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Igusa-Todorov Function on Path Rings

Gustavo Mata

ABSTRACT: The aim of this paper is to study the relation between the Igusa-Todorov functions for A, a finite dimensional algebra, and the algebra AQ. In particular it is proved that $\phi\dim(AQ) = \phi\dim(A) + 1$ when A is a Gorenstein algebra. As a consequence of the previous result, it is exhibited an example of a family of algebras $\{A_n\}_{n\in\mathbb{N}}$ such that $\phi\dim(A_n) = n$ and each A_n is of Ω^∞-infinite representation type.

Key Words: Igusa-Todorov Functions, Gorenstein Algebra, Path ring.

Contents

1. Introduction

One of the most important conjectures in Representation Theory of Artin algebras is the finitistic conjecture. It states that $\sup\{\text{pd}(M) : M$ is a finitely generated module of finite projective dimension$\}$ is finite. In an attempt to prove the conjecture Igusa and Todorov defined in [9] two functions from the objects of mod$A$ (the category of finitely generated right modules over an Artin algebra A) to the natural numbers, which generalizes the notion of projective dimension. Nowadays they are known as the Igusa-Todorov functions, ϕ and ψ. One of its nicest features is that they are finite for each module, and allow us to define the ϕ-dimension and the ψ-dimension of an algebra. These are new homological measures in the module category. In particular it holds that

$$\text{findim}(A) \leq \phi\dim(A) \leq \psi\dim(A) \leq \text{gldim}(A)$$

and they all agree in the case of algebras with finite global dimension.

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This article is organized as follows: after the introduction and the preliminary section devoted to fixing the notation and recalling the basic facts needed in this work, section 3 is devoted to Igusa-Todorov function for path rings. The main results in this section are the following

Theorem A: Let A be a finite dimensional $\Bbbk$-algebra. If Q is a finite acyclic quiver with at least two vertices, then the inequality below holds:

$$\phi\dim AQ \geq \phi\dim A + 1.$$

In section 4, using results given in *Cohen-Macaulay and Gorenstein Algebras* [2] and the previous theorem, we obtain the following result

Theorem B: Let A be a finite dimensional $\Bbbk$-algebra and Q be a finite acyclic quiver with at least two vertices. Then A is n-Gorenstein if and only if AQ is $(n+1)$-Gorenstein.

In [11] it was proved that if A is of Ω^n-finite representation type for some n then $\phi\dim(A)$ and $\psi\dim(A)$ are both finite. In this article, as a consequence of the above theorem, we give an example of a family of algebras of Ω^∞-infinite representation type with finite ϕ-dimension and ψ-dimension.

2. Preliminaries

2.1. Some notation and definitions

In this article, A denotes a finite dimensional algebra over a field $\Bbbk$ and $\text{mod}A$ the category of finitely generated right A-modules. For M in $\text{mod}A$, $\text{pd}M$ and $\text{id}M$ are the projective and injective dimension of M respectively.

2.2. Gorenstein projective modules and Gorenstein algebras

The concept of Gorenstein projective module goes back to a work of Auslander and Bridger [1]. In this work it was introduced the G-dimension for finitely generated modules over a two-sided noetherian ring. Later was proved by Avramov, Martisinkovsky, and Rieten that if M is a finitely generated module, M is Gorenstein projective if and only if the G-dimension of M is zero ([3, Theorem 4.2.6]).

Definition 2.1. *[5]*
*A finitely generated A-module G is **Gorenstein projective** if there exist a exact sequence:*

$$\ldots \longrightarrow P_{-2} \xrightarrow{p_{-2}} P_{-1} \xrightarrow{p_{-1}} P_0 \xrightarrow{p_0} P_1 \xrightarrow{p_1} P_2 \xrightarrow{p_2} \ldots$$

such that $G \cong Kerp_0$, P_i is a projective module for $i \in \mathbb{Z}$ and the following complex is exact:

$$\ldots \longrightarrow {P_2}^* \xrightarrow{{p_1}^*} {P_1}^* \xrightarrow{{p_0}^*} {P_0}^* \xrightarrow{{p_{-1}}^*} {P_{-1}}^* \xrightarrow{{p_{-2}}^*} {P_{-2}}^* \longrightarrow \ldots,$$

where $(\cdot)^* = \mathrm{Hom}_A(\cdot, A)$.

