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Marichev-Saigo-Maeda Fractional Integral Operators Involving the Product of Generalized Bessel-Maitland Functions

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ABSTRACT: The aim of this paper is to evaluate two theorems for fractional integration involving Appell's function $F_3(.)$ due to Marichev-Saigo-Maeda, to the product of the generalized Bessel-Maitland function. The results are expressed in terms of the multivariable generalized Lauricella functions. Corresponding assertions in terms of Saigo, Erdélyi-Kober, Riemann-Liouville, and Weyl type of fractional integrals are also presented. Some interesting special cases of our two main results are presented. Further, we point out also their relevance.

Key Words: Generalized Fractional integrals, Generalized Bessel-Maitland function, generalized Lauricella series in several variables, Appell function-$F_3(.)$.

Contents

1. Introduction , Definition and preliminaries

The fractional calculus is now a day's one of the most fast growing subject of mathematical analysis. It is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. The fractional integral operator involving various special functions has found considerable importance and applications in various sub-fields of applicable mathematical analysis. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamical system, plasma physics and nonlinear control theory, image processing, nonlinear biological systems, astrophysics, and in quantum mechanics. Since last four decades, a number of workers like Agarwal ([1]-[3]), Agarwal and Jain [5], Baleanu [8], Baleanu and Mustafa [9], Baleanu et al. ([10]-[11]), Kalla [14], Kalla and Saxena [15], Kilbas [16], Kilbas and Sebastian [17], Kiryakova ([18]-[19]), Love [20], McBride [22], Purohit and Kalla [25] and Saigo ([29]-[30]), so forth have studied, in depth, the properties, applications, and different extensions of various operators

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of fractional calculus. A detailed account of generalized fractional calculus operators along with their properties and applications can be found in the research monographs by Kiryakova [18], Miller and Ross [23], and so forth.

The computation of fractional derivatives (and fractional integrals) of special functions of one and more variables is important from the point of view of the usefulness of these results in the evaluation of generalized integrals and the solution of differential and integral equations. Motivated by these avenues of applications, a remarkably large number of fractional integral formulas involving a variety of special functions have been developed by many authors (see, e.g., ([7], [26]- [28], [34], [35]). Fractional integration formulae for the Bessel function and generalized Bessel functions are given recently by Kilbas and Sebastian [17], Malik et al. [21], Purohit et al. [28] and Saxena et al. [32].

Here, we aim at presenting composition formula of Marichev-Saigo-Maeda fractional integral operators and the product of generalized Bessel-Maitland function, which are expressed in terms of the multivariable generalized Lauricella functions. Some interesting special cases of our main results are also considered.

On account of success of the Saigo operators ([29]- [30]), in their study on various function spaces and their application in the integral equation and differential equations, Saigo and Maeda [31] introduced the following generalized fractional integral and differential operators of any complex order with Appell function $F_3(.)$ in the kernel, as follows:

Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ and $x > 0$, then the generalized fractional calculus operators (the Marichev-Saigo-Maeda operators) involving the Appell function, or Horn's F_3-function are defined by the following equations:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)}$$

$$\times \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt, \;\; (Re(\gamma) > 0), \quad (1.1)$$

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)}$$

$$\times \int_x^\infty (t-x)^{\gamma-1} t^{-\alpha} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)dt, \;\; (Re(\gamma) > 0), \quad (1.2)$$

where

$$F_3(\alpha, \alpha', \beta, \beta'; \gamma, x; y) = \sum_{m,n=0}^{\infty} \left(\frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n}{(\gamma)_{n+m}\, m!\, n!} x^m y^n \right), \;\; (\max\{|x|, |y| < 1\}),$$

is known as Appell function (known also as Horn function).
Following Saigo and Maeda [31], the image formulas for a power function, under operators (1.1) and (1.2), are given by:

$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}\right)(x) = \Gamma\left[\begin{array}{c} \rho,\, \rho+\gamma-\alpha-\alpha'-\beta,\, \rho+\beta'-\alpha' \\ \rho+\beta',\, \rho+\gamma-\alpha-\alpha',\, \rho+\gamma-\alpha'-\beta \end{array}\right] x^{\rho-\alpha-\alpha'+\gamma-1}, \quad (1.3)$$

where $\Re(\rho) > max\left\{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re(\alpha'-\beta')\right\}$ and $\Re(\gamma) > 0$.

