, then

$$\begin{aligned} d(f^m(x_0,y_0,z_0),f^n(x_0,y_0,z_0)) &\leq \Sigma_{j=n}^{m-1} d(f^{j+1}(x_0,y_0,z_0),f^j(x_0,y_0,z_0)) \\ &\leq \frac{\Sigma_{j=n}^{m-1}}{3}[d(f(x_0,y_0,z_0),x_0) \\ &\quad +d(f(y_0,x_0,y_0),y_0)+d(f(z_0,y_0,x_0),z_0)] \\ &\leq \frac{k^n-k^m}{3}[d(f(x_0,y_0,z_0),x_0) \\ &\quad +d(f(y_0,x_0,y_0),y_0)+d(f(z_0,y_0,x_0),z_0)] \\ &\leq \frac{k^n}{3}[d(f(x_0,y_0,z_0),x_0) \\ &\quad +d(f(y_0,x_0,y_0),y_0)+d(f(z_0,y_0,x_0),z_0)] \end{aligned}$$

$$\begin{aligned} d(f^m(y_0,x_0,y_0),f^n(y_0,x_0,y_0)) &\leq \Sigma_{j=n}^{m-1} d(f^{j+1}(y_0,x_0,y_0),f^j(y_0,x_0,y_0)) \\ &\leq \frac{\Sigma_{j=n}^{m-1}}{3}[d(f(y_0,x_0,y_0)),y_0) \\ &\quad +d(f(y_0,x_0,y_0),y_0)+d(f(x_0,y_0,z_0),x_0)] \\ &\leq \frac{k^n-k^m}{3}[d(f(y_0,x_0,y_0)),y_0) \\ &\quad +d(f(y_0,x_0,y_0),y_0)+d(f(x_0,y_0,z_0),x_0)] \\ &\leq \frac{k^n}{3}[d(f(y_0,x_0,y_0)),y_0) \\ &\quad +d(f(y_0,x_0,y_0),y_0)+d(f(x_0,y_0,z_0),x_0)]. \end{aligned}$$

$$\begin{aligned} d(fm(z_0,y_0,x_0),fn(z_0,y_0,x_0)) &\leq \Sigma_{j=n}^{m-1} d(f^{j+1}(x_0,y_0,z_0),f^j(x_0,y_0,z_0)) \\ &\leq \frac{\Sigma_{j=n}^{m-1}}{3}[d(f(x_0,y_0,z_0),x_0) \\ &\quad +d(f(y_0,x_0,y_0),y_0)+d(f(z_0,y_0,x_0),z_0)] \end{aligned}$$

$$\leq \frac{k^n - k^m}{3}[d(f(x_0, y_0, z_0), x_0)$$
$$+ d(f(y_0, x_0, y_0), y_0) + d(f(z_0, y_0, x_0), z_0)]$$
$$\leq \frac{k^n}{3}[d(f(x_0, y_0, z_0), x_0)$$
$$+ d(f(y_0, x_0, y_0), y_0) + d(f(z_0, y_0, x_0), z_0)].$$

Which shows that $\{f^n(x_0, y_0, z_0)\}_{n\in\mathbb{N}}$, $\{f^n(y, x, y)\}_{n\in\mathbb{N}}$ and $\{f^n(z, y, x)\}_{n\in\mathbb{N}}$ are Cauchy sequences in X. Since X is complete, there exist $x^*, y^*, z^* \in X$ such that $f^n(x_0, y_0, z_0) \to x^*$, $f^n(y_0, x_0, y_0) \to y^*$ and $f^n(z_0, y_0, x_0) \to z^*$. Now the conclusion of theorem follows from the orbitally continuous of f.

