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Finitely Generated Rings Obtain From Hyperrings Through the Fundamental Relations

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ABSTRACT: In this article, we introduce and analyze a strongly regular relation $\omega^*_{\mathcal{A}}$ on a hyperring R such that in a particular case we have $|R/\omega^*_{\mathcal{A}}| \leq 2$ or $R/\omega^*_{\mathcal{A}} =< \omega^*_{\mathcal{A}}(a) >$, i.e., $R/\omega^*_{\mathcal{A}}$ is a finite generated ring. Then, by using the notion of $\omega^*_{\mathcal{A}}$-parts, we investigate the transitivity condition of $\omega_{\mathcal{A}}$. Finally, we investigate a strongly regular relation $\chi^*_{\mathcal{A}}$ on the hyperring R such that $R/\chi^*_{\mathcal{A}}$ is a finitely generated commutative ring.

Key Words: Hyperring, Strongly regular relation, Ring.

Contents

1. Introduction

A hypergroup in the sense of Marty [9] is a non-empty set H endowed with a hyperoperation $\cdot : H \times H \longrightarrow \wp^*(H)$, the set of all non-empty subset of H, which satisfies the associative law and the reproduction axiom. If $(H, \cdot)$ is a hypergroup and $\rho \subseteq H \times H$ is an equivalence relation, then for all non-empty subsets A, B of H we set $A \,\overline{\overline{\rho}}\, B$ if and only if $a\rho b$, for all $a \in A,\ b \in B$. The relation ρ is called *strongly regular on the right* (*on the left*) if $x\ \rho\ y \Rightarrow a{\cdot}x \,\overline{\overline{\rho}}\, a{\cdot}y (x\ \rho\ y \Rightarrow x{\cdot}a \,\overline{\overline{\rho}}\, y{\cdot}a$, respectively), for all $(x, y, a) \in H^3$. Moreover, ρ is called strongly regular if it is strongly regular on the right and on the left. Let H be a hypergroup and ρ an equivalence relation on H. A hyperoperation $\otimes$ is defined on H/ρ by $\rho(a)\otimes\rho(b) = \{\rho(x)|x \in \rho(a)\cdot\rho(b)\}$. If ρ is strongly regular, then it readily follows that $\rho(a)\otimes\rho(b) = \{\rho(x) \mid x \in a{\cdot}b\}$. It is well known for ρ strongly regular that $\langle H/\rho, \otimes\rangle$ is a group, that is, $\rho(a)\otimes\rho(b) = \rho(c)$ for all $c \in a \cdot b$. Basic definitions and propositions about the hyperstructures can be found in [2,3,5]. Krasner [8] has studied the notion of *hyperfield, hyperring*, and then some researchers works on this subject. The more general structure that satisfies the ring-like axioms is the hyperring in the general sense: $(R, +, \cdot)$ is a hyperring if $+$ and $\cdot$ are two hyperoperations such that $(R, +)$ is a hypergroup and $\cdot$ is an associative hyperoperation, which is distributive with respect to $+$. We call

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$(R, +, \cdot)$ a *hyperfield* if $(R, +, \cdot)$ is a hyperring and $(R, \cdot)$ is a hypergroup. There are different notions of hyperrings. If only the addition $+$ is a hyperoperation and the multiplication $\cdot$ is a binary operation, then the hyperring is called Krasner additive hyperring [8]. Davvaz and Leoreanu-Fotea [5] published a book titled Hyperring Theory and Applications. The hyperrings were studied by many authors, for example see [4,7,11,12,13,15]. In [1], Babaeia et al. introduced the notion of $\Re$-parts in hyperrings as a generalization of complete parts in hyperrings. In [5] there are several types of hyperrings and hyperfields. In what follows we shall consider one of the most general types of hyperrings.

Definition 1.1. *[14] The triple $(R, +, \cdot)$ is a hyperring if (1) $(R, +)$ is a hypergroup; (2) $(R, \cdot)$ is a semihypergroup; (3) the hyperoperation "$\cdot$" is distributive over the hyperoperation "$+$", which means that for all x, y, z of R we have: $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x + y) \cdot z = x \cdot z + y \cdot z$. We call $(R, +, \cdot)$ a hyperfield if $(R, +, \cdot)$ is a hyperring and $(R, \cdot)$ is a hypergroup.*

Example 1.2. *Let $R = \{0, 1, 2, 3, 4\}$ be a set with the hyperoperations $+$ and $\cdot$ defined as follow:*

$+$	0	1	2	3	4
0	$\{0,4\}$	1	$\{2,3\}$	$\{2,3\}$	$\{0,4\}$
1	1	$\{2,3\}$	$\{0,4\}$	$\{0,4\}$	1
2	$\{2,3\}$	$\{0,4\}$	1	1	$\{2,3\}$
3	$\{2,3\}$	$\{0,4\}$	1	1	$\{2,3\}$
4	$\{0,4\}$	1	$\{2,3\}$	$\{2,3\}$	$\{0,4\}$

$\cdot$	0	1	2	3	4
0	$\{0,4\}$	$\{0,4\}$	$\{0,4\}$	$\{0,4\}$	$\{0,4\}$
1	$\{0,4\}$	1	$\{2,3\}$	$\{2,3\}$	$\{0,4\}$
2	$\{0,4\}$	$\{2,3\}$	1	1	$\{0,4\}$
3	$\{0,4\}$	$\{2,3\}$	1	1	$\{0,4\}$
4	$\{0,4\}$	$\{0,4\}$	$\{0,4\}$	$\{0,4\}$	$\{0,4\}$

Then $(R, +, \cdot)$ is a finite hyperring such that is not a ring.

Example 1.3. *Let $(R, +, \cdot)$ be a finite ring and S be a non-empty finite set such that $S \cap R = \emptyset$. Let $A = R \cup S$ and define two hyperoperations $\oplus$ and $\odot$ on A as follow: For all $x, y \in R$ and $s, t \in S$*

$$x \oplus y = \begin{cases} x + y & \text{if } x + y \neq 0 \\ S \cup \{0\} & \text{if } x + y = 0 \end{cases} \quad \text{and } x \odot y = \begin{cases} x \cdot y & \text{if } x \cdot y \neq 0 \\ S \cup \{0\} & \text{if } x \cdot y = 0 \end{cases}$$

and

$$x \oplus t = x \oplus 0, \;\; s \oplus y = 0 \oplus y, s \oplus t = S \cup \{0\}, \;\; x \odot t = s \odot y = s \odot t = S \cup \{0\}$$

It is not difficult to see that $(A, \oplus, \odot)$ is a proper finite hyperring.

Let us recall now some important equivalence relations and results of hypergroup and hyperring theory.

Definition 1.4. *[15] Let $(R, +, \cdot)$ be a hyperring. We define the relation γ as follows: $x \ \gamma \ y \Leftrightarrow \exists n \in \mathbb{N}, \exists (k_1, \cdots, k_n) \in \mathbb{N}^n$ and $[\exists (x_{i1}, \cdots, x_{ik_i}) \in R^{k_i}, (i = 1, \cdots, n)]$ such that*

$$x, y \in \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}).$$

Let γ^* be the transitive closure of γ. The fundamental relation γ^* on R can be considered as the smallest equivalence relation such that the quotient R/γ^* be a ring.

