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The q-difference Operator Associated with the Multivalent Function Bounded by Conical Sections

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ABSTRACT: In this paper we obtain some inclusion relations of k - starlike functions, k - uniformly convex functions and quasi-convex functions. Furthermore, we obtain coefficient bounds for some subclasses of fractional q-derivative multivalent functions together with generalized Ruscheweyh derivative.

Key Words: Multivalent function, Starlike function, Convex function, Subordination, k-uniformly starlike functions, k-uniformly convex functions, Quasi- convex functions, q-difference function.

Contents

1. Introduction

Let $\mathcal{A}_p$ denote the class of all analytic functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \tag{1.1}$$

defined on the open unit disk

$$\mathbb{U} = \{z \ : \ z \in \mathbb{C} \ : \ |z| < 1\},$$

and let $\mathcal{A}_1 := \mathcal{A}$. For $f(z)$ given by (1.1) and $g(z)$ given by

$$g(z) = z^p + \sum_{n=p+1}^{\infty} b_n z^n,$$

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their convolution (or Hadamard product), denoted by $(f * g)$, is defined as

$$(f * g)(z) = z^p + \sum_{n=p+1}^{\infty} a_n b_n z^n.$$

With a view to recalling the principle of subordination between analytic functions, let the functions $f(z)$ and $g(z)$ be analytic in $\mathbb{U}$. Then we say that the function $f(z)$ is subordinate to $g(z)$ if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0, \qquad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathbb{U}),$$

we denote this subordination by

$$f \prec g \quad \text{or} \quad f(z) \prec g(z) \qquad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0), \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

If $f \in \mathcal{A}_p$ satisfies

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) \geq k\left|\frac{zf'(z)}{f(z)} - p\right| \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some $\alpha(0 \leq \alpha < 1)$ and $k(0 \leq k < \infty)$, then we say that $f(z)$ is *k-uniformly starlike* of order α. We denote this class by $k - \mathcal{ST}_p(\alpha)$. If $f \in \mathcal{A}_p$ satisfies

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) \geq k\left|\frac{zf''(z)}{f'(z)} - (p-1)\right| \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some $\alpha(0 \leq \alpha < 1)$ and $k(0 \leq k < \infty)$, then we say that $f(z)$ is *k-uniformly convex* of order α. We denote this class by $k - \mathcal{UCV}_p(\alpha)$. (see Al- Kharsani and Al-Hajiry [3]). We observe that, if $p = 1$ and $k = 0$, the classes $k - \mathcal{ST}_p(\alpha)$ and $k - \mathcal{UCV}_p(\alpha)$ reduces to the well known classes $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ of starlike and convex functions of order α in $\mathbb{U}$, respectively. If we put $p = 1$ and $k = 1$ in the inequality (1.3), we get the class $\mathcal{UCV}$ introduced by Goodman [6] and studied extensively by Rønning [21] and independently by Ma and Minda [22]. Further, when $p = 1$ and $\alpha = 0$, the class $k - \mathcal{ST}_p(\alpha)$ reduces to the class $k - \mathcal{ST}$ studied by Rønning [21].

Conical sections introduced and studied by Kanas ([9], [10]), Kanas and Wisniowska ([13]–[15]), and recently studied by Kanas and Rodacanu [16] and Sim et al. [23].

Al-Kharsani [2] extended this conic sections for p-valent functions for $0 \leq k < \infty$ and defined the domain $\Omega_{k,\alpha}$ as follows:

$$\Omega_{k,\alpha} = \{u + iv : (u - \alpha)^2 > k^2(u - p)^2 + k^2v^2\}$$

with $p(z)=\frac{f'(z)}{f(z)}$ or $p(z)=1+\frac{f'(z)}{f(z)}$ and considering the functions which map $\mathbb{U}$ onto the conic domain $\Omega_{k,\alpha}$ such that $p(z)\in\Omega_{k,\alpha}$, we may rewrite the conditions (1.2) or (1.3) in the form $p(z)\prec Q_{k,\alpha}$. We note that the explicit form of function $Q_{k,\alpha}(z)$ for $k=0$ and $k=1$ are

$$Q_{0,\alpha}(z)=\frac{1+(1-2\alpha)z}{1-z}\quad (z\in\mathbb{U}),$$

and

$$Q_{1,\alpha}(z)=p+\frac{2(p-\alpha)}{\pi^2}\log^2\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\quad (z\in\mathbb{U}).$$