Definition 2.2. *[8]*
An Artin algebra A *is called* n*-Gorenstein if* $\mathrm{id}(A) \leq n$ *and* $\mathrm{pd}(DA^{op}) \leq n$ *with* $n \in \mathbb{N}$*, where* $D = \mathrm{Hom}_{\Bbbk}(\cdot, \Bbbk)$*. An Artin algebra* A *is called Gorenstein if it is* n*-Gorenstein for some* $n \in \mathbb{N}$.

Remark 2.3. *An Artin algebra* A *is* 0*-Gorenstein if and only if* A *is selfinjective.*

The following proposition can be seen in [13]:

Proposition 2.4. *[13, Corollary 3.4]*
Let A *be an* n*-Gorenstein algebra, and*

$$0 \longrightarrow K \longrightarrow P_{n-1} \xrightarrow{p_{n-1}} \dots \longrightarrow P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} M \longrightarrow 0$$

be an exact sequence with P_i *projective, then* K *is a Gorenstein projective* A*-module.*

2.3. Igusa-Todorov functions

In this section, we show some general facts about the Igusa-Todorov functions. Our objective is to introduce some properties that will be used in the following sections.

Definition 2.5. *[9]*
Let K_0 *be the abelian group generated by all symbols* $[M]$*, where* M *is a f.g.* A*-module, modulo the relations:*

1. $[M] = [M'] - [M'']$ *if* $M \cong M' \oplus M''$.
2. $[P] = 0$ *if* P *is projective.*

The group K_0 may also be described as the free abelian group with basis the set of symbols $[M]$, one for each isomorphism class of indecomposable non-projective module. Moreover every element in K_0 can be written in the form $[M] - [N]$, for some pair of, not necessarily indecomposable, M, N A-modules.

For any finitely generated A-module M let $\overline{\Omega}[M] = [\Omega M]$. Since Ω commutes with direct sums and takes projective modules to zero this gives a group homomorphism $\overline{\Omega} : K_0 \to K_0$.

For every finitely generated A-module M, let $\langle addM \rangle$ denotes the subgroup of K_0 generated by the classes of indecomposable summands of M.

Definition 2.6. *The* ***Igusa-Todorov function*** ϕ *is defined for* $M \in \mathrm{mod}A$*, as*

$$\phi(M) = \min \left\{ \overline{\Omega}|_{\overline{\Omega}^{l+s}\langle addM \rangle} \text{ is a monomorphism for all } s \in \mathbb{N} \right\}.$$

Definition 2.7. *The* ***Igusa-Todorov function*** ψ *is defined for* $M \in \mathrm{mod}A$*, as*

$$\psi(M) = \phi(M) + \sup \left\{ \mathrm{pd}(N) : \Omega^{\phi(M)}(M) = N \oplus N' \text{ and } \mathrm{pd}(N) < \infty \right\}.$$

The main properties of the Igusa-Todorov functions are summarised below. One can find the next propositions in [9] and [7].

Proposition 2.8. *[9], [7]*
Let A be an Artin algebra and M, $N \in \mathrm{mod}(A)$. Then, the following statements hold.

1. $\phi(M) = \mathrm{pd}(M)$ *if M has finite projective dimension.*
2. $\phi(M) = 0$ *if M is indecomposable and has infinite projective dimension.*
3. $\phi(M) \leq \phi(M \oplus N)$.
4. $\phi(M^k) = \phi(M)$ *for* $k \in \mathbb{N}$.
5. $\phi(M) \leq \phi(\Omega(M)) + 1$.

Proposition 2.9. *[9], [7]*
Let A be an Artin algebra and M, $N \in \mathrm{mod}(A)$. Then, the following statements hold.