Definition 1.5. *[6] Let $(R, +, \cdot)$ be a hyperring. We define the relation α as follows: $x \ \alpha \ y \Leftrightarrow \exists n \in \mathbb{N}, \exists (k_1, \cdots, k_n) \in \mathbb{N}^n, \exists \tau \in \mathbb{S}_n$ and $[\exists (x_{i1}, \cdots, x_{ik_i}) \in R^{k_i}, \exists \tau_i \in \mathbb{S}_{k_i}, (i = 1, \cdots, n)]$ such that*

$$x \in \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \text{ and } y \in \sum_{i=1}^{n} A_{\tau(i)},$$

where $A_i = \prod_{j=1}^{k_i} x_{i\tau_i(j)}$.

Let α^* be the transitive closure of α. Then, α^* is the smallest strongly regular relation on R such that R/α^* is a commutative ring.

Definition 1.6. *[5] Let $(R, +, \cdot)$ be a hyperring and M be a non-empty subset of R. We say that M is a α-part if for every $n \in \mathbb{N}, \exists (k_1, \cdots, k_n) \in \mathbb{N}^n, \exists \tau \in \mathbb{S}_n$ and $[\exists (x_{i1}, \cdots, x_{ik_i}) \in R^{k_i}, \exists \tau_i \in \mathbb{S}_{k_i}, (i = 1, \cdots, n)]$ such that*

$$\sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \cap M \neq \emptyset \Rightarrow \sum_{i=1}^{n} A_{\tau(i)} \subseteq M,$$

where $A_i = \prod_{j=1}^{k_i} x_{i\tau_i(j)}$. Also, M is said to be a complete part of R [10], if we have

$$\sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \cap M \neq \emptyset \Rightarrow \sum_{i=1}^{n} (\prod_{j=1}^{k_i} x_{ij}) \subseteq M.$$

2. The Relation $\omega_{\mathcal{A}}$

In this section, we introduce the relation $\omega_{\mathcal{A}}$ on a hyperring R, which we use in order to obtain a finite generated ring as a quotient structure of R.

Let $(R, +, \cdot)$ be a hyperring, $\mathcal{A}$ is a non-empty subset of R, $a_1, \cdots, a_m \in \mathcal{A}$ and

$$\mathcal{D} = \{t \mid t \in \sum_{i=1}^{m} z_i a_i + \sum_{i=1}^{m} s_i a_i + \sum_{i=1}^{m} a_i t_i + \sum_{i=1}^{m} (\sum_{i=1}^{n_i} u_{i,k} a_i v_{i,k}), \\ m, n_i \in \mathbb{N}, \ z_i \in \mathbb{Z}, \ s_i, t_i, u_{i,k}, v_{i,k} \in R\}.$$

For all $n \geq 1$ and $(k_1, \cdots, k_n) \in \mathbb{N}^n$ define $\mathcal{R}^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$ as follows:

$$\mathcal{R}^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n} := \gamma^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n} \cup \Im^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n} \cup \xi^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n},$$

where

$$\gamma^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n} := \{(\sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij}, \sum_{i=1}^{n}\prod_{j=1}^{k_i} y_{ij}) \mid \sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij} = \sum_{i=1}^{n}\prod_{j=1}^{k_i} y_{ij}\},$$

$$\begin{aligned}\Im^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n} &:= \{(\sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij}, \sum_{i=1}^{n}\prod_{j=1}^{k_i} y_{ij}) \mid \{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \\ &= \{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \neq \varnothing\}\end{aligned}$$

and

$$\begin{aligned}\xi^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n} &:= \{(\sum_{i=1}^{n}\prod_{j=1}^{k_i} x_{ij}, \sum_{i=1}^{n}\prod_{j=1}^{k_i} y_{ij}) \mid \{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \\ &= \{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \varnothing\}.\end{aligned}$$

Notice that $\mathcal{R}^{\mathcal{A}}_{1,1} := \{(\{x\}, \{y\}) \mid \{x, y\} \cap \mathcal{D} = \emptyset \ \text{ or } x = y\}$.

Definition 2.1. *We define the relation $\omega_{\mathcal{A}}$ on $(R, +, \cdot)$ as follows:*

$$x \ \omega_{\mathcal{A}} \ y \Leftrightarrow \ \exists (A, B) \in \mathcal{R}^{\mathcal{A}}_{n,k_1,k_2\cdots,k_n}, \text{ such that } \ x \in A, \ y \in B.$$

Notice that for $n = 1$ and $k_1 = 1$ we obtain $x \ \omega_{\mathcal{A}} \ y$ if and only if $(\{x\}, \{y\}) \in \mathcal{R}^{\mathcal{A}}_{1,1}$ or $x = y \in \mathcal{D}$.

Remark 2.1. *The relation $\omega_{\mathcal{A}}$ is reflexive and symmetric and $\beta \subseteq \omega_{\mathcal{A}}$ and $\gamma \subseteq \omega_{\mathcal{A}}$.*

Let $\omega^*_{\mathcal{A}}$ be the transitive closure of $\omega_{\mathcal{A}}$. In order to analyze the quotient hyperstructure with respect to this equivalence relation, we state the following lemma.

Lemma 2.2. *$\omega^*_{\mathcal{A}}$ is a strongly regular equivalence relation both on $(R, +)$ and on $(R, \cdot)$.*

Proof. Clearly, $\omega^*_{\mathcal{A}}$ is an equivalence relation. In order to prove that it is strongly regular, it is enough to show that

$$x \ \omega_{\mathcal{A}} \ y \Longrightarrow \begin{cases} x + a \ \overline{\overline{\omega}}_{\mathcal{A}} \ y + a, & a + x \ \overline{\overline{\omega}}_{\mathcal{A}} \ a + y, \\ x \cdot a \ \overline{\overline{\omega}}_{\mathcal{A}} \ y \cdot a, & a \cdot x \ \overline{\overline{\omega}}_{\mathcal{A}} \ a \cdot y, \end{cases}$$

for all $a \in R$. Since $x \ \omega_{\mathcal{A}} \ y$, it follows that there exists $(A, B) \in \mathcal{R}^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$ such that $x \in A$ and $y \in B$. We distinguish the following situations.

Case 1. Suppose that $(A, B) \in \gamma^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$ such that $x \in A = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} y_{ij})$. Then, we have $x + a \subseteq A + a = (\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})) + a$ and

$y + a \subseteq B + a = (\sum_{i=1}^{n}(\prod_{j=1}^{k_i} y_{ij})) + a$. Set $a = x_{n+1\ 1} = y_{n+1\ 1}$ and $k_{n+1} = 1$. Thus,

$$x + a \subseteq (\sum_{i=1}^{n+1}(\prod_{j=1}^{k_i} x_{ij}) \textit{ and } y + a \subseteq (\sum_{i=1}^{n+1}(\prod_{j=1}^{k_i} y_{ij})).$$

It is easy to see that the pair $(A + a, B + a)$ belongs to $\gamma^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n} \subseteq \mathcal{R}^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n}$. Therefore, for all $u \in x+a$ and $v \in y+a$, we have $u \in x+a \subseteq A+a$ and $v \in y + a \subseteq B + a$. So, $u\ \omega_{\mathcal{A}}\ v$. Thus, $x + a\ \overline{\overline{\omega}}_{\mathcal{A}}\ y + a$.