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} x^{\rho-1}\right)(x) = \Gamma\left[\begin{array}{c} 1-\rho-\beta,\, 1-\rho+\alpha+\beta'-\gamma,\, 1-\rho-\gamma+\alpha+\alpha' \\ 1-\rho,\, 1-\rho+\alpha-\beta,\, 1-\rho+\alpha+\alpha'+\beta'-\gamma, \end{array}\right] \times x^{\rho+\gamma-\alpha-\alpha'-1}, \tag{1.4}$$

where $\Re(\rho) < 1 + min\left\{\Re(-\beta), Re(\alpha+\alpha'-\gamma), \Re(\alpha+\beta'-\gamma)\right\}$ and $Re(\gamma) > 0$.

Here we used the symbol $\Gamma\left[\begin{array}{c}\cdots\\ \cdots\end{array}\right]$ representing the fraction of many Gamma functions.
We investigate composition of integral transforms (1.1) and (1.2) with the product of generalized Bessel-Maitland functions $J_{v,\,q}^{\mu,\,\gamma}(z)$, which is defined by Pathak [24] as follow:

$$J_{v,\,q}^{\mu,\,\gamma}(z) = \sum_{m=0}^{\infty} \frac{(\gamma)_{qm}\,(-z)^m}{\Gamma(v+\mu\, m+1)\; m!}, \tag{1.5}$$

where $\mu, v, \gamma \in \mathbb{C}$, $\Re(\mu) \geq 0$, $\Re(v) \geq -1$, $\Re(\gamma) \geq 0$ and $q \in (0,\,1) \bigcup \mathbb{N}$, $(\gamma)_{qm}$ is known as generalized Pochhammer symbol, is defined as $(\gamma)_{qm} = \frac{\Gamma(\gamma+qm)}{\Gamma(\gamma)}$.
Also, we show that the composition is expressed in terms of the multivariable generalized Lauricella functions due to Srivastava and Daoust [33] is a generalization of the Wright function ${}_p\psi_q$ in several variables [13] and defined by

$$S_{q,q_1,q_2,\ldots,q_n}^{p,p_1,p_2,\ldots,p_n}\left(\begin{array}{c} \left[(a)_p : (\alpha_p^1),...,(\alpha_p^n)\right] : \left[(c^1)_{p_1} : (\gamma_{p_1}^1)\right];...;(c^n)_{p_1} : (\gamma_{p_1}^n); \\ \left[(b)_q : (\beta_q^1),...,(\beta_p^n)\right] : \left[(d^1)_{q_1} : (\delta_{q_1}^1)\right];...;(d^n)_{q_1} : (\delta_{q_1}^n); \end{array} z_1,...,z_n\right)$$

$$= \sum_{m_1,m_2,...,m_n=0}^{\infty} A(m_1,m_2,...,m_n) \prod_{j=1}^{n} \frac{z_j^{m_j}}{m_j!}, \tag{1.6}$$

where

$$A(m_1,m_2,...,m_n) = \frac{\prod_{k=1}^{p}\Gamma\left[a_j + \sum_{j=1}^{n} m_k\alpha_j^k\right]}{\prod_{k=1}^{q}\Gamma\left[b_j + \sum_{j=1}^{n} m_k\beta_j^k\right]} \prod_{k=1}^{n}\left(\frac{\prod_{k=1}^{p_k}\Gamma\left[c_j + \sum_{j=1}^{n} m_k\gamma_j^k\right]}{\prod_{k=1}^{q_k}\Gamma\left[d_j + \sum_{j=1}^{n} m_k\delta_j^k\right]}\right), \tag{1.7}$$

The coefficient α_j^k (j = 1, 2, ..., p), β_j^k (j = 1, 2, ...,q), γ_j^k (j = 1, 2, ...,$\mathrm{p_k}$) and δ_j^k (j = 1, 2, ...,$\mathrm{q_k}$) for all $k = 1, ..., n$, are real and positive and (a_p) means the array of p-parameters $a_1, ..., a_p$; with similar interpretations for (b_p), $(\gamma_{p_1}^1)$, $(\alpha_{p_1}^1)$ and so forth.