1. $\psi(M) = \mathrm{pd}(M)$ *if M has finite projective dimension.*
2. $\psi(M) \leq \psi(M \oplus N)$.
3. $\psi(M^k) = \psi(M)$ *for* $k \in \mathbb{N}$.
4. *If N is direct summand of $\Omega^n(M)$ where $n \leq \phi(M)$ and $dp(N) < \infty$, then $dp(N) + n \leq \psi(M)$.*
5. $\psi(M) \leq \psi(\Omega(M)) + 1$.

The following definitions were introduced in [7].

Definition 2.10. *[7]*
Let A be an Artin algebra. We recall that the ϕ-dimension and ψ-dimension of A are defined as follows:

- $\phi\mathrm{dim}(A) = \sup\{\phi(M)$ *such that* $M \in \mathrm{mod}A\}$.
- $\psi\mathrm{dim}(A) = \sup\{\psi(M)$ *such that* $M \in \mathrm{mod}A\}$

The following results give properties of the Igusa-Todorov functions for an Artin algebra A with $\mathrm{id}(A) < \infty$. For the proof see [11].

Proposition 2.11. *[11, Corollary 3.17]*
Let A be an Artin algebra such that $\mathrm{id}(A) = n < \infty$, then

$$\mathrm{findim}(A) \leq \phi\mathrm{dim}(A) \leq \psi\mathrm{dim}(A) \leq n.$$

Proposition 2.12. *[11, Corollary 4.7]*
If A is a Gorenstein algebra, then

1. $\phi\dim(A) = \psi\dim(A) = \mathrm{findim}(A) < \infty$.

2. *Let m be the minimun natural number such that A is a m-Gorenstein algebra then:*

$$\phi\dim(A) = \psi\dim(A) = \mathrm{findim}(A) = m.$$

2.4. Path rings

A quiver Q consists of:

- The set Q_0 of vertices of Q.
- The set Q_1 of arrows of Q.
- Two functions: $s : Q_1 \to Q_0$ giving the start or source of the arrow, and $t : Q_1 \to Q_0$ giving the target of the arrow.

We say that a quiver Q is finite if Q_0 and Q_1 are finite. A path of length n in Q is a sequence of arrows $\alpha_n\alpha_{n-1}\ldots\alpha_3\alpha_2\alpha_1$ such that $t(\alpha_{i+1}) = s(\alpha_i)$. We also agree to associate with each point $a \in Q_0$ a path of length 0, called the trivial path at a, and denoted by e_a. For the composition of paths, we use the convention of concatenating paths from right to left. We denote by $\mathbb{P}(v, w)$ the set of paths with start v and target w.

By the previous facts, we observe that Q can be considered as a category.

Note 1. *Given a finite dimensional $\mathbb{k}$-algebra A and a finite acyclic quiver Q, we denote by $Rep_A(Q)$ the category of funtors from Q to* $\mathrm{mod}A$.

Note 2. *We denote by AQ the path algebra with quiver Q and coefficients over A, i.e. $A \otimes_{\mathbb{k}} \mathbb{k}Q$.*

Remark 2.13. *The categories $Rep_A(Q)$ and $Rep_{\mathbb{k}}(AQ)$ are equivalent for any finite dimensional $\mathbb{k}$-algebra A and any finite acyclic quiver Q.*

Definition 2.14. *Let A be a finite dimensional $\mathbb{k}$-algebra and a finite acyclic quiver Q. Given M an A-module and v a vertex of Q, then we denote by M^v the AQ-module such that:*

- *$M^v(v) = M$, $M^v(w) = 0$ if $w \neq v$, and*
- *$M^v(\alpha) = 0$ for every arrow in Q_1.*

Definition 2.15. *Let Q be a finite acyclic quiver. We denote by:*

- *$\bar{P}^v$, if P is a projective A-module, the following AQ module:*

- $\bar{P}^v(w) = \begin{cases} P & \text{if } w = v; \\ \oplus_{\lambda \in \mathbb{P}(v,w)} P_\lambda & \text{if } w \neq v \text{ and } \mathbb{P}(v,w) \neq \emptyset; \\ 0 & \text{otherwise,} \end{cases}$

  where $P_\lambda = P$ *for all* $\lambda \in \mathbb{P}(v,w)$.