Example 3.1. Let $X = \mathbb{R}$ with $d(x, y) = |x - y|$ and consider the relation R on X by $xRy \Longleftrightarrow x^2 + x = y^2 + y$. Let $f : X^3 \to X$ be defined by $f(x, y) = x^2 + x - 1$. then for any $(x, y, z) \in X^3$, $X_R(x, y, z) = \{(x, y, z), (x, -y-1, -z-1), (-x-1, -y-1, z), (-x-1, y, -z-1), (-x-1, y, z), (x, -y-1, z), (x, y, -z-1), (-x-1, -y-1, -z-1)\}$

$$f \times f \times f(XR(x, y, z)) = f \times f \times f(x, y, z) \subseteq X_R(f \times f \times f(x, y, z)).$$

Thus, f having the mixed $R-$monotone property on X. Moreover, f is continuous and there exists point $(1, -2, 1) \in X^3$ such that $f \times f \times f(1, -2, 1) \in X_R(1, -2, 1)$. So, the hypothesis of Theorem 3.6 is satisfies. Therefore, we conclude that f has a tripled fixed point in X^3. This tripled fixed points are $(x, y, z) = \{(1, 1, 1), (-1, -1, 1), (-1, 1, -1), (1, -1, -1), (-1, 1, 1), (1, -1, 1), (1, 1, -1), (-1, -1, -1)\}$.

Theorem 3.7. *In addition to the hypothesis of Theorem 3.6, suppose that for every $(x, y, z), (p, q, r) \in X^3$ there exists a $(u, v, w) \in X^3$ such that $(x, y, z), (p, q, r) \in X_R(u, v, w)$. Then f is a Picard operator.*

Proof. According to the proof of Theorem 3.6, there exist $x^*, y^*, z^* \in X$ such that $f(x^*, y^*, z^*) = x^*$, $f(y^*, x^*, y^*) = y^*$ and $f(z^*, y^*, x^*) = z^*$. Now, we show that $A_f(x^*, y^*, z^*) = X^3$. Let $(x, y, z) \in X^3$ be arbitrary, then (i) implies that there exists $(u, v, w) \in X^3$ such that $(x, y, z), (x_0, y_0, z_0) \in X_R(u, v, w)$. From $(x_0, y_0, z_0) \in X_R(u, v, w)$ and (ii) it follows that for $n \in \mathbb{N}$

$$(f^n(x_0, y_0, z_0), f^n(y_0, x_0, y_0), f^n(z_0, y_0, x_0)) \in X_R(f^n(u, v, w), f^n(v, u, v), f^n(w, v, u)).$$

Also by using (v) we have

$$d(f^n(x_0, y_0, z_0), f^n(u, v, w)) \leq \frac{k^n}{3}[d(x_0, u) + d(y_0, v) + d(z_0, w)]$$

$$d(f^n(y_0, x_0, y_0), f^n(v, u, v)) \leq \frac{k^n}{3}[d(y_0, v) + d(x_0, u) + d(y_0, v)]$$

$$d(f^n(x_0, y_0, x_0), f^n(w, v, u)) \leq \frac{k^n}{3}[d(x_0, u) + d(y_0, v) + d(z_0, w)].$$

From this and the fact that $(x_0, y_0, z_0) \in A_f(x^*, y^*, z^*)$, which this implies that $A_f(x^*, y^*, z^*) = X^3$. Now as the proof of the Theorem 3.4 we obtain that f is a Picard operator.

Theorem 3.6 is still valid for a mapping without the orbitally continuous property, assuming an additional hypothesis on X.

Theorem 3.8. *Let (X, d) be a metric space and R be a reflexive relation on X. If $f : X^3 \to X$ is a mapping such that:*

(i) f having the mixed $R-$monotone property on X.

(ii) (X, d) be a complete metric space.

(iii) f having a $R-$tripled fixed point. i.e., there exists $(x_0, y_0, z_0) \in X^3$ such that $f \times f \times f(x_0, y_0, z_0) \in X_R(x_0, y_0, z_0)$.