The paper is organized as follow. The composition formula of Marichev-Saigo-Maeda fractional integral operators (1.1) and (1.2) with the product of generalized Bessel-Maitland function (1.5) is proved in terms of the multivariable generalized

Lauricella functions (1.6) in Sections 2 and 3, respectively. The corresponding results for the corresponding assertions in terms of Saigo, Erdélyi-Kober, Riemann-Liouville and Weyl type of fractional integrals are also presented in Sections 2 and 3. Special cases giving compositions of fractional integrals with the product of Bessel functions and concluding remarks are considered in Section 4.

2. Left-Side Fractional Integration of Generalized Bessel-Maitland Functions

In this section, we establish image formulas for the product of generalized Bessel-Maitland function involving left sided of Marichev-Saigo-Meada fractional integral operator, in term of the multivariable generalized Lauricella functions. These formulas are given by the following theorem:

Theorem 2.1. *Let* $\alpha, \alpha', \beta, \beta', \sigma, \lambda, \gamma, \nu_j,\ q_j,\ \mu_j,\ k_j,\ b \in \mathbb{C}$ *and satisfying the inequalities,* $\Re(\gamma) > 0$, $\Re(\sigma) > 0$, $\Re(\sigma) > \max\left\{0, \Re(\alpha+\alpha'+\beta-\gamma), \Re\left(\alpha'-\beta'\right)\right\}$ *and* $\Re\left(\sum_{j=1}^{n}\nu_j + 1\right) > 0$ *then for* $x > 0$, $\left|\frac{ax}{b}\right| < 1$ *one has*

$$\left(\mathrm{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}(b-at)^{-\lambda}\prod_{j=1}^{n}J_{\nu_j,\,q_j}^{\mu_j,\,k_j}\left(a_j t^{\rho_j}\right)\right]\right)(x) = \frac{x^{\sigma+\gamma-\alpha-\alpha'-1}}{\Gamma(\lambda)\ b^{\lambda}}$$

$$\times\prod_{j=1}^{n}\frac{1}{\Gamma(k_j)}\,S_{3;\,1;\,\ldots\,;\,1;\,0}^{3;\,1;\,\ldots\,;\,1;\,1}\left[\begin{array}{cccc} A, & C, & E: & G; \\ B, & D, & F: & H; \end{array} -a_1x^{\rho_1},\ldots,-a_nx^{\rho_n},\frac{ax}{b}\right], \tag{2.1}$$

where A, B, C, D, E, F, G *and* H *are given by the following*

$$A = \left[(\sigma):(\rho_1),(\rho_2),\ldots,(\rho_n),(1)\right], \tag{2.2}$$

$$B = \left[\left(\sigma+\beta'\right):(\rho_1),(\rho_2),\ldots,(\rho_n),(1)\right], \tag{2.3}$$

$$C = \left[(\sigma+\gamma-\alpha-\alpha'-\beta):(\rho_1),(\rho_2),\ldots,(\rho_n),(1)\right], \tag{2.4}$$

$$D = \left[(\sigma+\gamma-\alpha-\alpha'):(\rho_1),(\rho_2),\ldots,(\rho_n),(1)\right], \tag{2.5}$$

$$E = \left[\left(\sigma+\beta'-\alpha'\right):(\rho_1),(\rho_2),\ldots,(\rho_n),(1)\right], \tag{2.6}$$

$$F = \left[(\sigma+\gamma-\alpha'-\beta):(\rho_1),(\rho_2),\ldots,(\rho_n),(1)\right], \tag{2.7}$$

$$G = \left[\left[(k_1:q_1);\ldots;(k_n:q_n)\right];(\lambda:1)\right], \tag{2.8}$$

$$H = \left[(v_1+1:\ \mu_1);\ldots;(v_n+1:\mu_n)\right];0. \tag{2.9}$$

Proof: For the sake of convenience, let the left-hand side of the (2.1) be denoted by $\mathcal{F}$. Using definition (1.5) and binomial expansion, namely,