Case 2. Suppose that $(A, B) \in \Im^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Then, we have $x \in A = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \neq \varnothing$. If $a \notin \mathcal{D}$, then $(A+a, B+a) \in \Im^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n}$ and if $a \in \mathcal{D}$, then $(A + a, B + a) \in \Im^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n}$. Thus, according to *Case* 1, $(A + a, B + a) \in \mathcal{R}^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n}$. So, $u\ \omega_{\mathcal{A}}\ v$. Thus, $x + a\ \overline{\overline{\omega}}_{\mathcal{A}}\ y + a$.

Case 3. Suppose that $(A, B) \in \xi^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Then, we have $x \in A = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \varnothing$. If $a \notin \mathcal{D}$, then $(A+a, B+a) \in \xi^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n}$ and if $a \in \mathcal{D}$, then $(A + a, B + a) \in \Im^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n}$. Thus, according to *Case* 1, $(A + a, B + a) \in \mathcal{R}^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n}$. So, $u\ \omega_{\mathcal{A}}\ v$. This implies that $x + a\ \overline{\overline{\omega}}_{\mathcal{A}}\ y + a$.

In the same way, we can show that $a + x\ \overline{\overline{\omega}}_{\mathcal{A}}\ a + y$. It is easy to see that

$$a + x\ \overline{\overline{\omega}}^{*}_{\mathcal{A}}\ a + y \quad \text{and} \quad x + a\ \overline{\overline{\omega}}^{*}_{\mathcal{A}}\ y + a.$$

Notice that for $(R, \cdot)$ we have

Case 1. Suppose that $(A, B) \in \gamma^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$ such that $x \in A = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} y_{ij})$. Then, we obtain $x \cdot a \subseteq A \cdot a = (\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})) \cdot a$ and $y \cdot a \subseteq B \cdot a = (\sum_{i=1}^{n}(\prod_{j=1}^{k_i} y_{ij})) \cdot a$. Set $k'_i = k_i + 1$, $x_{ik'_i} = a$ and $y_{ik'_i} = a$. Thus,

$$x \cdot a \subseteq (\sum_{i=1}^{n}(\prod_{j=1}^{k'_i} x_{ij})) \text{ and } y \cdot a \subseteq (\sum_{i=1}^{n}(\prod_{j=1}^{k'_i} y_{ij})).$$

It is easy to see that the pair $(A \cdot a, B \cdot a)$ belongs to $\gamma^{\mathcal{A}}_{n,k'_1,k'_2,\cdots,k'_n} \subseteq \mathcal{R}^{\mathcal{A}}_{n,k'_1,k'_2,\cdots,k'_n}$. Therefore, for all $u \in x \cdot a$ and $v \in y \cdot a$, we have $u \in x \cdot a \subseteq A \cdot a$ and $v \in y \cdot a \subseteq B \cdot a$. So, $u\ \omega_{\mathcal{A}}\ v$. This implies that $x \cdot a\ \overline{\overline{\omega}}_{\mathcal{A}}\ y \cdot a$.

Case 2. Suppose that $(A, B) \in \Im^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Then, we have $x \in A = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} =$

$\{y_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \neq \varnothing$. If $a \notin \mathcal{D}$, then $(A \cdot a, B \cdot a) \in \Im^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n}$ and if $a \in \mathcal{D}$, then $(A \cdot a, B \cdot a) \in \Im^{\mathcal{A}}_{n,k'_1,k'_2,\cdots,k'_n}$. Thus, according to *Case* 1, $(A \cdot a, B \cdot a) \in \mathcal{R}^{\mathcal{A}}_{n,k'_1,k'_2,\cdots,k'_n}$. So, $u\ \omega_{\mathcal{A}}\ v$. We conclude that $x \cdot a\ \overline{\overline{\omega}}_{\mathcal{A}}\ y \cdot a$.

Case 3. Suppose that $(A, B) \in \xi^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Then, we have $x \in A = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \{y_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \varnothing$. If $a \notin \mathcal{D}$, then $(A \cdot a, B \cdot a) \in \xi^{\mathcal{A}}_{n,k'_1,k'_2,\cdots,k'_n}$ and if $a \in \mathcal{D}$, then $(A \cdot a, B \cdot a) \in \Im^{\mathcal{A}}_{n,k'_1,k'_2,\cdots,k'_n}$. Thus, according to *Case*1, $(A \cdot a, B \cdot a) \in \mathcal{R}^{\mathcal{A}}_{n,k'_1,k'_2,\cdots,k'_n}$. So, $u\ \omega_{\mathcal{A}}\ v$. This implies that $x \cdot a\ \overline{\overline{\omega}}_{\mathcal{A}}\ y \cdot a$.

In the same way, we can show that $a \cdot x\ \overline{\overline{\omega}}_{\mathcal{A}}\ a \cdot y$. It is easy to see that $a \cdot x\ \overline{\overline{\omega}}^*_{\mathcal{A}}\ a \cdot y$ and $x \cdot a\ \overline{\overline{\omega}}^*_{\mathcal{A}}\ y \cdot a$. □

Theorem 2.3. *The quotient $R/\omega^*_{\mathcal{A}}$ is a ring with generators*

$$\{\omega^*_{\mathcal{A}}(b), \omega^*_{\mathcal{A}}(a_1), \omega^*_{\mathcal{A}}(a_2), \cdots, \omega^*_{\mathcal{A}}(a_m) \mid b \in (R - \mathcal{D}), a_1, \cdots, a_m \in \mathcal{A}\}$$

where $\omega^*_{\mathcal{A}}(a_1), \omega^*_{\mathcal{A}}(a_2), \cdots, \omega^*_{\mathcal{A}}(a_m) \in R/\omega^*_{\mathcal{A}}$ necessarily are not distinct.