(iv) There exists $k \in [0, 1)$ such that

$$d(f(x, y, z), f(p, q, r)) \leq \frac{k}{3}[d(x, u) + d(y, v) + d(z, r)].$$

(v) If a $R-$monotone sequence $\{(x_n, y_n, z_n)\}_{n\in\mathbb{N}} \to (x, y, z)$, then $(x_n, y_n, z_n) \in X_R(x, y, z)$ for all $n \in \mathbb{N}$. Then,

(a) There exists $x^*, y^*, z^* \in X$ such that $f(x^*, y^*, z^*) = x^*$, $f(y^*, x^*, y^*) = y^*$ and $f(z^*, y^*, x^*) = z^*$.

(b) The sequences $\{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ defined by

$$x_{n+1} = f(x_n, y_n, z_n),$$

$$y_{n+1} = f(y_n, x_n, y_n)$$

and

$$z_{n+1} = f(z_n, y_n, x_n)$$

converges respectively to x^*, y^* and z^*.

(c) The error estimation is given by

$$\max_{n\in\mathbb{N}}\{d(x_n, x^*), d(y_n, y^*), d(z_n, z^*)\}$$
$$\leq \frac{k^n}{3(1-k)}[d(f(x_0, y_0, z_0), x_0) + d(f(y_0, x_0, y_0), y_0) + d(f(z_0, y_0, x_0), z_0)].$$

Proof. Following the proof of Theorem 3.6, we only have to show that

$$f(x^*, y^*, z^*) = x^*,$$

$$f(y^*, x^*, y^*) = y^*$$

and

$$f(z^*, y^*, x^*) = z^*.$$

Since $f^n(x_0, y_0, z_0) \to x^*$, $f^n(y_0, x_0, y_0) \to y^*$ and $f^n(z_0, y_0, x_0) \to z^*$, using (v), we get

$$\begin{aligned}
d(f(x^*, y^*, z^*), x^*) &\leq d(f(x^*, y^*, z^*), f^{n+1}(x_0, y_0, z_0)) + d(f^{n+1}(x_0, y_0, z_0), x^*) \\
&\leq d(f(x^*, y^*, z^*), f(f^n(x_0, y_0, z_0), f^n(y_0, x_0, y_0), f^n(z_0, y_0, x_0))) \\
&\quad + d(f^{n+1}(x_0, y_0, z_0), x^*) \\
&\leq \frac{k}{3}[d(x^*, f^n(x_0, y_0, z_0)) + d(y^*, f^n(y_0, x_0, y_0)) \\
&\quad + d(z^*, fn(z_0, y_0, x_0))] + d(fn + 1(x_0, y_0, z_0), x^*) \qquad (3.3)
\end{aligned}$$

$$\begin{aligned}
&\frac{k}{3}(x^*, f^n(x_0, y_0, z_0)) + d(y^*, f^n(y_0, x_0, y_0)) \\
&+ d(z^*, f^n(z_0, y_0, x_0))] + d(f^{n+1}(x_0, y_0, z_0), x^*) \to 0.
\end{aligned}$$

as $n \to \infty$, we have This implies that $f(x^*, y^*, z^*) = x^*$. Similarly to the previous case, we can prove that $f(y^*, x^*, y^*) = y^*$ and $f(z^*, y^*, x^*) = z^*$.

Alternatively, if we know that in Theorem 3.6 (resp. Theorem 3.8), the element $(x_0, y_0, z_0) \in X^3$ is such that $(x_0, y_0, z_0) \in R$, then we can also demonstrate that the components x^*, y^*, z^* of the tripled fixed point are indeed the same.