$$(b-at)^{-\lambda} = b^{-\lambda}\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!}\left(\frac{at}{b}\right)^m, \qquad \left|\frac{at}{b}\right| < 1. \tag{2.10}$$

we find

$$\mathfrak{F}=\left(\mathrm{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}\left(b-at\right)^{-\lambda}\prod_{j=1}^{n}J_{\nu_j,q_j}^{\mu_j,k_j}\left(a_jt^{\rho_j}\right)\right]\right)(x)$$
$$=\left(\mathrm{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}\left(b^{-\lambda}\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!}\left(\frac{at}{b}\right)^m\right)\right.\right.$$
$$\left.\left.\times\prod_{j=1}^{n}\left(\sum_{m_j=0}^{\infty}\frac{(k_j)_{q_jm_j}\left(-a_jt^{\rho_j}\right)^{m_j}}{m_j!\Gamma(v_j+\mu_jm_j+1)}\right)\right]\right)(x), \tag{2.11}$$

Following the convergence condition of Theorem 2.1, for any $k\in\mathbb{N}_0$,

$$\Re\left(\sigma+m+\sum_{j=1}^{n}\rho_jm_j\right)\geq\Re\left(\sigma+\sum_{j=1}^{n}\rho_jm_j\right)$$
$$>\max\left\{0,\Re(\alpha+\alpha'+\beta-\gamma),\Re\left(\alpha'-\beta'\right)\right\}, \tag{2.12}$$

and changing the order of integration and summation, we obtain

$$\mathfrak{F}=b^{-\lambda}\sum_{m,m_1,m_2,\ldots,m_n=0}^{\infty}\frac{(k_1)_{q_1m_1}(-a_1)^{m_1}}{\Gamma(v_1+\mu_1m_1+1)}\frac{(\lambda)_m}{m!\,m_1!}\left(\frac{a}{b}\right)^m$$
$$\times\ldots\times\frac{(k_n)_{q_nm_n}(-a_n)^{m_n}}{\Gamma(v_n+\mu_nm_n+1)\,m_n!}\left(\mathrm{I}_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\sigma+m+\rho_1m_1+\ldots+\rho_nm_n-1}\right)(x), \tag{2.13}$$

Now on applying (1.3) with ρ replaced by $(\sigma+m+\rho_1m_1+\ldots+\rho_nm_n)$, we obtain

$$\mathfrak{F}=\frac{x^{\sigma+\gamma-\alpha-\alpha'-1}}{\Gamma(\lambda)\,b^{\lambda}}\prod_{j=1}^{n}\frac{1}{\Gamma(k_j)}\sum_{m,m_1,m_2,\ldots,m_n=0}^{\infty}\frac{\Gamma\left(\sigma+m+\sum_{j=1}^{n}\rho_jm_j\right)}{\Gamma\left(\sigma+m+\beta'+\sum_{j=1}^{n}\rho_jm_j\right)}$$
$$\times\frac{\Gamma\left(\sigma+m+\gamma-\alpha-\alpha'-\beta+\sum_{j=1}^{n}\rho_jm_j\right)\Gamma\left(\sigma+m-\alpha'+\beta'+\sum_{j=1}^{n}\rho_jm_j\right)}{\Gamma\left(\sigma+m+\gamma-\alpha-\alpha'\sum_{j=1}^{n}\rho_jm_j\right)\Gamma\left(\sigma+m+\gamma-\alpha'-\beta+\sum_{j=1}^{n}\rho_jm_j\right)}$$
$$\times\frac{\Gamma(k_1+q_1m_1)}{\Gamma(v_1+\mu_1m_1+1)}\times\ldots\times\frac{\Gamma(k_n+q_nm_n)}{\Gamma(v_n+\mu_nm_n+1)}\Gamma(\lambda+m)$$
$$\times\frac{\left(-a_1x^{\rho_1}\right)^{m_1}}{m_1!}\times\ldots\times\frac{\left(-a_nx^{\rho_n}\right)^{m_n}}{m_n!}\frac{\left(\frac{ax}{b}\right)^m}{m!}. \tag{2.14}$$

In accordance with (1.6), gives the required result (2.1). This completed the proof of the Theorem 2.1. □

Now, we present some special cases of Theorem 2.1 are given below:

On setting $\alpha'=0$ in Theorem 2.1, we get the following Saigo fractional integral image of the product of generalized Bessel-Maitland function.