- $\bar{P}^v(\alpha) = \begin{cases} f_\alpha & \text{if } \alpha \in \mathbb{P}(w_1,w_2) \text{ and } \mathbb{P}(v,w_1) \neq \emptyset; \\ 0 & \text{otherwise,} \end{cases}$

  where $f_\alpha = \sum_{\lambda \in \mathbb{P}(v,w_1)} \bar{P}^v(w_1) \overset{\pi_\lambda}{\to} P_\lambda \overset{1_P}{\to} P_{\lambda\alpha} \hookrightarrow \bar{P}^v(w_2)$.

- $\tilde{I}^v$, *if* I *is an injective* A*-module, the following* AQ *module:*

  - $\tilde{I}^v(w) = \begin{cases} I & \text{if } w = v; \\ \oplus_{\lambda \in \mathbb{P}(w,v)} I_\lambda, & \text{if } w \neq v \text{ and } \mathbb{P}(w,v) \neq \emptyset; \\ 0 & \text{otherwise,} \end{cases}$

    where $I_\lambda = I$ *for all* $\lambda \in \mathbb{P}(v,w)$.

  - $\tilde{I}^v(\alpha) = \begin{cases} f_\alpha & \text{if } \alpha \in \mathbb{P}(w_1,w_2) \text{ and } \mathbb{P}(w_2,v) \neq \emptyset; \\ 0, & \text{otherwise,} \end{cases}$

    where $f_\alpha = \sum_{\lambda \in \mathbb{P}(w_2,v)} \tilde{I}^v(w_1) \overset{\pi_\lambda}{\to} I_{\alpha\lambda} \overset{1_I}{\to} I_\lambda \hookrightarrow \tilde{I}^v(w_2)$.

Definition 2.16. *Let* A *be a finite dimensional* $\Bbbk$*-algebra and* Q *be a finite acyclic quiver. If* $\iota : M \hookrightarrow P$ *is a monomorphism of* A*-modules where* P *is a projective module and* v *a vertex in* Q*, then we denote by* MP^v *the following* AQ*-module:*

- $MP^v(w) = \begin{cases} M & \text{if } w = v; \\ \oplus_{\lambda \in \mathbb{P}(v,w)} P_\lambda & \text{if } w \neq v \text{ and } \mathbb{P}(v,w) \neq \emptyset; \\ 0 & \text{otherwise,} \end{cases}$

  where $P_\lambda = P$ *for all* $\lambda \in \mathbb{P}(v,w)$.

- $MP^v(\alpha) = \begin{cases} \iota_\alpha & \text{if } \alpha \text{ starts in } v; \\ f_\alpha & \text{if } \alpha \in \mathbb{P}(w_1,w_2) \text{ and } \mathbb{P}(v,w_1) \neq \emptyset; \\ 0 & \text{otherwise,} \end{cases}$

  where:

  - $\iota_\alpha = M \hookrightarrow P_\alpha \hookrightarrow \oplus_{\lambda \in \mathbb{P}(v,w)} P_\lambda$
  - $f_\alpha = \sum_{\lambda \in \mathbb{P}(v,w_1)} MP^v(w_1) \overset{\pi_\lambda}{\to} P_\lambda \overset{1_P}{\to} P_{\lambda\alpha} \hookrightarrow MP^v(w_2)$.

Remark 2.17. *Given* M, N, P *and* P' A*-modules where* P *and* P' *are projectives and the morphisms* $\iota : M \to P$ *and* $\iota' : N \to P'$ *are inclusions, then* $MP^v \cong NP'^v$ *if and only if the following commutative diagram exists*

$$\begin{array}{ccc} M & \xrightarrow{\cong} & N \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle\iota'} \\ P & \xrightarrow{\cong} & P' \end{array}$$

The following theorem can be found in [4] in a more general version:

Theorem 2.18. *Let Q be a finite quiver without oriented cycles and A a finite dimensional $\mathbb{k}$-algebra. A representation P of AQ is projective if and only if the following conditions are satisfied:*