For $0<k<1$, we have

$$Q_{k,\alpha}(z)=\frac{(p-\alpha)}{1-k^2}\cos\{A(k)i\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\}-\frac{k^2p-\alpha}{1-k^2}\quad (z\in\mathbb{U}),$$

and

$$Q_{k,\alpha}(z)=\frac{(p-\alpha)}{k^2-1}\sin^2\left(\frac{\pi}{2\kappa(t)}\mathcal{F}\left(\frac{\sqrt{z}}{\sqrt{t}},t\right)\right)+\frac{k^2p-\alpha}{k^2-1}\quad (z\in\mathbb{U}),$$

for $k>1$, where $A(k)=\frac{2}{\pi}\arccos k$, and $\mathcal{F}(\omega,t)$ is the Legendre elliptic integral of the first kind

$$\mathcal{F}(\omega,t)=\int_0^{\omega}\frac{dx}{\sqrt{1-x^2}\sqrt{1-t^2x^2}},\quad \kappa(t)=\mathcal{F}(1,t)$$

and $t\in(0,1)$ is chosen such that $k=\cosh\frac{\pi\kappa'(t)}{4\kappa(t)}$ which maps the unit disk $\mathbb{U}$ onto the conic domains are, respectively for $0<k<1$,

$$\Omega_{k,\alpha}=\left\{u+iv:\left(\frac{u+\frac{k^2p-\alpha}{1-k^2}}{k(\frac{p-\alpha}{1-k^2})}\right)^2-\left(\frac{v}{\frac{p-\alpha}{\sqrt{1-k^2}}}\right)^2>1\right\},$$

and for $k>1$,

$$\Omega_{k,\alpha}=\left\{u+iv:\left(\frac{u+\frac{k^2p-\alpha}{k^2-1}}{k(\frac{p-\alpha}{k^2-1})}\right)^2+\left(\frac{v}{\frac{p-\alpha}{\sqrt{k^2-1}}}\right)^2<1\right\}.$$

By virtue of

$$p(z)=\frac{zf'(z)}{f(z)}\prec Q_{k,\alpha}(z)\text{ and }p(z)=1+\frac{zf''(z)}{f'(z)}\prec Q_{k,\alpha}(z)\tag{1.4}$$

and the properties of the domains, we have

$$\Re(p(z))>\Re(Q_{k,\alpha}(z))>\frac{kp+\alpha}{k+1}.\tag{1.5}$$

Define $\mathcal{UCC}(k,\alpha,\beta)$ to be the family of functions $f \in \mathcal{A}_p$ such that

$$\frac{zf'(z)}{g(z)} \prec Q_{k,\alpha}(z) \quad \text{for some} \quad g(z) \in k - \mathcal{ST}(\alpha). \tag{1.6}$$

Similarly, we define $\mathcal{UQC}(k,\alpha,\beta)$ to be the family of functions $f \in \mathcal{A}_p$ such that

$$\frac{(zf'(z))'}{g'(z)} \prec Q_{k,\alpha}(z), \quad \text{for some} \quad g(z) \in k - \mathcal{UCV}(\alpha). \tag{1.7}$$

We note that $\mathcal{UCC}(0,\alpha,\beta)$ is the class of close-to-convex of univalent functions of order α and type β and $\mathcal{UQC}(0,\alpha,\beta)$ is the class of quasi-convex univalent functions of order α and type β.

2. Preliminaries

The theory of relativity, or simply relativity in physics, usually encompasses two theories by Albert Einstein: special relativity and general relativity. Special relativity applies to elementary particles and their interactions, whereas general relativity applies to the cosmological and astrophysical realm, including astronomy. Special relativity theory rapidly became a significant and necessary tool for theorists and experimentalists in the new fields of atomic physics, nuclear physics, and quantum mechanics. In our q-difference operator plays an vital role in the theory of hypergeometric series and quantum physics.

Now we study about some subclasses of fractional q-derivative multivalent function together with Ruscheweyh derivative introduced and studied by Wongsaijai and Sukantamala [24].