Corollary 2.2. *Let* $\alpha, \beta, \beta', \sigma, \lambda, \gamma, \nu_j, q_j, \mu_j, k_j, b \in \mathbb{C}$ *and* $x > 0$, $\left|\frac{ax}{b}\right| < 1$ *satisfying the inequalities,* $\Re(\gamma) > 0, \Re(\sigma) > 0, \Re\left(\sum_{j=1}^{n} \nu_j + 1\right) > 0$ *then there holds the results:*

$$\left(I_{0+}^{\gamma,\alpha-\gamma,-\beta}\left[t^{\sigma-1}(b-at)^{-\lambda}\prod_{j=1}^{n} J_{\nu_j,q_j}^{\mu_j,k_j}\left(a_j t^{\rho_j}\right)\right]\right)(x)$$

$$= \frac{x^{\sigma+\gamma-\alpha-1}}{\Gamma(\lambda)\, b^{\lambda}}\prod_{j=1}^{n}\frac{1}{\Gamma(k_j)} S_{2;\,1;\,\ldots;\,1;\,0}^{2;\,1;\,\ldots;\,1;\,1}\left[\begin{matrix} A, & C: & G; \\ D, & F: & H; \end{matrix}\; -a_1 x^{\rho_1}, \ldots, -a_n x^{\rho_n}, \frac{ax}{b}\right]. \tag{2.15}$$

where A, C, D, F, G *and* H *are given by* (2.2), (2.4), (2.5), (2.7), (2.8) *and* (2.9) *respectively.*

Again, on letting $\alpha' = 0$ and $\alpha = 0$ in Theorem 2.1, we get the Riemann-Liouville fractional image of the product of generalized Bessel-Maitland function; asserted by the following corollary.

Corollary 2.3. *Let* $\alpha, \beta, \beta', \sigma, \lambda, \gamma, \nu_j, q_j, \mu_j, k_j, b \in \mathbb{C}$ *and* $x > 0$, $\left|\frac{ax}{b}\right| < 1$ *satisfying the inequalities,* $\Re(\gamma) > 0, \Re(\sigma) > 0, \Re\left(\sum_{j=1}^{n} \nu_j + 1\right) > 0$ *then there holds the results:*

$$\left(I_{0+}^{\gamma}\left[t^{\sigma-1}(b-at)^{-\lambda}\prod_{j=1}^{n} J_{\nu_j,q_j}^{\mu_j,k_j}\left(a_j t^{\rho_j}\right)\right]\right)(x)$$

$$= \frac{x^{\sigma+\gamma-1}}{\Gamma(\lambda)\ b^{\lambda}}\prod_{j=1}^{n}\frac{1}{\Gamma(k_j)}\ S_{1;\,1;\,\ldots;\,1;\,0}^{1;\,1;\,\ldots;\,1;\,1}\left[\begin{matrix} A: & G; \\ D: & H; \end{matrix}\; -a_1 x^{\rho_1}, \ldots, -a_n x^{\rho_n}, \frac{ax}{b}\right]. \tag{2.16}$$

where A, D, G *and* H *are given by* (2.2), (2.5), (2.8) *and* (2.9) *respectively.*

3. Right-Side Fractional Integration of Generalized Bessel-Maitland Functions

In this section, we establish image formulas for the product of generalized Bessel-Maitland function involving right sided of Marichev-Saigo-Meada fractional integral operator, in term of the multivariable generalized Lauricella functions. These formulas are given by the following theorem:

Theorem 3.1. *Let* $\alpha, \alpha', \beta, \beta', \sigma, \lambda, \gamma, \nu_j, q_j, \mu_j, k_j, b \in \mathbb{C}$ *and satisfying the inequalities,* $\Re(\gamma) > 0$, $\Re(\sigma) < 1 + \min\left\{\Re(-\beta), \Re(\alpha+\alpha'-\gamma), \Re\left(\alpha+\beta'-\gamma\right)\right\}$ $\Re(\sigma) > 0$, *and* $\Re\left(\sum_{j=1}^{n} \nu_j + 1\right) > 0$ *then for* $x > 0$, $\left|\frac{a}{b\,x}\right| < 1$ *one has*

$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}(b-(a/t))^{-\lambda}\prod_{j=1}^{n} J_{\nu_j,q_j}^{\mu_j,k_j}\left(a_j/t^{\rho_j}\right)\right]\right)(x) = \frac{x^{\sigma+\gamma-\alpha-\alpha'-1}}{\Gamma(\lambda)\, b^{\lambda}}$$

$$\times \prod_{j=1}^{n} \frac{1}{\Gamma(k_j)} S^{3;\,1;\,\ldots\,;\,1;\,1}_{3;\,1;\,\ldots\,;\,1;\,0} \left[\begin{array}{lllll} A_0, & C_0, & E_0: & G_0; & \frac{-a_1}{x^{\rho_1}}, \ldots, \frac{-a_n}{x^{\rho_n}}, \frac{a}{b\,x} \\ B_0, & D_0, & F_0: & H_0; & \end{array} \right], \tag{3.1}$$

where A_0, B_0, C_0, D_0, E_0 *and* F_0 *are given by the following:*

$$A_0 = [(1-\sigma-\beta):(\rho_1),(\rho_2),\ldots,(\rho_n),(1)], \tag{3.2}$$

$$B_0 = [(1-\sigma):(\rho_1),(\rho_2),\ldots,(\rho_n),(1)], \tag{3.3}$$

$$C_0 = [(1-\sigma-\gamma+\alpha+\alpha'):(\rho_1),(\rho_2),\ldots,(\rho_n),(1)], \tag{3.4}$$

$$D_0 = [(1-\sigma+\alpha-\beta):(\rho_1),(\rho_2),\ldots,(\rho_n),(1)], \tag{3.5}$$

$$E_0 = \left[\left(1-\sigma+\alpha+\beta'-\gamma\right):(\rho_1),(\rho_2),\ldots,(\rho_n),(1)\right], \tag{3.6}$$

$$F_0 = \left[\left(1-\sigma+\alpha+\alpha'+\beta'-\gamma\right):(\rho_1),(\rho_2),\ldots,(\rho_n),(1)\right], \tag{3.7}$$

and G, H *are given by* (2.8) *and* (2.9), *respectively.*

Proof: For convenience, Let the left-hand side of the (3.1) be denoted by $\mathcal{F}$. Using definition (1.5) and binomial expansion, we find

$$\mathcal{F} = \left(\mathrm{I}_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}\left(b-(a/t)\right)^{-\lambda}\prod_{j=1}^{n} J_{\nu_j,q_j}^{\mu_j,k_j}\left(a_j/t^{\rho_j}\right)\right]\right)(x)$$

$$= \left(\mathrm{I}_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\left[t^{\sigma-1}\left(b^{-\lambda}\sum_{m=0}^{\infty}\frac{(\lambda)_m}{m!}\left(\frac{a}{bt}\right)^m\right)\right.\right.$$

$$\left.\left.\times\prod_{j=1}^{n}\left(\sum_{m_j=0}^{\infty}\frac{(k_j)_{q_j m_j}\left(-a_j/t^{\rho_j}\right)^{m_j}}{m_j!\Gamma(v_j+\mu_j m_j+1)}\right)\right]\right)(x), \tag{3.8}$$