1. *For every $v \in Q$, $P(v)$ is a projective A-module.*
2. *For every $v \in Q$, the morphism $\oplus_{t(\alpha)=v} P(s(\alpha)) \to P(v)$ (where $P(s(\alpha)) \to P(v)$ is $P(\alpha)$) is a split monomorphism.*

The next theorem is the dual version of the previous one

Theorem 2.19. *Let Q be a finite quiver without oriented cycles and A be a finite dimensional $\mathbb{k}$-algebra. A representation I of AQ is injective if and only if the following conditions are satisfied:*

1. *For every $v \in Q$, $I(v)$ is an injective A-module.*
2. *For every $v \in Q$, the morphism $I(v) \to \oplus_{s(\alpha)=v} I(t(\alpha))$ (where $I(v) \to I(t(\alpha))$ is $I(\alpha)$) is a split epimorphism.*

Remark 2.20. *From Theorem 2.18 we can see that every indecomposable projective module is of the form $\bar{P}^v$ where P is an indecomposable projective module and $v \in Q_0$. Dually from Theorem 2.19 one can see that every indecomposable injective module is of the form $\tilde{I}^v$ where I is an indecomposable injective module and $v \in Q_0$.*

3. Igusa-Todorov functions for path rings

The objective for this section is to relate the ϕ-dimension and ψ-dimension of A and AQ.
Let us begin, by computing the syzygies for a particular class of AQ-modules.

Proposition 3.1. *Let A be a $\mathbb{k}$-algebra and Q be a finite acyclic quiver. If M is a finitely generated A-module and v a vertex of Q, then following results holds:*

1. *If v is a sink, then $\Omega^k_{AQ}(M^v) = (\Omega^k_A(M))^v$.*
2. *If v is not a sink, then $\Omega^k_{AQ}(M^v) = \Omega^k_A(M) {P_{k-1}}^v$, where*

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

is the minimal projective resolution of M.

Proof:

1. *Is clear.*

2. *Consider the AQ-module* $(\Omega_A M)P_0^v$. *By Definition 2.16, it has the following shape:*

$$(\Omega_A M)P_0^v(w) = \begin{cases} \Omega_A M, & \text{if } w = v; \\ \oplus_{\lambda \in \mathbb{P}(v,w)} P_\lambda, & \text{if } w \neq v \text{ and } \mathbb{P}(v,w) \neq \emptyset; \\ 0, & \text{otherwise.} \end{cases}$$

where $P_\lambda = P_0$ *for all* $\lambda \in \mathbb{P}(v,w)$.

Let α *be an arrow starting at* v. *The following diagram shows locally the syzygy of* M^v *at the arrow* α*:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega(M) & \xrightarrow{i} & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle 1} & & \downarrow & & \\ 0 & \longrightarrow & P_0 & \xrightarrow{1} & P_0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

If α *is an arrow that does not start at* v, *but it belongs to any path starting at* v, *then there exists the following commutative diagram with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_0 & \xrightarrow{1} & P_0 & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow & & \\ 0 & \longrightarrow & P_0 & \xrightarrow{1} & P_0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

and therefore $\Omega_{AQ}(M^v) = (\Omega_A M)P_0^v$.

Suppose that $\Omega^k(M^v) = (\Omega^k(M)P_{k-1})^v$.

If α *is an arrow that start at* v, *we obtain the following commutative diagram with exact rows, such that the first one locally represents* $\Omega^k_{AQ}(M^v)$ *at* α*:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^{k+1}(M) & \xrightarrow{i_k} & P_k & \xrightarrow{f_k} & \Omega^k M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i'_k} & & \downarrow{\scriptstyle j_k} & & \downarrow{\scriptstyle i'_{k-1}} & & \\ 0 & \longrightarrow & P_k & \xrightarrow{j'_k} & P_k \oplus P_{k-1} & \xrightarrow{(i'_{k-1} \circ f_k, 1)} & P_{k-1} & \longrightarrow & 0 \end{array}$$

where $j_k = (1_{P_k}, 0)$. *We deduce that the following diagram commutes:*

$$\begin{array}{ccc} \Omega^{k+1}(M) & \xrightarrow{1} & \Omega^{k+1}(M) \\ \downarrow{\scriptstyle i'_k} & & \downarrow{\scriptstyle i_k} \\ P_k & \xrightarrow{h_k} & P_k \end{array}$$

where h_k is the first coordenate of the map j'_k. The second coordenate of $(i'_{k-1} \circ f_k, 1_{P_{k-1}})$ is an isomorphism, then h_k must be a monomorphism. Therefore the rows of the previous diagrams are isomorphisms. This implies that the left column can be changed by the right one in the diagram above.