Proof. By Lemma 2.2, $\omega^*_{\mathcal{A}}$ is a strongly regular equivalence relation, so the quotient structure $R/\omega^*_{\mathcal{A}}$ is a ring with respect to the following operations:

$$\omega^*_{\mathcal{A}}(x) \oplus \omega^*_{\mathcal{A}}(y) = \omega^*_{\mathcal{A}}(z), \text{ for all } z \in x + y,$$

$$\omega^*_{\mathcal{A}}(x) \otimes \omega^*_{\mathcal{A}}(y) = \omega^*_{\mathcal{A}}(t), \text{ for all } t \in x \cdot y.$$

For all $(x, y) \in (R - \mathcal{D})^2$ since $\{x, y\} \cap \mathcal{D} = \emptyset$, we have $(\{x\}, \{y\}) \in \mathcal{R}^{\mathcal{A}}_{1,1}$ and hence $x\omega^*_{\mathcal{A}}y$ then $\omega^*_{\mathcal{A}}(x) = \omega^*_{\mathcal{A}}(y)$. If $b \in (R - \mathcal{D})$, then for every $x \in (R - \mathcal{D})$ we have $\omega^*_{\mathcal{A}}(x) = \omega^*_{\mathcal{A}}(b)$. Now, suppose that $\omega^*_{\mathcal{A}}(h)$ is given. If $h \in (R - \mathcal{D})$, then $\omega^*_{\mathcal{A}}(h) = \omega^*_{\mathcal{A}}(b)$ and if $h \in \mathcal{D}$ then $\omega^*_{\mathcal{A}}(h) \in \langle\omega^*_{\mathcal{A}}(a_1), \cdots, \omega^*_{\mathcal{A}}(a_m)\rangle$. Therefore, $R/\omega^*_{\mathcal{A}} = \{\omega^*_{\mathcal{A}}(b)\} \cup \langle\omega^*_{\mathcal{A}}(a_1), \cdots, \omega^*_{\mathcal{A}}(a_m)\rangle$. □

Example 2.4. *Let $R = \mathbb{Z}_6$ and $\mathcal{A} = \{\bar{2}, \bar{4}\}$. Then, $\mathcal{D} = \{\bar{0}, \bar{2}, \bar{4}\}$ and $R - \mathcal{D} = \{\bar{1}, \bar{3}, \bar{5}\}$. Hence, $R/\omega^*_{\mathcal{A}} \cong \mathbb{Z}_2$ where $\omega^*_{\mathcal{A}}(\bar{1}) = \omega^*_{\mathcal{A}}(\bar{3}) = \omega^*_{\mathcal{A}}(\bar{5})$ and $\omega^*_{\mathcal{A}}(\bar{0}) = \omega^*_{\mathcal{A}}(\bar{2}) = \omega^*_{\mathcal{A}}(\bar{4})$ and also $R/\gamma^* \cong \mathbb{Z}_6$. So, $\omega^*_{\mathcal{A}} \neq \gamma^*$. If $\mathcal{A} = \{\bar{3}, \bar{5}\}$ then $R/\omega^*_{\mathcal{A}} = \langle\omega^*_{\mathcal{A}}(\bar{3}), \omega^*_{\mathcal{A}}(\bar{5})\rangle$.*

Indeed, this example shows that in general, $\omega^*_{\mathcal{A}} \neq \gamma^*$.

Now, suppose that $\mathcal{A} = \{a\}$, so $\mathcal{D} = \{t \mid t \in ra + as + na + \sum_{i=1}^{m} r_i a s_i,\ r, s, r_i, s_i \in R, m \in \mathbb{N}, n \in \mathbb{Z}\}$. Put $\rho_a := \omega_{\mathcal{A}}$ and $\rho^*_a := \omega^*_{\mathcal{A}}$. Then, we have the following corollary.

Corollary 2.5. *The quotient R/ρ^*_a is a ring generated by $\rho^*_a(a)$ i. e, $R/\rho^*_a = \langle\rho^*_a(a)\rangle$ or $|R/\rho^*_a| \leq 2$.*

Proof. By the proof of Theorem 2.3, we conclude that the equivalence classes determined by $\omega^*_{\mathcal{A}}$ of all elements of $(R - \mathcal{D})$ coincide and the equivalence class of every element of $\mathcal{D}$ is generated by $\omega^*_{\mathcal{A}}(a_1), \omega^*_{\mathcal{A}}(a_2), \cdots, \omega^*_{\mathcal{A}}(a_m) \in R/\omega^*_{\mathcal{A}}$. If $t \in \mathcal{D}$, then

$$\begin{aligned}\rho^*_a(t) &= [\rho^*_a(r) \otimes \rho^*_a(a)] \oplus [\rho^*_a(a) \otimes \rho^*_a(s)] \oplus n\rho^*_a(a) \oplus \sum_{i=1}^{m} (\rho^*_a(r_i) \otimes \rho^*_a(a) \otimes \rho^*_a(s_i)) \\ &\in \langle \rho^*_a(a) \rangle.\end{aligned}$$

So, $R/\rho^*_a = \{\rho^*_a(b)\} \cup \langle \rho^*_a(a) \rangle$, where $b \in R - \mathcal{D}$. Now, we have $\rho^*_a(b) \oplus \rho^*_a(a) \in R/\rho^*_a$. Then, $\rho^*_a(b) + \rho^*_a(a) = \rho^*_a(b)$ or $\rho^*_a(b) + \rho^*_a(a) \in < \rho^*_a(a) >$. If $\rho^*_a(b) \oplus \rho^*_a(a) = \rho^*_a(b)$ then $\rho^*_a(a) = 0_{R/\rho^*_a}$ and so $R/\rho^*_a = \{0_{R/\rho^*_a}, \rho^*_a(b)\}$. This implies that $|R/\rho^*_a| \leq 2$. If $\rho^*_a(b) \oplus \rho^*_a(a) \in < \rho^*_a(a) >$, then there exist $n \in \mathbb{Z}$, $m \in \mathbb{N}$ and $r, s, r_i, s_i \in R$ such that $\rho^*_a(b) \oplus \rho^*_a(a) = [\rho^*_a(r) \otimes \rho^*_a(a)] \oplus [\rho^*_a(a) \otimes \rho^*_a(s)] \oplus n\rho^*_a(a) \oplus \sum_{i=1}^{m} (\rho^*_a(r_i) \otimes \rho^*_a(a) \otimes \rho^*_a(s_i))$ and this implies that $\rho^*_a(b) \in \langle \rho^*_a(a) \rangle$. □

Corollary 2.6. *If $R - \mathcal{D} = \emptyset$, which means that R is a hyperring generated by the element a, then $R/\rho^*_a = \langle \rho^*_a(a) \rangle$.*

Corollary 2.7. *If the hyperring R has an identity element and a is in the center of R, then $R/\rho^*_a = \langle \rho^*_a(a) \rangle = \{\rho^*_a(a) \otimes \rho^*_a(r) \mid \rho^*_a(r) \in R/\rho^*_a\}$ or $|R/\rho^*_a| \leq 2$.*

Example 2.8. *Let $R = \{b, c, d, e, f\}$. Consider the hyperring $(R, +, \cdot)$, where hyperoperations $+$ and $\cdot$ are defined on R as follows:*

$+$	f	b	c	d	e
f	f	$\{b, e\}$	c	d	$\{b, e\}$
b	$\{b, e\}$	c	d	f	c
c	c	d	$\{b, e\}$	c	d
d	d	f	$\{b, e\}$	c	f
e	$\{b, e\}$	c	d	f	c

$\cdot$	f	b	c	d	e
f	f	f	f	f	f
b	f	$\{b, e\}$	c	d	$\{b, e\}$
c	f	c	f	c	c
d	f	d	c	$\{b, e\}$	d
e	f	$\{b, e\}$	c	d	$\{b, e\}$

Suppose that $\mathcal{A} = \{d\}$. Then, $\mathcal{D} = \{b, c, d, e, f\}$ and $R - \mathcal{D} = \emptyset$. Thus, $R/\rho^*_a = \langle \rho^*_a(d) \rangle = \{\rho^*_a(f), \rho^*_a(b), \rho^*_a(c), \rho^*_a(d)\}$.