Following the convergence condition of Theorem 3.1, for any $k\in\mathbb{N}_0$,

$$\Re\left(\sigma-m-\sum_{j=1}^{n}\rho_j m_j\right) \le \Re\left(\sigma-\sum_{j=1}^{n}\rho_j m_j\right)$$

$$< 1+\min\left\{\Re(-\beta),\Re(\alpha+\alpha'-\gamma),\Re\left(\alpha+\beta'-\gamma\right)\right\}, \tag{3.9}$$

and changing the order of integration and summation, we obtain

$$\mathcal{F} = b^{-\lambda}\sum_{m,m_1,m_2,\ldots,m_n=0}^{\infty}\frac{(k_1)_{q_1 m_1}(-a_1)^{m_1}}{\Gamma(v_1+\mu_1 m_1+1)}\frac{(\lambda)_m}{m!\,m_1!}\left(\frac{a}{b}\right)^m$$

$$\times\ldots\times\frac{(k_n)_{q_n m_n}(-a_n)^{m_n}}{\Gamma(v_n+\mu_n m_n+1)\,m_n!}\left(\mathrm{I}_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\,t^{\sigma-m-\rho_1 m_1-\ldots-\rho_n m_n-1}\right)(x), \tag{3.10}$$

Now on applying (1.3) with ρ replaced by $(\sigma+m+\rho_1 m_1+...+\rho_n m_n)$, we obtain

$$\mathcal{F}=\frac{x^{\sigma+\gamma-\alpha-\alpha'-1}}{\Gamma(\lambda)\,b^{\lambda}}\prod_{j=1}^{n}\frac{1}{\Gamma(k_j)}\sum_{m,m_1,m_2,\ldots,m_n=0}^{\infty}\frac{\Gamma\left(1-\sigma-\beta+m+\sum_{j=1}^{n}\rho_j m_j\right)}{\Gamma\left(1-\sigma+m+\sum_{j=1}^{n}\rho_j m_j\right)}$$

$$\times\frac{\Gamma\left(1-\sigma+\alpha+\beta'-\gamma+m+\sum_{j=1}^{n}\rho_j m_j\right)}{\Gamma\left(1-\sigma+\alpha+\alpha'+\beta'-\gamma+m+\sum_{j=1}^{n}\rho_j m_j\right)}$$

$$\times\frac{\Gamma\left(1-\sigma-\gamma+\alpha+\alpha'+m+\sum_{j=1}^{n}\rho_j m_j\right)}{\Gamma\left(1-\sigma+\alpha-\beta+m+\sum_{j=1}^{n}\rho_j m_j\right)}$$

$$\times\frac{\Gamma(k_1+q_1 m_1)}{\Gamma(v_1+\mu_1 m_1+1)}\times\ldots\times\frac{\Gamma(k_n+q_n m_n)}{\Gamma(v_n+\mu_n m_n+1)}\Gamma(\lambda+m)$$

$$\times\frac{(-a_1/x^{\rho_1})^{m_1}}{m_1!}\times\ldots\times\frac{(-a_n/x^{\rho_n})^{m_n}}{m_n!}\frac{\left(\frac{a}{bx}\right)^m}{m!}. \tag{3.11}$$

In accordance with (1.6), gives the required result (3.1). This completed the proof of the Theorem 3.1. □

Now, we present some special cases of Theorem 3.1 are given below:

On setting $\alpha'=0$ in Theorem 3.1, we get the following Saigo fractional integral image of the product of generalized Bessel-Maitland function.

Corollary 3.2. *Let $\alpha,\beta,\beta',\sigma,\lambda,\gamma,\nu_j,\,q_j,\,\mu_j,\,k_j,\,b\in\mathbb{C}$ and $x>0$, $\left|\frac{a}{b\,x}\right|<1$ satisfying the inequalities, $\Re(\gamma)>0,\Re(\sigma)>0,\Re\left(\sum_{j=1}^{n}\nu_j+1\right)>0$ then there holds the results:*

$$\left(\mathrm{I}_{-}^{\gamma,\,\alpha-\gamma,\,-\beta}\left[t^{\sigma-1}\left(b-(a/t)\right)^{-\lambda}\prod_{j=1}^{n}J_{\nu_j,\,q_j}^{\mu_j,\,k_j}\left(a_j/t^{\rho_j}\right)\right]\right)(x)$$

$$=\frac{x^{\sigma+\gamma-\alpha-1}}{\Gamma(\lambda)\,b^{\lambda}}\prod_{j=1}^{n}\frac{1}{\Gamma(k_j)}S_{2;\,1;\,\ldots\,;\,1;\,0}^{2;\,1;\,\ldots\,;\,1;\,1}\left[\begin{array}{ll}A_0, & C_0: \quad G; \\ B_0, & D_0: \quad H;\end{array}\;\frac{-a_1}{x^{\rho_1}},\ldots,\frac{-a_n}{x^{\rho_n}},\frac{a}{b\,x}\right]. \tag{3.12}$$

where A_0, B_0, C_0 and D_0 are given by (3.2), (3.3), (3.4) and (3.5) respectively.