If α is an arrow that does not start at v, but it belongs to a path that start at v, then the following diagram with exact rows commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_k & \xrightarrow{j'_k} & P_k \oplus P_{k-1} & \xrightarrow{(i'_{k-1}\circ f_k, 1)} & P_{k-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \\
0 & \longrightarrow & P_k & \xrightarrow{j'_k} & P_k \oplus P_{k-1} & \xrightarrow{(i'_{k-1}\circ f_k, 1)} & P_{k-1} & \longrightarrow & 0
\end{array}$$

therefore the result follows.

□

Remark 3.2. *Assume the hypothesis of Proposition 3.1. Let M be a finitely generated A-module. By using the fact that $0P^v$ is a projective AQ-module, if M is a finite dimensional A-module, it follows that:*

1. *If v is a sink, then $\mathrm{pd}(M^v) = \mathrm{pd}(M)$.*
2. *If v is not a sink, then $\mathrm{pd}(M^v) = \mathrm{pd}(M) + 1$ and $\mathrm{pd}(MP^v) = \mathrm{pd}(M) + 1$.*

In the following proposition we compute the Igusa-Todorov function ϕ for AQ-modules of the form M^v.

Proposition 3.3. *Let A be a $\mathbb{k}$-algebra and Q a finite acyclic quiver. If M is an A-module and v a vertex of Q, then the following results hold:*

1. *If v is a sink, then $\phi(M^v) = \phi(M)$.*
2. *If v is not a sink, then $\phi(M^v) = \phi(M) + 1$.*

Proof:

1. *It is clear by the first part of Proposition 3.1.*
2. *Let M be an A-module such that $\phi(M) = k$ and $M = \oplus_{i\in I} M_i^{l_i}$ the decomposition into indecomposables modules of M.*

   Consider the linear combination $\sum_{i\in I} \alpha_i \overline{\Omega}^k([M_i]) = 0$ for $k \geq 1$, such that $\sum_{i\in I} \alpha_i \overline{\Omega}^{k-1}([M_i]) \neq 0$. If we prove that the previous condition implies that

$\sum_{i\in I}\alpha_i\Omega^{k+1}([M^v{}_i]) = 0$ and $\sum_{i\in I}\alpha_i\overline{\Omega}^k([M^v{}_i]) \neq 0$, we obtain the inequality $\phi(M^v) \geq \phi(M) + 1$.

<u>Claim:</u>

$$\text{If } \sum_{i\in I}\alpha_i\overline{\Omega}^k([M_i]) = 0 \text{ and } \sum_{i\in I}\alpha_i\overline{\Omega}^{k-1}([M_i]) \neq 0 \text{ then}$$

$$\sum_{i\in I}\alpha_i\overline{\Omega}^{k+1}([M^v{}_i]) = 0 \text{ with } \sum_{i\in I}\alpha_i\overline{\Omega}^k([M^v{}_i]) \neq 0.$$

Suppose that $\sum\alpha_i\overline{\Omega}^k([M_i]) = 0$. Then there exist projectives P_1 and P_2 such that $\oplus_{i\in I_1}\Omega^k(M_i)^{\beta_i}\oplus P_1 \cong \oplus_{i\in I_2}\Omega^k(M_i)^{\beta_i}\oplus P_2$, where $I = I_1 \cup I_2$ is a disjoint union and the exponents $\beta_i = |\alpha_i|$ for every $i \in I$. Then it follows the next commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \oplus_{i\in I_1}\Omega^{k+1}(M_i)^{\beta_i} & \xrightarrow{\iota} & P & \longrightarrow & \oplus_{i\in I_1}\Omega^k(M_i)^{\beta_i}\oplus P_1 & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
0 & \longrightarrow & \oplus_{i\in I_2}\Omega^{k+1}(M_i)^{\beta_i} & \xrightarrow{\iota} & P' & \longrightarrow & \oplus_{i\in I_2}\Omega^k(M_i)^{\beta_i}\oplus P_2 & \longrightarrow & 0
\end{array}$$