Suppose that $\mathcal{A} = \{c\}$. Then, $\mathcal{D} = \{f, c\}$ and $R - \mathcal{D} = \{b, d, e\}$. Hence, $\rho^*_a(b) = \rho^*_a(d) = \rho^*_a(e)$ and $\rho^*_a(f) = \rho^*_a(c)$. This implies that $R/\rho^*_a \cong \mathbb{Z}_2$ and $R/\gamma^* \cong \mathbb{Z}_4$. Therefore, $\rho^*_a \neq \gamma^*$.

Example 2.9. *Let $R = \mathbb{Z}$ be the set of all integers and $a = 2$. Then, $\mathcal{D} = \{\cdots, 4, 2, 0, -2, -4, \cdots\}$ and $R - \mathcal{D} = \{\cdots, 3, 1, -1, 3, \cdots\}$. Then, $R/\rho^*_a \cong \mathbb{Z}_2$ and $R/\gamma^* \cong \mathbb{Z}$. This implies that $\rho^*_a \neq \gamma^*$. If $a = 1$, then $R/\rho^*_a \cong \mathbb{Z}$ and $R/\gamma^* \cong \mathbb{Z}$. Thus, $\rho^*_a = \gamma^*$.*

Theorem 2.10. *The relation ρ^*_a is the smallest strongly regular relation such that the quotient R/ρ^*_a is a ring generated by $\rho^*_a(a)$ or $|R/\rho^*_a| \leq 2$, where the equivalence classes of all elements of $R - \mathcal{D}$ are equal.*

Proof. Suppose that θ is a strongly regular relation such that the quotient R/θ is a ring generated by $\theta^*(a)$ or $|R/\theta^*| \leq 2$, and the equivalence classes of θ of all elements of $(R - \mathcal{D})$ are equal. Suppose that $\phi : R \to R/\theta$ is the canonical projection. Clearly, ϕ is a good homomorphism. We show that $\rho_a^* \subseteq \theta$. Let $x \ \rho_a \ y$. Then, there exists $(A, B) \in \mathcal{R}^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$ such that $x \in A$ and $y \in B$. We have three cases.

Case 1. Suppose that $(A, B) \in \gamma^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Then, we have $A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ such that $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij}) = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$. Thus,

$$\phi(x) = \oplus \sum_{i=1}^{n}(\otimes \prod_{j=1}^{k_i} \theta(x_{ij})) = \phi(y) = \oplus \sum_{i=1}^{n}(\otimes \prod_{j=1}^{k_i} (\theta(y_{ij}))).$$

Therefore, $x \ \theta \ y$.

Case 2. Suppose that $(A, B) \in \Im^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Then, we have $A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \{y_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \neq \varnothing$. Renumber the elements of the sets $\{x_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\}$ and $\{y_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\}$ such that $\{y_{i'j'}|1 \leq i \leq n', 1 \leq j \leq k'_i\} \subseteq \mathcal{D}$, where $1 \leqslant n' \leqslant n$ and $1 \leqslant k'_i \leqslant k_i$ and $x_{ts}, y_{ts} \notin \mathcal{D}$ for all $n'+1 \leqslant t \leqslant n$ and $k'_i+1 \leqslant s \leqslant k_i$. Then, we have $\phi(x) = (\oplus \sum\limits_{i=1}^{n'}(\otimes \prod\limits_{j=1}^{k'_i} \theta(x_{ij}))) \bigoplus (\oplus \sum\limits_{t=n'+1}^{n}(\otimes \prod\limits_{s=k'_i+1}^{k_i} \theta(x_{ij}))) = \phi(y) = (\oplus \sum\limits_{i=1}^{n'}(\otimes \prod\limits_{j=1}^{k'_i} \theta(y_{ij}))) \bigoplus (\oplus \sum\limits_{t=n'+1}^{n}(\otimes \prod\limits_{s=k'_i+1}^{k_i} \theta(y_{ij})))$. So, $\phi(x) = \phi(y)$ and $x \ \theta \ y$.

Case 3. Suppose that $(A, B) \in \xi^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Then, we have $A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \{y_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \varnothing$. Hence, for all $1 \leq i \leq n$ and $1 \leq j \leq k_i$ we have $\phi(x_{ij}) = \phi(y_{ij})$. Therefore, $\phi(x) = \phi(y)$ which implies that and $x \ \theta \ y$.

In the all cases we have $x \ \theta \ y$ and hence $x \ \rho_a \ y$ implies that $x \ \theta \ y$ and hence $x \ \rho_a^* \ y$ implies that $x \ \theta \ y$ by transitivity of θ. Therefore, we have $\rho_a^* \subseteq \theta$. □

3. The transitivity condition of $\omega_{\mathcal{A}}$

In this section, we introduce the concept of $\omega^*_{\mathcal{A}}$-part of a hyperring R and we determine necessary and sufficient conditions such that the relation $\omega_{\mathcal{A}}$ to be transitive. Let M be a non-empty subset of a hyperring $(R, +, \cdot)$.

Definition 3.1. *We say that M is a $\omega^*_{\mathcal{A}}$-part of R if the following conditions hold.*

(F_1) $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij}) \subseteq M$;

(F_2) *If* $\{x_{ij}|1 \le i \le n, 1 \le j \le k_i\} \cap \mathcal{D} \neq \varnothing$ *then for all* $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ *such that* $\{x_{ij}|1 \le i \le n, 1 \le j \le k_i\} \cap \mathcal{D} = \{y_{ij}|1 \le i \le n, 1 \le j \le k_i\} \cap \mathcal{D}$, *we have* $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij}) \subseteq M$;

(F_3) *If* $\{x_{ij}|1 \le i \le n, 1 \le j \le k_i\} \cap \mathcal{D} = \varnothing$ *then for all* $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ *such that* $\{y_{ij}|1 \le i \le n, 1 \le j \le k_i\} \cap \mathcal{D} = \varnothing$, *we have* $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij}) \subseteq M$.

By using the above notion we obtain the following characterization.

Proposition 3.2. *The following conditions are equivalent.*

(1) M *is a* $\omega^*_{\mathcal{A}}$*-part;*

(2) $x \in M$, $x\ \omega_{\mathcal{A}}\ y \Longrightarrow y \in M$;

(3) $x \in M$, $x\ \omega^*_{\mathcal{A}}\ y \Longrightarrow y \in M$.

Proof. (1⇒2): Let $(x, y) \in R^2$ such that $x \in M$ and $x\omega_{\mathcal{A}}y$. Then, there exists $(A, B) \in \mathcal{R}^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$ such that $x \in A$ and $y \in B$. Hence, we have three cases.

Case 1. Suppose that $(A, B) \in \gamma^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Then, we have $A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ such that $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij}) = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$. Since $x \in \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij}) \cap M$, by (F_1) we obtain $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij}) \subseteq M$ and hence $y \in M$.