In the theory of q-calculus, the q-shifted factorial is defined for α, $q \in \mathbb{C}$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as a product of n factors by

$$(\alpha;q)_n = \begin{cases} 1, & n = 0 \\ (1-\alpha)(1-\alpha q)\dots(1-\alpha q^{n-1}), & n \in \mathbb{N} \end{cases} \tag{2.1}$$

and in terms of the basic analogue of the gamma function

$$(q^{\alpha};q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \quad (n > 0)$$

where the q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty(1-q)^{1-x}}{(q^x;q)_\infty}, \quad (0 < q < 1).$$

We note that, if $|q| < 1$, the q-shifted factorial (2.1) remains meaningful for $n = \infty$ as a convergent infinite product:

$$(\alpha : q)_\infty = \prod_{k=0}^{\infty}(1-\alpha q^k).$$

It is well known that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1$, where $\Gamma(x)$ is the ordinary Euler gamma function. In view of the relation

$$lim_{q\to -1}\frac{(q^\alpha;q)_n}{(1-q)^n}=(\alpha)_n,$$

we observe that the q-shifted factorial (2.1) reduces to the familiar Pochhammer symbol $(\alpha)_n=\alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)$.

Let $\mu\in\mathbb{C}$ be fixed. A set $A\subset\mathbb{C}$ is called a μ-geometric set if $z\in\mathbb{C}$, $\mu z\in A$. Let f be a function defined on a q-geometric set. The Jackson's q-derivative and q-integral of a function on subset of $\mathbb{C}$ are, respectively, given by (see Gasper and Rahman [5])

$$D_q f(z)=\frac{f(z)-f(zq)}{z(1-q)},\quad (z\neq 0, q\neq 0), \tag{2.2}$$

and

$$\int_0^z f(t)d_qt = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k). \tag{2.3}$$

Definition 2.1. *[20] The fractional q-integral operator $I^\delta_{q,z}$ of a function $f(z)$ of order $\delta(\delta>0)$ is defined by*

$$I^\delta_{q,z}=D^{-\delta}_{q,z}=\frac{1}{\Gamma_q(\delta)}\int_0^z (z-tq)_{1-\delta}f(t)d_qt, \tag{2.4}$$

where $f(z)$ is analytic in a simply connected region of the z-plane containing the origin. Here, the term $(z-tq)_{1-\delta}$ is a binomial function defined by

$$(z-tq)_{1-\delta}=z^{\delta-1}\prod_{k=0}^{\infty}\left[\frac{1-(tq/z)q^k}{1-(tq/z)q^{\delta+k-1}}\right]=z^\delta_1\Phi_0\left[q^{-\delta+1};-;q;\frac{tq^\delta}{z}\right]. \tag{2.5}$$

According to [5], the series ${}_1\Phi_0[\delta;-;q,z]$ is single-valued when $|arg(z)|<\pi$, therefore, the function $(z-tq)_{1-\delta}$ in (2.5) is single-valued when $|arg(-tq^\delta/z)|<\pi$, $|arg(tq^\delta/z)|<1$ and $|arg(z)|<\pi$.

Definition 2.2. *[20] The fractional q-integral operator $D^\delta_{q,z}$ of a function $f(z)$ of order $\delta(0\le\delta<1)$ is defined by*

$$D^\delta_{q,z}=D_{q,z}I^{1-\delta}_{q,z}f(z)=\frac{1}{\Gamma_q(1-\delta)}D_q\int_0^z (z-tq)_{-\delta}f(t)d_qt, \tag{2.6}$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\alpha}$ is removed as in Definition 2.5 above. In particular, for $\delta=1$, we have $D^1_{q,z}f(z)\equiv D_qf(z)$.

Definition 2.3. *Under the Definition 2.6, the The fractional q-derivative for a function $f(z)$ of order δ is defined by*

$$D^\delta_{q,z}=D^m_{q,z}I^{m-\delta}_{q,z}f(z). \tag{2.7}$$

In addition, the q-differ-integral operator $\Omega^{\delta}_{q,p} : \mathcal{A}_p \to \mathcal{A}_p$, for $\delta < p+1$, $0 < q < 1$ and $n \in \mathbb{N}$, defined as follows:

$$\begin{aligned} \Omega^{\delta}_{q,p} &= \frac{\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)} z^{\delta} D^{\delta}_{q,z} \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)} a_k z^k, \end{aligned} \tag{2.8}$$

where $D^{\delta}_{q,z}$ represents, respectively, the fractional q-integral of $f(z)$ of order δ when $-\infty < \delta < 0$ and the fractional q-derivative for a function $f(z)$ of order δ when $0 \le \delta < p+1$.Clearly, we have $\Omega^{0}_{q,p} f(z) = f(z)$. Wongsaijai and Sukantamala [24] defined the generalized Ruschweyh type differential operator $\mathcal{R}^{\delta,m}_{q,p} : \mathcal{A}_p \to \mathcal{A}_p$, for $m \in \mathbb{N}$, as follows:

$$\mathcal{R}^{\delta,0}_{q,p} f(z) = f(z)$$

$$\mathcal{R}^{\delta,1}_{q,p} f(z) = \frac{z}{[p]_q} D_q(\Omega^{\delta}_{q,p} f(z)),$$

$$2\mathcal{R}^{\delta,2}_{q,p} f(z) = \frac{z}{[p]_q} D_q(\mathcal{R}^{\delta,1}_{q,p} f(z)) + \mathcal{R}^{\delta,1}_{q,p} f(z),$$

in general

$$(m+1)\mathcal{R}^{\delta,m+1}_{q,p} f(z) = \frac{z}{[p]_q} D_q(\mathcal{R}^{\delta,m}_{q,p} f(z)) + m\mathcal{R}^{\delta,m}_{q,p} f(z). \tag{2.9}$$

For $f \in \mathcal{A}_p$ given by (1.1), the equation (2.9) yields

$$\mathcal{R}^{\delta,m}_{q,p} f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)} \right)^m C_m \left(\frac{[k]_q}{[p]_q} \right) a_k z^k, \tag{2.10}$$

where

$$C_m(t) = \frac{t(t+1)\dots(t+m)}{m!}.$$

Lemma 2.4. *[4] Let γ, β be complex constants and $h(z)$ be univalently convex in the unit disc $\mathbb{U}$ with $h(0) = p$ and $\Re(\beta h(z)+\gamma) > 0$. Let $g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}$ be analytic in $\mathbb{U}$. Then*

$$g(z) + \frac{zg'(z)}{\beta g(z) + \gamma} \prec h(z) \quad \textit{implies} \quad g(z) \prec h(z).$$

Lemma 2.5. *[18] Let h be convex in the open unit disk $\mathbb{U}$ and let $E \ge 0$. Suppose $B(z)\,(z \in \mathbb{U})$ is analytic in $\mathbb{U}$ with $\Re(B(z)) > 0$. If $g(z)$ is analytic in $\mathbb{U}$ and $h(0) = g(0)$. Then*

$$Ez^2 g''(z) + B(z)g(z) \prec h(z) \quad \Rightarrow \quad g(z) \prec h(z). \tag{2.11}$$

Definition 2.6. *Let $Q_{k,\alpha}(z)$ be a univalent starlike function with respect to 1 which maps the open unit disk $\mathbb{U}$ onto a region in the conical domain and is symmetric with respect to the real axis, $Q_{k,\alpha}(0) = p$, $Q'_{k,\alpha}(0) > 0$. A function $f \in \mathcal{A}_p$ is in the class $\mathcal{S}^{\beta}_{p,b}(Q_{k,\alpha})$ if*

$$1 + \frac{1}{b}\left[(1 + i\tan\beta)\frac{1}{p}\left(z\frac{D_q(\mathcal{R}^{\delta,m}_{q,p} f(z))}{\mathcal{R}^{\delta,m}_{q,p} f(z)}\right) - i\tan\beta - 1\right] \prec Q_{k,\alpha}(z) \tag{2.12}$$

$(z \in \mathbb{U}; b \neq 0; -\frac{\pi}{2} < \beta < \frac{\pi}{2})$.

Let Ω be the class of analytic functions of the form

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \cdots \tag{2.13}$$

in the open unit disk $\mathbb{U}$ satisfying $|w(z)| < 1$. To prove our main result, we need the following.

Lemma 2.7. *[1] If $w \in \Omega$, then*

$$\left|w_2 - t w_1^2\right| \leq \begin{cases} -t & if \quad t < -1 \\ 1 & if \quad -1 \leq t \leq 1 \\ t & if \quad t > 1. \end{cases} \tag{2.14}$$

When $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then equality holds if and only if $w(z) = z^2$ or one of its rotations. Equality holds for $t = -1$ if and only if

$$w(z) = z\frac{\lambda + z}{1 + \lambda z} \qquad (0 \leq \lambda \leq 1) \tag{2.15}$$

or one of its rotations, while for $t = 1$, the equality holds if and only if

$$w(z) = -z\frac{\lambda + z}{1 + \lambda z} \qquad (0 \leq \lambda \leq 1) \tag{2.16}$$

or one of its rotations.