Again, on letting $\alpha'=0$ and $\alpha=0$ in Theorem 3.1, we get the Riemann-Liouville fractional image of the product of generalized Bessel-Maitland function; asserted by the following corollary.

Corollary 3.3. *Let* $\alpha, \beta, \beta', \sigma, \lambda, \gamma, \nu_j, q_j, \mu_j, k_j, b \in \mathbb{C}$ *and* $x > 0$, $\left|\frac{a}{b\,x}\right| < 1$ *satisfying the inequalities,* $\Re(\gamma) > 0, \Re(\sigma) > 0, \Re\left(\sum_{j=1}^{n} \nu_j + 1\right) > 0$ *then there holds the results:*

$$\left(\mathrm{I}_{-}^{\gamma}\left[t^{\sigma-1}\left(b-(a/t)\right)^{-\lambda}\prod_{j=1}^{n} J_{\nu_j,\,q_j}^{\mu_j,\,k_j}\left(a_j/t^{\rho_j}\right)\right]\right)(x)$$

$$= \frac{x^{\sigma+\gamma-1}}{\Gamma(\lambda)\,b^{\lambda}}\prod_{j=1}^{n}\frac{1}{\Gamma(k_j)} S_{1;\,1;\,\ldots\,;\,1;\,0}^{1;\,1;\,\ldots\,;\,1;\,1}\left[\begin{array}{ll} C_0: & G;\\ B_0: & H;\end{array} \frac{-a_1}{x^{\rho_1}}, \ldots, \frac{-a_n}{x^{\rho_n}}, \frac{a}{b\,x}\right]. \tag{3.13}$$

where B_0, C_0, G *and* H *are given by* (3.3), (3.4), (2.8) *and* (2.9) *respectively.*

4. Consequence Results and Concluding Remarks

In this section, we briefly consider another variation of the results derived in the preceding sections. Bessel-Maitland functions are important special functions that appear widely in science and engineering. Bessel-Maitland functions are oscillatory and may be regarded as generalizations of Bessel functions. Further, it can be easily seen that for $q = 1$, $\gamma = 1$ and v is replaced by $v + \sigma$ and z is replaced by $z^2/4$, the generalized Bessel-Maitland function (1.5) reduces to $J_{v,\,\sigma}^{\mu}(z)$ which is defined by Agarwal et al. [6], and when $q = 0$ the function reduces to generalized Bessel function $J_v^{\mu}(z)$ defined by Agarwal [4]. Similarly when $q = 0$, $\mu = 1$ and z is replaced by $z^2/4$, the generalized Bessel-Maitland function (1.5) reduces to Bessel's function $J_v(z)$ of the first kind of order v, and when $q = 0$ and v is replaced by $v-1$ and z is replaced by $-z$ reduces to Wright function $\phi(\mu, v; z)$ which is defined by defined by Choi et al. [12]. Therefore, the results presented in this paper are easily converted in terms of the various special Bessel functions after some suitable parametric replacement.

The generalized Bessel-Maitland function defined by (1.5), possess the advantage that a number of Bessel functions, Mittag-Leffler function, trigonometric functions and hyperbolic functions happen to be the particular cases of this function. Therefore, we conclude this paper with the remark that, the results deduced above are significant and can lead to yield numerous other fractional integrals involving various Bessel functions and trigonometric functions by the suitable specializations of arbitrary parameters in the theorems. More importantly, they are expected to find some applications to the solutions of fractional differential and integral equations. The results thus derived in this paper are general in character and likely to find certain applications in the theory of special functions.

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