Using Remark 2.17 in the left square of the previous diagram, we obtain the following

$$(\oplus_{i\in I_1}\Omega^{k+1}(M_i)^{\beta_i})P^v \cong (\oplus_{i\in I_2}\Omega^{k+1}(M_i)^{\beta_i})P'^v.$$

Therefore by Proposition 3.1 $\sum\alpha_i\overline{\Omega}^{k+1}([M_i^v]) = 0$.

Suppose that $\sum\alpha_i\overline{\Omega}^k([M^v{}_i]) = 0$. This implies, by Remark 2.17, that the following diagram is commutative with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \oplus_{i\in I_1}\Omega^{k}(M_i)^{\beta_i} & \xrightarrow{\iota} & P & \longrightarrow & \oplus_{i\in I_1}\Omega^{k-1}(M_i)^{\beta_i}\oplus P_1' & \longrightarrow & 0 \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \\
0 & \longrightarrow & \oplus_{i\in I_2}\Omega^{k}(M_i)^{\beta_i} & \xrightarrow{\iota} & P' & \longrightarrow & \oplus_{i\in I_2}\Omega^{k-1}(M_i)^{\beta_i}\oplus P_2' & \longrightarrow & 0
\end{array}$$

and we obtain the relation $\sum\alpha_i\overline{\Omega}^{k-1}([M_i]) = 0$, which is a contradiction.

□

As a consequence of the previous results, we obtain the following theorem that is trivial in the case Q has one vertex:

Theorem 3.4. *Let A be a finite dimensional $\Bbbk$-algebra. If Q is a finite acyclic quiver with at least two vertices, then the inequality below holds:*

$$\phi\text{dim}AQ \geq \phi\text{dim}A + 1.$$

Proof:
It follows from Proposition 3.3. □

4. Special path rings

In this section we study the ϕ-dimension for algebras AQ when A is a Gorenstein algebra. Using these computations, we show an example of an algebra of Ω^{∞}-infinite representation type with finite ϕ-dimension.

The following theorem can be found in [2].

Theorem 4.1. *[2, Proposition 2.2] If A and B are finite dimensional algebras over a field $\Bbbk$, then*

$$\max(\text{id}_A A, \text{id}_B B) \leq \text{id}_{A\otimes B}(A \otimes_{\Bbbk} B) \leq \text{id}_A A + \text{id}_B B.$$

The last theorem shows that tensor products of Gorenstein algebras are Gorenstein algebras. Now, by the above theorem and Theorem 3.4 we obtain the next result:

Theorem 4.2. *Let A be a finite dimensional $\Bbbk$-algebra and Q be a finite acyclic quiver with at least two vertices. Then A is n-Gorenstein if and only if AQ is $(n+1)$-Gorenstein.*

Proof:
Let n be the minimum non-negative integer such that A is an n-Gorenstein algebra. By Corollary 2.12 and Theorem 3.4 we obtain $\phi\dim AQ \geq \phi\dim A + 1 = n+1$. Again by Corollary 2.12 we have that $\text{id}_{AQ}AQ = \phi\dim AQ \geq n+1$. On the other hand, the inequality $\text{id}_{AQ}AQ \leq \text{id}_A A + 1 < \infty$ follows from Theorem 4.1. This proves that $\text{id}_{AQ}AQ = n+1$, thus AQ is $(n+1)$-Gorenstein with minimum n.
Now suppose that AQ is an $(n+1)$-Gorenstein algebra. Let I be an indecomposable injective A-module with $\text{pd}(\tilde{I}^v) = k \leq n+1$ for v a source (therefore it is not a sink of Q). This implies that $\text{pd}I = k-1$, by Remark 3.2. The proof is analoguos if we consider the injective dimension of an indecomposable projective A-module. Thus A is a t-Gorenstein algebra with $t \leq n$. □

As a consequence of the previous theorem we obtain another proof for the following result of [12].