Case 2. Let $(A, B) \in \Im^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Then, we have $A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij}|1 \le i \le n, 1 \le j \le k_i\} \cap \mathcal{D} = \{y_{ij}|1 \le i \le n, 1 \le j \le k_i\} \cap \mathcal{D} \neq \varnothing$. Since $x \in \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij}) \cap M$, by (F_2) we obtain $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij}) \subseteq M$ and hence $y \in M$.

Case 3. Suppose that $(A, B) \in \xi^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Then, we have $A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij}|1 \le i \le n, 1 \le j \le k_i\} \cap \mathcal{D} = \{y_{ij}|1 \le i \le n, 1 \le$

$j \leq k_i\} \cap \mathcal{D} = \varnothing$. Since $x \in \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij}) \cap M$, by (F_3) we obtain $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij}) \subseteq M$ and hence $y \in M$.

(2⇒3): Let $(x, y) \in R^2$ such that $x \in M$ and $x\ \omega^*_{\mathcal{A}}\ y$. Then, there exist $t \in \mathbb{N}$ and $(v_0 = x, v_1, \cdots, v_t = y) \in R^{t+1}$ such that

$$x = v_0\ \omega_{\mathcal{A}}\ v_1\ \omega_{\mathcal{A}}\ v_2 \cdots \omega_{\mathcal{A}}\ v_{t-1}\ \omega_{\mathcal{A}}\ v_t = y.$$

Since $x \in M$, by applying (2) t times, we obtain $y \in M$.

(3⇒1): Let $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij}) \cap M \neq \emptyset$. We shall check the conditions (F_1), (F_2) and (F_3).

(F_1) Suppose that $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij}) = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$. Then,

$$(\sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \sum_{i=1}^{n}(\prod_{j=1}^{k_i} y_{ij}) \in \gamma^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}.$$

So, for all $y \in \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij}$ we have $x\ \omega^*_{\mathcal{A}}\ y$ and $y \in M$. Therefore, $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij}) \subseteq M$.

(F_2) Let $\{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \neq \emptyset$. Now, suppose that for all $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ if $\{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D}$ then $(\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij}), \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})) \in \Im^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Hence, for all $y \in \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ we have $x\ \omega^*_{\mathcal{A}}\ y$ and $y \in M$. Therefore, $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij}) \subseteq M$.

(F_3) Let $\{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \emptyset$ and $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ be such that $\{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \emptyset$. Then, $(\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij}), \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})) \in \xi^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Hence, for all $y \in \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ we have $x\ \omega^*_{\mathcal{A}}\ y$ and $y \in M$. Therefore, $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij}) \subseteq M$. □

Before proving the next theorem, we introduce the following notions.

Definition 3.3. *Let x be an arbitrary element of a hyperring R. For all $n \geqslant 1$, set:*

(N_1) $P_{\gamma^{\mathcal{A}}_n}(x) = \cup\{\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij}) \mid x \in \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij}) = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})\}$;

(N_2) $P_{\Im_n^{\mathcal{A}}}(x) = \cup\{\sum_{i=1}^{n}(\prod_{j=1}^{k_i} y_{ij}) \mid x \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \{x_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \{y_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \neq \varnothing\}$;

(N_3) $P_{\xi_n^{\mathcal{A}}}(x) = \cup\{\sum_{i=1}^{n}(\prod_{j=1}^{k_i} y_{ij}) \mid x \in \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij}), \{x_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \{y_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \varnothing\}$;

(N_4) $P(x) = \bigcup_{n\geq 1}(P_{\gamma_n^{\mathcal{A}}}(x) \cup P_{\Im_n^{\mathcal{A}}}(x) \cup P_{\xi_n^{\mathcal{A}}}(x))$.

Notice that if $x \notin \mathcal{D}$, then $P_{1,1}(x) = R - \mathcal{D}$.

Proposition 3.4. *For all $x \in R$, $P(x) = \{y \in R \mid x\ \omega_{\mathcal{A}}\ y\}$.*

Proof. Suppose that $x \in R$ and $y \in P(x)$. Then, there exist A, B such that $x \in A$, $y \in B$ and

(1) $y \in P_{\gamma^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}}(x) \Rightarrow (A, B) \in \gamma^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$;

(2) $x \in P_{\Im^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}}(x) \Rightarrow (A, B) \in \Im^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$;

(3) $x \in P_{\xi^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}}(x) \Rightarrow (A, B) \in \xi^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$.

Therefore, $x\ \omega_{\mathcal{A}}\ y$ and $P(x) \subseteq \{y \in R \mid x\ \omega_{\mathcal{A}}\ y\}$.

The proof of the reverse of the inclusion is obvious. □

Lemma 3.5. *Let $(R, +, \cdot)$ be a hyperring and let M be a $\omega^*_{\mathcal{A}}$-part of R. If $x \in M$, then $P(x) \subseteq M$.*

Proof. It follows by Definition 3.1. □

Theorem 3.6. *Let $(R, +, \cdot)$ be a hyperring. The following conditions are equivalent.*

(1) *$\omega_{\mathcal{A}}$ is transitive;*

(2) *For any $x \in R$, $\omega^*_{\mathcal{A}}(x) = P(x)$;*

(3) *For any $x \in R$, $P(x)$ is a $\omega^*_{\mathcal{A}}$-part of R.*

Proof. (1⇒2) By Proposition 3.4, for all pairs $(x, y) \in R^2$ we have

$$y \in \omega^*_{\mathcal{A}}(x) \Leftrightarrow x\ \omega_{\mathcal{A}}\ y \Leftrightarrow y \in P(x).$$

(2⇒1) By Proposition 3.2, if M is a non-empty subset of R, then M is a $\omega^*_{\mathcal{A}}$-part of R if and only if it is a union of equivalence classes modulo $\omega^*_{\mathcal{A}}$. In particular, every equivalence class modulo $\omega^*_{\mathcal{A}}$ is a $\omega^*_{\mathcal{A}}$-part of R.

(3⇔1) Let $x\ \omega_{\mathcal{A}}\ y$ and $y\ \omega_{\mathcal{A}}\ z$. Then, $x \in P(y)$ and $y \in P(z)$ by Proposition 3.4. Since $P(z)$ is a $\omega^*_{\mathcal{A}}$-part, by Lemma 3.5, we obtain $P(y) \subseteq P(z)$ and hence, $x \in P(z)$. Therefore, $x\ \omega_{\mathcal{A}}\ z$ by Proposition 3.4 and the proof is completed. □

Example 3.7. *In Example 2.8, if* $\mathcal{A} = \{d\}$ *then* $\{b, e\}, \{c\}, \{d\}$ *and* $\{f\}$ *are* $\omega^*_{\mathcal{A}}$*-parts of the hyperring R.It is not difficult to see that* $P(b) = \{b, e\}$, $P(c) = \{c\}$, $P(d) = \{d\}$ *and* $P(f) = \{f\}$ *and so by Theorem 3.6,* $\omega_{\mathcal{A}} = \rho_{\mathcal{A}}$ *is transitive.*