Although the above upper bound is sharp, it can be improved as follows when $-1 < t < 1$:

$$\begin{aligned} &\left|w_2 - t w_1^2\right| + (1 + t)\left|w_1\right|^2 \leq 1 \qquad (-1 < t \leq 0), \\ &\left|w_2 - t w_1^2\right| + (1 - t)\left|w_1\right|^2 \leq 1 \qquad (0 < t < 1). \end{aligned} \tag{2.17}$$

Lemma 2.8. *[17] If $w \in \Omega$, then for any complex number t,*

$$\left|w_2 - t w_1^2\right| \leq \max\{1; |t|\}. \tag{2.18}$$

The result is sharp for the functions $w(z) = z$ or $w(z) = z^2$.

3. Inclusion Relation Problems

Theorem 3.1. *Let $f(z) \in \mathcal{A}(p)$. If $\mathcal{R}_{q,p}^{\delta,m} f(z) \in k - \mathcal{ST}_p(\alpha)$. Then $\mathcal{R}_{q,p}^{\delta,m+1} f(z) \in k - \mathcal{ST}_p(\alpha)$.*

Proof: Let

$$s(z) = z\frac{D_q(\mathcal{R}_{q,p}^{\delta,m} f(z))}{\mathcal{R}_{q,p}^{\delta,m} f(z)} \quad (z \in \mathcal{U}) \tag{3.1}$$

where s is an analytic function in $\mathcal{U}$ with $s(0) = p$. By using (2.9), we get

$$s(z) + m[p]_q = (m+1)[p]_q \frac{\mathcal{R}_{q,p}^{\delta,m+1} f(z)}{\mathcal{R}_{q,p}^{\delta,m} f(z)}.$$

Differentiating logarithmically with respect to z and multiplying by z, we obtain

$$s(z) + \frac{zs'(z)}{s(z) + m[p]_q} = \frac{zD_q(\mathcal{R}_{q,p}^{\delta,m+1} f(z))}{\mathcal{R}_{q,p}^{\delta,m+1} f(z)}.$$

From this argument, we may write

$$s(z) + \frac{zs'(z)}{s(z) + m[p]_q} \prec Q_{k,\alpha}(z).$$

Therefore, the theorem follows Lemma 2.4 and the condition (1.5) since $Q_{k,\alpha}(z)$ is univalent and convex in $\mathcal{U}$ and $\Re(Q_{k,\alpha}(z)) > \frac{kp+\alpha}{k+1}$. □

Theorem 3.2. *Let $f \in \mathcal{A}_p$. If $\mathcal{R}_{q,p}^{\delta,m} f(z) \in k - \mathcal{UCV}_p(\alpha)$, then $\mathcal{R}_{q,p}^{\delta,m+1} f(z) \in k - \mathcal{UCV}_p(\alpha)$.*

Proof: From the equations (1.2), (1.3) and the Theorem 3.1 we have

$$\begin{aligned}
\mathcal{R}_{q,p}^{\delta,m} f(z) \in k - \mathcal{UCV}_p(\alpha) &\Leftrightarrow zD_q(\mathcal{R}_{q,p}^{\delta,m} f(z)) \in k - \mathcal{ST}_p(\alpha) \\
&\Leftrightarrow \mathcal{R}_{q,p}^{\delta,m} zD_q(f(z)) \in k - \mathcal{ST}_p(\alpha) \\
&\Rightarrow \mathcal{R}_{q,p}^{\delta,m+1} zD_q(f(z)) \in k - \mathcal{ST}_p(\alpha) \\
&\Leftrightarrow \mathcal{R}_{q,p}^{\delta,m+1} f(z) \in k - \mathcal{UCV}_p(\alpha)
\end{aligned}$$

and the proof is complete. □

Theorem 3.3. *Let $f \in \mathcal{A}$. If $\mathcal{R}_{q,p}^{\delta,m} f(z) \in \mathcal{UCC}_p(k, \alpha, \beta)$, then $\mathcal{R}_{q,p}^{\delta,m+1} f(z) \in \mathcal{UCC}_p(k, \alpha, \beta)$.*

Proof: Since

$$\mathcal{R}_{q,p}^{\delta,m} f(z) \in \mathcal{UCC}_p(k, \alpha, \beta)$$

$$\frac{zD_q(\mathcal{R}_{q,p}^{\delta,m} f(z))}{k(z)} \prec Q_{k,\alpha}(z), \text{for} \quad \text{some} \quad k(z) \in k - \mathcal{ST}_p(\beta).$$

For $g(z)$ such that $\mathcal{R}_{q,p}^{\delta,m} g(z) = k(z)$ we have

$$\frac{zD_q(\mathcal{R}_{q,p}^{\delta,m} f(z))}{\mathcal{R}_{q,p}^{\delta,m} g(z)} \prec Q_{k,\alpha}(z). \tag{3.2}$$