Theorem 3.9. *In addition to the hypothesis of Theorem 3.6 (resp. Theorem 3.8), suppose that $(x_0, y_0, z_0) \in X3$ is such that $(x_0, y_0, z_0) \in R$. Then $x^* = y^* = z^*$.*

Proof. If $(x_0, y_0, z_0) \in R$, then $(x_0, y_0, z_0) \in X_R(y_0, x_0, y_0) \cap X_R(z_0, y_0, x_0)$, so from the mixed $R-$monotone of f , it follows that

$$(f(x_0, y_0, z_0), f(y_0, x_0, y_0), f(z_0, y_0, x_0)) \in X_R(f(x_0, y_0, z_0), f(y_0, x_0, y_0), f(z_0, y_0, x_0)).$$

Further, we can easily verify that for any $n \in \mathbb{N}$,

$$(f^{n-1}(x_0, y_0, z_0), f^{n-1}(y_0, x_0, y_0), f^{n-1}(z_0, y_0, x_0))$$

in

$$X_R(f^{n-1}(x_0, y_0, z_0), f^{n-1}(y_0, x_0, y_0), f^{n-1}(z_0, y_0, x_0)).$$

Also by using the contractivity property of f , we obtain

$$\begin{aligned}
& d(f^n(x_0, y_0, z_0), f^n(y_0, x_0, y_0), f^n(z_0, y_0, x_0)) \\
= \; & d(f(f^{n-1}(x_0, y_0, z_0)), f(f^{n-1}(y_0, x_0, y_0)), f(f^{n-1}(z_0, y_0, x_0))) \\
\leq \; & kd(f^{n-1}(x_0, y_0, z_0), f^{n-1}(y_0, x_0, y_0), f^{n-1}(z_0, y_0, x_0)) \\
\leq \; & k^2 d(f^{n-2}(x_0, y_0, z_0), f^{n-2}(y_0, x_0, y_0), f^{n-2}(z_0, y_0, x_0)) \\
& \vdots \\
\leq \; & k^n d(x_0, y_0, z_0) \to 0(n \to \infty).
\end{aligned}$$

This implies that

$$x^* = \lim_{n\to\infty} f^n(x_0, y_0, z_0),$$

$$y^* = \lim_{n\to\infty} f^n(y_0, x_0, y_0)$$

and

$$z^* = \lim_{n\to\infty} f^n(z_0, y_0, x_0).$$

4. An Application

In this section, on the basis of the tripled fixed point theorems in section 3, we study the nonlinear matrix equation

$$X = Q + \Sigma_{i=1}^m A_i^* G(X) A_i - \Sigma_{j=1}^n B_j^* K(X) B_j - \Sigma_{r=1}^t C_r^* L(X) C_r \tag{4.1}$$

where Q is a positive definite matrix, A_i, B_j, C_r are arbitrary $n \times n$ matrices and G, K, L are three continuous order preserving maps from $H(n)$ into $P(n)$ such that $G(0) = K(0) = L(0) = 0$.

In this section we will use the following notation: $M(n)$ denotes the set of all $n \times n$ complex matrices. $H(n) \subset M(n)$ the set of all $n \times n$ Hermitian matrices and $P(n) \subset H(n)$ is the set of all $n \times n$ positive matrices. Instead of $X \in P(n)$ we will also write $X > 0$. Furthermore, $X \geq 0$ means that X is positive semi-definite. Moreover, in $H(n)$, if we define $X \leq Y$ then $H(n)$ is partially ordered set and for any $X, Y \in H(n)$ there is a greatest lower bound and least upper bound. Therefore for any $(X, Y, Z), (A, B, C) \in (H(n))^3$ there exists $(U, V, W) \in (H(n))^3$ such that $(X, Y, Z), (A, B, C) \in (H(n))^3 \leq (U, V, W)$. We also denote $\|.\|$ the spectral norm i.e., $\|A\| =, \sqrt{\lambda - (A * A)}$ where $\lambda - (A * A)$ is the largest eigenvalue of $A * A$. We will use the metric induced by the trace norm $\|.\|_1$ defined by $\|A\|_1 = \Sigma_{j=1}^n s_j(A)$, where $s_j(A), j = 1, 2, \ldots, n$ are the singular values of A. The set $H(n)$ endowed with this norm is a complete metric space. In [12,13,28], the authors considered matrix equations and established the existence and uniqueness of positive definite solutions. Matrix equations of type Eq.4.1 often arise from many areas, such as ladder networks [2,3], dynamic programming [17,27], control theory [15,18].