Example 3.8. *In Example 2.9,* $P(1) = \rho^*_a(1) = \{\cdots, 3, 1, -1, 3, \cdots\}$ *and* $P(0) = \rho^*_a(0) = \{\cdots, 4, 2, 0, -2, -4, \cdots\}$ *are* $\omega^*_{\mathcal{A}}$*-parts of the hyperring* $\mathbb{Z}$*. Then by Theorem 3.6,* $\omega_{\mathcal{A}} = \rho_{\mathcal{A}}$ *is transitive.*

4. New strongly regular relation $\chi_{\mathcal{A}}$

In this section, we introduce the relation $\chi_{\mathcal{A}}$ on a hyperring R, which we use in order to obtain a finite generated commutative ring as a quotient structure of R. Moreover, we determine some necessary and sufficient conditions for the relation $\chi_{\mathcal{A}}$ to be transitive. Let $(R, +, \cdot)$ be a hyperring, $\varnothing \neq \mathcal{A} \subseteq R$, $a_1, \cdots, a_m \in \mathcal{A}$ and

$$\begin{array}{c}\mathcal{D} = \{t \mid t \in \sum_{i=1}^{m} z_i a_i + \sum_{i=1}^{m} s_i a_i + \sum_{i=1}^{m} a_i t_i + \sum_{i=1}^{m} (\sum_{i=1}^{n_i} u_{i,k} a_i v_{i,k}), \\ m, n_i \in \mathbb{N}, z_i \in \mathbb{Z},\ \ s_i, t_i, u_{i,k}, v_{i,k} \in R\}.\end{array}$$

For all $n \geq 1$ and $(k_1, \cdots, k_n) \in \mathbb{N}^n$ define $\Re^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$ as follows.

$$\Re^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n} := \alpha^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n} \cup \wp^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n} \cup \jmath^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n},$$

where

$$\alpha^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n} := \{(\sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij}, \sum_{i=1}^{n} \prod_{j=1}^{k_i} y_{ij}) | \exists \sigma \in \mathbb{S}_n, \exists \sigma_i \in \mathbb{S}_{k_i}, \sum_{i=1}^{n} \prod_{j=1}^{k_i} y_{ij} = \sum_{i=1}^{n} A_{\sigma(i)},$$

with $A_i = \prod_{j=1}^{k_i} x_{i\sigma_i(j)}$,

$$\begin{aligned}\wp^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n} \quad := \{(\sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij}, \sum_{i=1}^{n} \prod_{j=1}^{k_{\sigma(i)}} y_{ij}) \mid \{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \\ = \{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \neq \varnothing, \exists \sigma \in \mathbb{S}_n, \sigma_i \in \mathbb{S}_{k_i}, \\ y_{ts} \in \{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \Rightarrow y_{ts} = x_{\sigma(t)\sigma_{\sigma(t)}(s)}\}\end{aligned}$$

and

$$\begin{aligned}\jmath^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n} \quad := \{(\sum_{i=1}^{n} \prod_{j=1}^{k_i} x_{ij}, \sum_{i=1}^{n} \prod_{j=1}^{k_i} y_{ij}) \mid \{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \\ = \{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \varnothing\}.\end{aligned}$$

Notice that $\Re^{\mathcal{A}}_{1,1} = \{(\{x\}, \{y\}) \mid \{x, y\} \cap \mathcal{D} = \emptyset \text{ or } x = y\}$.

Definition 4.1. *We define the relation* $\chi_{\mathcal{A}}$ *on* $(R, +, \cdot)$ *as follows:*

$$x\ \chi_{\mathcal{A}}\ y \Leftrightarrow\ \exists (A, B) \in \Re^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n},\ \ x \in A,\ \ y \in B.$$

Notice that for $n = 1$ and $k_1 = 1$ we obtain $x\ \chi_{\mathcal{A}}\ y \Leftrightarrow\ (\{x\}, \{y\}) \in \Re^{\mathcal{A}}_{1,1}$ or $x = y \in \mathcal{D}$.

Remark 4.1. *The relation* $\chi_{\mathcal{A}}$ *is reflexive and symmetric and* $\beta, \gamma \subseteq \chi_{\mathcal{A}}$ *and* $\gamma, \alpha \subseteq \chi_{\mathcal{A}}$.

Let $\chi^*_{\mathcal{A}}$ be the transitive closure of $\chi_{\mathcal{A}}$. In order to analyze the quotient hyperstructure with respect to this equivalence relation, we check that:

Lemma 4.2. $\chi^*_{\mathcal{A}}$ *is a strongly regular equivalence relation both on* $(R, +)$ *and on* $(R, \cdot)$.

Proof. Clearly $\chi^*_{\mathcal{A}}$ is an equivalence relation. In order to prove that it is strongly regular, it is enough to show that

$$x\ \chi_{\mathcal{A}}\ y \Rightarrow \begin{cases} x + z\ \overline{\overline{\chi}}_{\mathcal{A}}\ y + z, & z + x\ \overline{\overline{\chi}}_{\mathcal{A}}\ z + y, \\ x \cdot z\ \overline{\overline{\chi}}_{\mathcal{A}}\ y \cdot z, & z \cdot x\ \overline{\overline{\chi}}_{\mathcal{A}}\ z \cdot y, \end{cases}$$

for all $z \in R$. Since $x\ \chi_{\mathcal{A}}\ y$, it follows that there exists $(A, B) \in \Re^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$ such that $x \in A$ and $y \in B$. We distinguish the following situations.

Case 1. If $(A, B) \in \alpha^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$, then $x \in A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ and there exist $\sigma \in \mathbb{S}_n$ and $\sigma_i \in \mathbb{S}_{k_i}$ such that $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij} = \sum\limits_{i=1}^{n} A_{\sigma(i)}$, where $A_i = \prod\limits_{j=1}^{k_i} x_{i\sigma_i(j)}$. Now, let $k_{n+1} = 1$, $x_{n+1\ 1} = y_{n+1\ 1} = z$, $\sigma_{n+1} = id$ and τ be the permutation of $\mathbb{S}_{n+1}$ such that

$$\tau(i) = \sigma(i),\ \text{for all } i = 1, \cdots, n \text{ and } \tau(n + 1) = n + 1.$$

Thus,

$$x + z \subseteq (\sum_{i=1}^{n+1}(\prod_{j=1}^{k_i} x_{ij})) \text{ and } y + z \subseteq (\sum_{i=1}^{n+1}(\prod_{j=1}^{k_{\tau(i)}} y_{ij})).$$

It is easy to see that the pair $(A + z, B + z)$ belongs to $\alpha^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n,k_{n+1}} \subseteq \Re^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n,k_{n+1}}$. Therefore, for all $u \in x + z$ and $v \in y + z$, we have $u \in x + z \subseteq A + z$ and $v \in y + z \subseteq B + z$. So, $u\ \chi_{\mathcal{A}}\ v$. Thus, $x + z\ \overline{\overline{\chi}}_{\mathcal{A}}\ y + z$.