Letting

$$h(z) = \frac{zD_q(\mathcal{R}_{q,p}^{\delta,m} f(z))}{\mathcal{R}_{q,p}^{\delta,m} g(z)} \quad \text{and } H(z) = \frac{zD_q(\mathcal{R}_{q,p}^{\delta,m} g(z))}{\mathcal{R}_{q,p}^{\delta,m} g(z)}.$$

We observe that $h(z)$ and $H(z)$ are analytic in $\mathbb{U}$ and $h(0) = H(0) = p$.
Now, by Theorem 3.1,

$$\mathcal{R}_{q,p}^{\delta,m} g(z) \in k - \mathcal{ST}_p(\beta) \quad \text{and} \quad \Re(H(z)) > \frac{kp+\beta}{k+1}.$$

Also note that

$$zD_q(\mathcal{R}_{q,p}^{\delta,m} f(z)) = (\mathcal{R}_{q,p}^{\delta,m} g(z))h(z). \tag{3.3}$$

Differentiating both sides of (3.3), we obtain

$$\frac{zD_q(zD_q(\mathcal{R}_{q,p}^{\delta,m} f(z)))}{\mathcal{R}_{q,p}^{\delta,m} g(z)} = z\frac{D_q(\mathcal{R}_{q,p}^{\delta,m} g(z))}{\mathcal{R}_{q,p}^{\delta,m} g(z)} h(z) + zD_q h(z) = H(z).h(z) + zD_q h(z). \tag{3.4}$$

Now using the identity (2.9),we obtain

$$\begin{aligned}
\frac{zD_q(\mathcal{R}_{q,p}^{\delta,m+1} f(z))}{\mathcal{R}_{q,p}^{\delta,m+1} g(z)} &= \frac{\mathcal{R}_{q,p}^{\delta,m+1}(zD_q(f(z)))}{\mathcal{R}_{q,p}^{\delta,m+1} g(z)} \\
&= \frac{zD_q(\mathcal{R}_{q,p}^{\delta,m} zD_q(f(z))) + m[p]_q \mathcal{R}_{q,p}^{\delta,m}(zD_q(f(z)))}{zD_q(\mathcal{R}_{q,p}^{\delta,m} g(z)) + m[p]_q \mathcal{R}_{q,p}^{\delta,m} g(z)} \\
&= \frac{\frac{zD_q(\mathcal{R}_{q,p}^{\delta,m} zD_q(f(z))}{\mathcal{R}_{q,p}^{\delta,m} g(z)} + m[p]_q \frac{\mathcal{R}_{q,p}^{\delta,m}(zD_q(f(z)))}{\mathcal{R}_{q,p}^{\delta,m} g(z)}}{\frac{zD_q(\mathcal{R}_{q,p}^{\delta,m} g(z))}{\mathcal{R}_{q,p}^{\delta,m} g(z)} + m[p]_q} \\
&= \frac{H(z)h(z) + zD_q h(z) + m[p]_q h(z)}{H(z) + m[p]_q} \\
&= h(z) + \frac{zD_q h(z)}{H(z) + m[p]_q}.
\end{aligned} \tag{3.5}$$

From (3.2), (3.4) and above equation, we conclude that

$$h(z) + \frac{zD_q h(z)}{H(z) + m[p]_q} \prec Q_{k,\alpha}(z).$$

On letting $E = 0$ and $B(z) = \frac{1}{H(z)+m[p]_q}$, we obtain

$$\Re(B(z)) = \frac{\Re(H(z) + m[p]_q)}{|H(z) + m[p]_q|^2} > 0$$

and the above inequality satisfies the conditions required by Lemma 2.5. Hence

$$h(z) \prec Q_{k,\alpha}(z)$$

and so the proof is complete. $\square$

Using similar argument as in Theorem 3.3, we can prove the following theorem.