The following lemmas will be useful in the study of the matrix equations, which is generalized form of Lemma- 3.1 in [28].

Lemma 4.1. *Let $A \geq 0$, $B \geq 0$ and $C \geq 0$ be $n \times n$ matrices, then $0 \leq tr(ABC) \leq \|A\| tr(BC) \leq \|A\|\|B\| tr(C)$.*

Proof. It is well known that the eigenvalues of the product of three positive semi-definite matrices are nonnegative. In particular, $tr(ABC) \geq 0$. Furthermore, since $A \leq \|A\| I_n$, we have

$$0 \leq tr((\|A\| - A)BC) = tr((\|A\|B - AB)C) = \|A\| tr(BC) - tr(ABC),$$

which completes the proof.

Lemma 4.2. *Let $A \in H(n)$ satisfy $A < I$, then $\|A\| < 1$.*

In total of this section if, we define the mapping $F : (H(n))^3 \to H(n)$ by

$$F(X, Y, Z) = Q + \Sigma_{i=1}^m A_i^* G(X) A_i - \Sigma_{j=1}^n B_j^* K(X) B_j - \Sigma_{r=1}^t C_r^* L(X) C_r$$

where $Q \in P(n)$, $A_i, B_j, C_r \in M(n)$ and G, K, L are three continuous order preserving maps. Then F is well defined and and having the mixed $R-$monotone property on $H(n)$ and the fixed points of F are the solutions of Eq.4.1. In the following theorem we first discuss existence of a tripled fixed point of F in $(H(n))^3$.

Theorem 4.3. *Let $Q \in P(n)$. Assume there is a positive number M such that:*
(i) For every $(X, Y, Z) \in H(n)_{\leq}(U, V, W)$,

$$|tr(G(U) - G(X))| \leq \frac{1}{M} |tr(U - X)|,$$

$$|tr(K(Y) - K(V))| \leq \frac{1}{M} |tr(Y - V)|,$$

$$|tr(L(Z) - L(W))| \leq \frac{1}{M} |tr(Z - W)|.$$

(ii) $\Sigma_{i=1}^m A_i^* A_i < \frac{M}{3} I_n$, $\Sigma_{j=1}^n B_j^* B_j < \frac{M}{3} I_n$, and $\Sigma_{r=1}^t C_r^* C_r < \frac{1}{M} I_n$,
(iii) $\Sigma_{i=1}^m A_i^*(3Q) A_i < Q$, $\Sigma_{j=1}^n B_j^*(3Q) B_j < Q$, and $\Sigma_{r=1}^t C_r^*(3Q) C_r < Q$.
Then there exist $X^*, Y^*, Z^* \in H(n)$ such that

$$F(X^*, Y^*, Z^*) = X^*,$$

$$F(Y^*, X^*, Y^*) = Y^*$$

and

$$F(Z^*, Y^*, X^*) = Z^*.$$

Proof. Let $(X, Y, Z) \in H(n)_{\leq}(U, V, W)$. Then $G(X) \leq G(U)$, $K(Y) \geq K(V)$ and $L(Z) \leq L(W)$. Therefore