Corollary 4.3. *[12, Lemma 4.1] The algebra $T_n(A)$ with $n \geq 2$ is $(m+1)$-Gorenstein if and only if A is m-Gorenstein, where:*

$$T_n(A) = \begin{pmatrix} A & 0 & \cdots & 0 & 0 \\ A & A & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A & A & \cdots & A & 0 \\ A & A & \cdots & A & A \end{pmatrix}$$

Proof:

It is easy to see that $T_n(A) \cong AQ$ where Q is an A_n Dynkin type quiver with its arrows in the same orientation. Then by Theorem 4.2 we obtain the thesis. □

Definition 4.4. *Given an Artin algebra A, we define K_n as*

$$K_n = \{[M] : \text{ there exist } N \in \operatorname{mod} A \text{ such that } \Omega^n(N) = M\}$$

Definition 4.5. *Let A be an Artin algebra. The class K_n is of* **finite type** *if there exists an A-module N such that $M \in \operatorname{add} N$ for all $[M] \in K_n$. In this case A is called of* Ω^n**-finite representation type**. *If A is not of Ω^n finite representation type for any $n \in \mathbb{N}$ we say that A is* Ω^∞**-infinite representation type**.

In [11] was proved that if A is of Ω^n-finite representation type for some n then $\phi\dim(A)$ and $\psi\dim(A)$ are both finite. A natural question to ask is which is the behaviour of these dimensions in the case that A is of Ω^∞-infinite representation type. we show in Example 4.1, that they can be finite.

Proposition 4.6. *[11, Proposition 4.2]*
Given G a Gorenstein projective module, $[G] \in K_i$ for every $i \in \mathbb{N}$.

Therefore if A has infinite indecomposable Gorenstein projective modules up to isomorphism (A is CM-infinite), A is of Ω^∞-infinite representation type.

Proposition 4.7. *Let A be a Gorenstein algebra and Q be a finite acyclic quiver with at least two vertices. If M is a Gorenstein indecomposable A-module and v a sink of Q, then M^v is a Gorenstein indecomposable AQ-module.*

Proof:

Suppose that A is an n-Gorenstein algebra. By Theorem 4.2 AQ is an $(n+1)$-Gorenstein algebra. Since M is Gorenstein, there exists N a A-module such that $\Omega_A^{n+1}(N) = M$. Using Proposition 3.1 it follows that $M^v = \Omega_{AQ}^{n+1}(N^v)$, and by Proposition 2.4 the result is obtained. □

The following example exhibits a family of algebras $\{A_n\}_{n\in\mathbb{N}}$ such that $\phi\dim(A_n) = n$ and every A_n is Ω^∞-infinite representation type.

Example 4.1. *Let A be the radical square zero algebra with associated quiver Q as follows:*

$$1\cdot \rightleftarrows \cdot 2$$

It is clear that A is a selfinjective $\Bbbk$-algebra.

Now consider $A^e = A^{op} \otimes A$. It is not difficult to see that A^e is a selfinjective algebra of infinite representation type. Consider Γ a finite acyclic quiver, with a sink v. A_k denotes the following algebras:

$$A_k = \begin{cases} A_k = A^e & \text{if } k = 0 \\ A_k = A_{k-1}\Gamma & \text{if } k \geq 1 \end{cases}$$

Then A_k is a k-Gorenstein algebra, but is not $(k-1)$-Gorenstein for $k \geq 1$, in particular $\phi\dim(A_k) = \psi\dim(A_k) = k$ and all of them are of Ω^∞-infinite representation type because $\mathcal{G}P_A$ is of infinite representation type by Proposition 4.7.

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Gustavo Mata,
IMERL,
Facultad de Ingeniería-Universidad de la República,
Uruguay.
E-mail address: gmata@fing.edu.uy