Theorem 3.4. *Let $f \in \mathcal{A}$. If $\mathcal{R}_{q,p}^{\delta,m} f(z) \in \mathcal{UQC}_p(k,\eta,\beta)$, then $\mathcal{R}_{q,p}^{\delta,m+1} f(z) \in \mathcal{UQC}_p(k,\eta,\beta)$.*

4. Coefficient Bounds Problems

Theorem 4.1. *Let $Q_{k,\alpha}(z) = 1 + Q_1 z + Q_2 z^2 + \dots$, where Q_n's are real with $Q_1 > 0$ and $Q_2 \geq 0$. Let $m \in \mathbb{N}_0$, and*

$$\begin{aligned}
\sigma_1 &= \frac{A}{BpbQ_1^2}\left(-Q_1 + Q_2 - \frac{([p+1]_q + p)(1 + i\tan\beta)bQ_1^2}{A}\right),\\
\sigma_2 &= \frac{A}{BpbQ_1^2}\left(Q_1 + Q_2 - \frac{([p+1]_q + p)(1 + i\tan\beta)bQ_1^2}{A}\right),\\
\sigma_3 &= \frac{A}{BpbQ_1^2}\left(Q_2 - \frac{([p+1]_q + p)(1 + i\tan\beta)bQ_1^2}{A}\right).
\end{aligned}$$

where

$$A = (1 + i\tan\beta)^2([p+1] - p)^2 \left(\frac{\Gamma_q(p+1-\delta)\Gamma_q(p+2)}{\Gamma_q(p+1)\Gamma_q(p+3-\delta)}\right)^{2m} C_m^2 \left(\frac{[p+1]_q}{[p]_q}\right)$$

$$B = (1+i\tan\beta)([p+2]-p)([p+2]_q+p)\left(\frac{\Gamma_q(p+1-\delta)\Gamma_q(p+3)}{\Gamma_q(p+1)\Gamma_q(p+4-\delta)}\right)^{m} C_m \left(\frac{[p+2]_q}{[p]_q}\right).$$

If $f(z)$ given by (1.1) belongs to $S_{p,b}^{\beta}(Q_{k,\alpha})$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leq \begin{cases} \frac{bpQ_1}{B}\left[\frac{Q_2}{Q_1} - \frac{pQ_1 b([p+1]+p)(1+i\tan\beta)}{A} - \frac{\mu p Q_1 b}{A}\right] & if \quad \mu < \sigma_1 \\ \frac{bpQ_1}{B} & if \quad \sigma_1 \leq \mu \leq \sigma_2 \\ -\frac{bpQ_1}{B}\left[\frac{Q_2}{Q_1} - \frac{pQ_1 b([p+1]+p)(1+i\tan\beta)}{A} - \frac{\mu p Q_1 b}{A}\right] & if \quad \mu > \sigma_2. \end{cases} \tag{4.1}$$

Further, if $\sigma_1 \leq \mu \leq \sigma_3$, then

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{bp}{B}\left[Q_1 - Q_2 + \frac{\mu p Q_1^2 b(1 + i\tan\beta)}{A} + \frac{\mu p b Q_1^2}{A}\right]\left|a_{p+1}\right|^2 \leq \frac{bpQ_1}{B}. \tag{4.2}$$

Further, if $\sigma_3 \leq \mu \leq \sigma_2$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{bp}{B}\left[Q_1 + Q_2 - \frac{\mu p Q_1 b(1+i\tan\beta)}{A} - \frac{\mu p b Q_1^2}{A}\right]\left|a_{p+1}\right|^2 \leq \frac{bpQ_1}{B}. \tag{4.3}$$

For any complex number μ,

$$\begin{aligned}\left|a_{p+2} - \mu a_{p+1}^2\right| \leq &\frac{pQ_1 b}{(1+i\tan\beta)} \\ &. \max\left\{1, \left|\frac{Q_2}{Q_1} - \frac{pQ_1 b([p+1]+p)(1+i\tan\beta)}{A} - \frac{\mu p B_1 b}{A}\right|\right\}.\end{aligned} \tag{4.4}$$

Proof: If $f(z) \in S_{b,p}^{\beta}(Q_{k,\alpha})$, then there is a Schwarz function $w(z) \in \Omega$ of the form (2.13) such that

$$1 + \frac{1}{b}\left[\frac{(1+i\tan\beta)}{p}\left(z\frac{D_q(\mathcal{R}_{q,p}^{\delta,m} f(z))}{\mathcal{R}_{q,p}^{\delta,m} f(z)}\right) - i\tan\beta - 1\right] = Q_{k,\alpha}(w(z)). \tag{4.5}$$