$$\begin{aligned}
& \|F(U,V,W)-F(X,Y,Z)\|_1 \\
= \; & tr(F(U,V,W)-F(X,Y,Z)) \\
= \; & \Sigma_{i=1}^m tr(A^*(G(U)-G(X))A_i)+\Sigma_{j=1}^n trB^*(K(Y)-K(V))B_j) \\
& +\Sigma_{r=1}^n tr(C^*(L(Z)-L(W))C_r) \\
= \; & \Sigma_{i=1}^m tr(A^*A_i(G(U)-G(X)))+\Sigma_{j=1}^n trB^*B_j(K(Y)-K(V))) \\
& +\Sigma_{r=1}^n tr(C^*C_r(L(Z)-L(W))) \\
= \; & tr\Sigma_{i=1}^m(A^*A_i(G(U)-G(X)))+tr\Sigma_{j=1}^n B^*B_j(K(Y)-K(V))) \\
& +tr\Sigma_{r=1}^n(C^*C_r(L(Z)-L(W))) \\
\leq \; & \|\Sigma_{i=1}^m(A^*A_i\|\|(G(U)-G(X)))\|_1+\|\Sigma_{j=1}^n B^*B_j\|\|(K(Y)-K(V)))\|_1 \\
& +\|\Sigma_{r=1}^n(C^*C_r\|\|(L(Z)-L(W)))\|_1 \\
\leq \; & \frac{\|\Sigma_{i=1}^m(A^*A_i\|}{3}\|U-X\|_1+\frac{\|\Sigma_{j=1}^n B^*B_j\|}{3}\|Y-V\|_1 \\
& +\frac{\|\Sigma_{r=1}^n(C^*C_r\|}{3}\|Z-W\|_1 \\
\leq \; & \frac{\lambda}{3}(\|U-X\|_1+\|Y-V\|_1+\|Z-W\|_1)
\end{aligned}$$

where $\lambda=\max\{\frac{\|\Sigma_{i=1}^m(A^*A_i\|}{M},\frac{\|\Sigma_{j=1}^n B^*B_j\|}{M},\frac{\|\Sigma_{r=1}^n(C^*C_r\|}{M}\}$. From Lemma 4.2, we have $\lambda<1$. Thus, the contractive condition of Theorem 3.6 is satisfied for all $(X,Y,Z)\in H(n)_{\leq}(U,V,W)$. Moreover, F has the mixed $\leq -$monotone property of $H(n)$ and from (iii), we have $F\times F\times F(3Q,0,0)\in H(n)_{\leq}(3Q,0,0)$. Now from the Theorem 3.6 there exist $X^*,Y^*,Z^*\in H(n)$ such that $F(X^*,Y^*,Z^*)=X^*$, $F(Y^*,X^*,Y^*)=Y^*$ and $F(Z^*,Y^*,X^*)=Z^*$.

Theorem 4.4. *Let$Q\in P(n)$ also $\Sigma_{i=1}^m A_i^*(3Q)A_i<Q$, $\Sigma_{j=1}^n B_j^*(3Q)B_j<Q$, and $\Sigma_{r=1}^t C_r^*(3Q)C_r<Q$.*
Then 4.1 has atleast one positive definite solution in

$$[\min\{F(3Q,0,0),F(0,3Q,0),F(0,0,3Q)\},\max\{F(3Q,0,0),F(0,3Q,0),F(0,0,3Q)\}].$$

Proof. Define a mapping $S:P(n)\to P(n)$ by

$$S(X)=Q+\Sigma_{i=1}^m A_i^*G(X)A_i-\Sigma_{j=1}^n B_j^*K(X)B_j-\Sigma_{r=1}^t C_r^*L(X)C_r$$

Now we claim that

$$S([\min\{F(3Q,0,0),F(0,3Q,0),F(0,0,3Q)\},\max\{F(3Q,0,0),F(0,3Q,0),F(0,0,3Q)\}])$$

is subset of

$$[\min\{F(3Q,0,0),F(0,3Q,0),F(0,0,3Q)\},\max\{F(3Q,0,0),F(0,3Q,0),F(0,0,3Q)\}].$$

Indeed, if

$$min\{F(3Q,0,0),F(0,3Q,0),F(0,0,3Q)\}\leq X\leq\max\{F(3Q,0,0),F(0,3Q,0),F(0,0,3Q)\},$$

then we have $X \leq 3Q$. Applying G, K, L, we can easily show that, S maps the compact convex set

$$[\min\{F(3Q,0,0), F(0,3Q,0), F(0,0,3Q)\}, \max\{F(3Q,0,0), F(0,3Q,0), F(0,0,3Q)\}]$$

into itself. Since S is continuous, it follows from Schauder's fixed point theorem that S has at least one fixed in this set. However, fixed points of S are solutions of eq. 4.1.

Theorem 4.5. *Under the assumption Theorem 4.3, the eq.4.1 has an unique solution* $\hat{X}$.

Proof. Since for every $X, Y, Z \in H(n)$ there is a greatest lower bound and a least upper bound, for any $(X, Y, Z), (A, B, C) \in (H(n))^3$ there exists $(U, V, W) \in (H(n))^3$ such that $(X, Y, Z), (A, B, C) \in H(n)_{\leq}(U, V, W)$. Therefore, we deduce from Theorem 3.7 that $X^*, Y^*, Z^* \in H(n)$ in Theorem 4.3 is unique and $X^* = Y^* = Z^* = \hat{X}$.

Theorem 4.6. *Let $Q \in P(n)$. Then under the assumption Theorem 4.3,*
(i) Eq. 4.1 has an unique positive definite solution

$$\hat{H} \in [\min\{F(3Q,0,0), F(0,3Q,0), F(0,0,3Q)\}, \max\{F(3Q,0,0), F(0,3Q,0), F(0,0,3Q)\}].$$

(ii) The sequences $\{X_n\}_{n\in\mathbb{N}}, \{Y_n\}_{n\in\mathbb{N}}$ and $\{Z_n\}_{n\in\mathbb{N}}$ defined by $X_0 = 3Q, Y_0 = 0, Z_0 = 0$ and

$$X_{n+1} = Q + \Sigma_{i=1}^{m} A_i^* G(X_n) A_i - \Sigma_{j=1}^{n} B_j^* K(Y_n) B_j - \Sigma_{r=1}^{t} C_r^* L(Z_n) C_r,$$

$$Y_{n+1} = Q + \Sigma_{i=1}^{m} A_i^* G(Y_n) A_i - \Sigma_{j=1}^{n} B_j^* K(X_n) B_j - \Sigma_{r=1}^{t} C_r^* L(Y_n) C_r,$$

$$Z_{n+1} = Q + \Sigma_{i=1}^{m} A_i^* G(Z_n) A_i - \Sigma_{j=1}^{n} B_j^* K(Y_n) B_j - \Sigma_{r=1}^{t} C_r^* L(X_n) C_r,$$

converges to $\hat{H}$ and the error estimation is given by

$$\max_{n\in\mathbb{N}}\{\|X_n - \hat{X}\|_1, \|Y_n - \hat{X}\|_1, \|Z_n - \hat{X}\|_1\} \leq \frac{\lambda^n}{3(1-\lambda)}(\|X_1 - X_0\|_1 + \|Y_1 - Y_0\|_1 + \|Z_1 - Z_0\|_1),$$

for all $n \in \mathbb{N}$, where $\lambda = \max\{\frac{\|\Sigma_{i=1}^{m}(A^* A_i\|}{M}, \frac{\|\Sigma_{j=1}^{n} B^* B_j\|}{M}, \frac{\|\Sigma_{r=1}^{n}(C^* C_r\|}{M}\}$.

Proof. By Theorem 4.4, eq.4.1 has atleast one positive definite solution in

$$[\min\{F(3Q,0,0), F(0,3Q,0), F(0,0,3Q)\}, \max\{F(3Q,0,0), F(0,3Q,0), F(0,0,3Q)\}].$$

and by Theorem 26 this equation having a unique solution in $H(n)$. Thus this solution must be in the set. further the proof of (ii) follows from part (c) of Theorem 3.6.

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