Since

$$z\frac{D_q(\mathcal{R}_{q,p}^{\delta,m} f(z))}{\mathcal{R}_{q,p}^{\delta,m} f(z)} = p + A_{p+1}([p+1]-p)z + (A_{p+2}([p+2]-p) + A_{p+1}^2([p+1]+p))z^2 + \dots \tag{4.6}$$

$$Q_{k,\alpha}(w(z)) = 1 + Q_1 w_1 + (Q_1 w_2 + Q_2 w_1^2)z^2 + (Q_1 w_3 + 2Q_2 w_1 w_2 + Q_3 w_1^3)z^3 + \dots \tag{4.7}$$

we have from (4.6) and (4.7)

$$A_{p+1} = \frac{pQ_1 b w_1}{(1+i\tan\beta)([p+1]-p)}, \tag{4.8}$$

$$\begin{aligned}&(A_{p+2}([p+2]-p) + A_{p+1}^2([p+1]+p)) \\ &= \frac{(Q_1 w_2 + Q_2 w_1^2)pb}{([p+2]+p)(1+i\tan\beta)} - \frac{p^2 Q_1^2 b^2 w_1^2([p+1]+p)}{(1+i\tan\beta)^2([p+2]-p)([p+1]-p)^2}\end{aligned} \tag{4.9}$$

where

$$A_{p+c} = \left(\frac{\Gamma_q(p+1-\delta)\Gamma_q(p+c)}{\Gamma_q(p+1)\Gamma_q(p+c-\delta)}\right)^m C_m\left(\frac{[p+c]_q}{[p]_q}\right) a_{p+c}. \tag{4.10}$$

From (4.8), (4.9) and (4.10), we get the following

$$a_{p+1} = \frac{pQ_1 b w_1}{\sqrt(A)}$$

$$a_{p+2} = \frac{pQ_1 b w_2}{B} + \frac{pQ_2 b w_1^2}{B} - \frac{p^2 Q_1^2 b^2([p+1]+p)(1+i\tan\beta)w_1^2}{AB}.$$

Therefore we have

$$a_{p+2} - \mu a_{p+1}^2 = \frac{pQ_1 b}{B}[w_2 - \nu w_1^2] \tag{4.11}$$

where

$$\nu = -\frac{Q_2}{Q_1} + \frac{pQ_1 b([p+1]+p)(1+i\tan\beta)}{A} + \frac{\mu p Q_1 b}{A}.$$

The results (4.1)– (4.4) are established by an application of Lemma 2.7, inequality (4.5) by Lemma 2.7 and (4.2) follows from Lemma 2.8.
To show that the bounds in (4.1)-(4.4)are sharp, we define the function $K_{\phi n}$ $(n = 2, 3, \ldots)$ by

$$1 + \frac{1}{b}\left[\frac{(1+i\tan\beta)}{p}\left(z\frac{D_q(\mathcal{R}_{q,p}^{\delta,m} K_{\phi n}(z))}{\mathcal{R}_{q,p}^{\delta,m} K_{\phi n}(z)}\right) - i\tan\beta - 1\right] = Q_{k,\alpha}(z^{n-1})$$

$$K_{\phi n}(0) = 0 = [K_{\phi n}]'(0) - 1$$

and the function F_λ and G_λ $(0 \le \lambda \le 1)$ by

$$1 + \frac{1}{b}\left[\frac{(1+i\tan\beta)}{p}\left(z\frac{D_q(\mathcal{R}_{q,p}^{\delta,m} F_\lambda(z))}{\mathcal{R}_{q,p}^{\delta,m} F_\lambda(z)}\right) - i\tan\beta - 1\right] = Q_{k,\alpha}\left(z\frac{\lambda + z}{1+\lambda z}\right)$$

$$F_\lambda(0) = 0 = [F_\lambda]'(0) - 1$$

and

$$1 + \frac{1}{b}\left[\frac{(1+i\tan\beta)}{p}\left(z\frac{D_q(\mathcal{R}_{q,p}^{\delta,m} G_\lambda(z))}{\mathcal{R}_{q,p}^{\delta,m} G_\lambda(z)}\right) - i\tan\beta - 1\right] = Q_{k,\alpha}\left(-z\frac{\lambda + z}{1+\lambda z}\right)$$

$$G_\lambda(0) = 0 = [G_\lambda]'(0) - 1.$$

Clearly the functions $K_{\phi n}$, F_λ, $G_\lambda \in S_{p,b}^{\beta}(\Omega_{k,\alpha})$. Also we write $K_\phi := K_{\phi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if f is K_ϕ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if and only if f is $K_{\phi 3}$ or one of its rotations. If $\mu = \sigma_1$ then the equality holds if and only if f is F_λ or one of its rotations. If $\mu = \sigma_2$ then the equality holds if and only if f is G_λ or one of its rotations. □

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