Case 2. If $(A, B) \in \wp^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$, then $x \in A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \neq \varnothing$ and $\sigma \in \mathbb{S}_n$ and $\sigma_i \in \mathbb{S}_{k_i}$ such that for all $1 \leqslant t \leqslant n, 1 \leqslant s \leqslant k_n$ if $y_{ts} \in \mathcal{D}$, then $y_{ts} = x_{\sigma(t)\sigma_{\sigma(t)}(s)}$. If $z \notin \mathcal{D}$, then it is easy to sea that $(A+z, B+z) \in \wp^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$. Let $z \in \mathcal{D}$. Set $k_{n+1} = 1$, $x_{n+1\ 1} = y_{n+1\ 1} = z$, $\sigma_{n+1} = id$ and τ be the permutation of $\mathbb{S}_{n+1}$ such that

$$\tau(i) = \sigma(i),\ \text{for all } i = 1, \cdots, n \text{ and } \tau(n + 1) = n + 1.$$

So, for $1 \leqslant t \leqslant n+1$ and $1 \leqslant s \leqslant k_{n+1}$ if $y_{ts} \in \mathcal{D}$, then $y_{ts} = x_{\sigma(t)\sigma_{\sigma(t)}(s)}$. Therefore, $(A+z, B+z) \in \wp^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n,k_{n+1}}$. Hence, $(A+z, B+z) \in \Re^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n,k_{n+1}}$. So, $u \ \chi_{\mathcal{A}} \ v$. Thus, $x+z \ \overline{\overline{\chi}}_{\mathcal{A}} \ y+z$.

Case 3. If $(A, B) \in \jmath^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$, then $x \in A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij} | 1 \le i \le n, 1 \le j \le k_i\} \cap \mathcal{D} = \{y_{ij} | 1 \le i \le n, 1 \le j \le k_i\} \cap \mathcal{D} = \varnothing$. If $z \notin \mathcal{D}$, then $(A+z, B+z) \in \jmath^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n,k_{n+1}}$ and if $z \in \mathcal{D}$, then $(A+z, B+z) \in \wp^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n,k_{n+1}}$. Thus, $(A+z, B+z) \in \Re^{\mathcal{A}}_{n+1,k_1,k_2,\cdots,k_n,k_{n+1}}$. Hence, $u \ \chi_{\mathcal{A}} \ v$. This implies that $x+z \ \overline{\overline{\chi}}_{\mathcal{A}} \ y+z$.

In the same way, we can show that $z+x \ \overline{\overline{\chi}}_{\mathcal{A}} \ z+y$. It is easy to see that

$$z+x \ \overline{\overline{\chi}}^{*}_{\mathcal{A}} \ z+y \text{ and } x+z \ \overline{\overline{\chi}}^{*}_{\mathcal{A}} \ y+z.$$

Notice that for $(R, \cdot)$ we have

Case 1. If $(A, B) \in \alpha^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$, then $x \in A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ and there exist $\sigma \in \mathbb{S}_n$ and $\sigma_i \in \mathbb{S}_{k_i}$ such that $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij} = \sum\limits_{i=1}^{n} A_{\sigma(i)}$, where $A_i = \prod\limits_{j=1}^{k_i} x_{i\sigma_i(j)}$. We set $f'_i = k_i + 1$, $x_{if_i} = z$ and we define

$$\tau_i(r) = \sigma_i(r) \text{ (for all } r = 1, \cdots, f'_i) \text{ and } \tau_i(f'_i + 1) = f'_i + 1.$$

Hence, $\tau_i \in \mathbb{S}_{f'_i}$ $(i = 1, \cdots, n)$. It is easy to see that the pair $(A \cdot z, B \cdot z)$ belongs to $\alpha^{\mathcal{A}}_{n,f'_1,f'_2,\cdots,f'_n} \subseteq \Re^{\mathcal{A}}_{n,f'_1,f'_2,\cdots,f'_n}$. Therefore, for all $u \in x \cdot z$ and $v \in y \cdot z$, we have $u \in x \cdot z \subseteq A \cdot z$ and $v \in y \cdot z \subseteq B \cdot z$. So, $u \ \chi_{\mathcal{A}} \ v$. Thus, $x \cdot z \ \overline{\overline{\chi}}_{\mathcal{A}} \ y \cdot z$.

Case 2. If $(A, B) \in \wp^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$, then $x \in A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij} | 1 \le i \le n, 1 \le j \le k_i\} \cap \mathcal{D} = \{y_{ij} | 1 \le i \le n, 1 \le j \le k_i\} \cap \mathcal{D} \neq \varnothing$, $\sigma \in \mathbb{S}_n$ and $\sigma_i \in \mathbb{S}_{k_i}$ such that for all $1 \leqslant t \leqslant n, 1 \leqslant s \leqslant k_n$, if $y_{ts} \in \mathcal{D}$, then $y_{ts} = x_{\sigma(t)\sigma_{\sigma(t)}(s)}$. If $z \notin \mathcal{D}$, then it is easy to sea that $(A \cdot z, B \cdot z) \in \wp^{\mathcal{A}}_{n,f'_1,f'_2,\cdots,f'_n}$, where $f'_i = k_i + 1$, $x_{if_i} = z$. Let $z \in \mathcal{D}$. Set $f'_i = k_i + 1$, $x_{if'_i} = y_{if'_i} = z$, and we define

$$\tau(r) = \sigma_i(r), \text{ for all } i = 1, \cdots, k_i \text{ and } \tau_i(k_i + 1) = k_i + 1.$$

So, for $1 \leqslant t \leqslant n$ and $1 \leqslant s \leqslant f'_i$ if $y_{ts} \in \mathcal{D}$, then $y_{ts} = x_{\sigma(t)\sigma_{\sigma(t)}(s)}$. Therefore, $(A \cdot z, B \cdot z) \in \wp^{\mathcal{A}}_{n,f'_1,f'_2,\cdots,f'_n}$. This implies that $(A \cdot z, B \cdot z) \in \Re^{\mathcal{A}}_{n,f'_1,f'_2,\cdots,f'_n}$. Hence, $u \ \chi_{\mathcal{A}} \ v$. Thus, $x \cdot z \ \overline{\overline{\chi}}_{\mathcal{A}} \ y \cdot z$.

Case 3. If $(A,B) \in \jmath^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$, then $x \in A = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} x_{ij})$ and $y \in B = \sum_{i=1}^{n}(\prod_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \varnothing$. If $z \notin \mathcal{D}$, then $(A \cdot z, B \cdot z) \in \jmath^{\mathcal{A}}_{n,f'_1,f'_2,\cdots,f'_n}$ and if $z \in \mathcal{D}$, we set $f'_i = k_i + 1$, $x_{if'_i} = z$ and we define

$$\tau_i(r) = \sigma_i(r), \text{for all } r = 1, \cdots, k_i \text{ and } \tau_i(k_i + 1) = k_i + 1.$$

Hence, $\tau_i \in \mathbb{S}_{f'_i}$ $(i = 1, \cdots, n)$. Thus, $(A \cdot z, B \cdot z) \in \Re^{\mathcal{A}}_{n,f'_1,f'_2,\cdots,f'_n}$. This implies that $u\ \chi_{\mathcal{A}}\ v$. Therefore, $x \cdot z\ \overline{\overline{\chi}}_{\mathcal{A}}\ y \cdot z$.

In the same way, we can show that $z \cdot x\ \overline{\overline{\chi}}_{\mathcal{A}}\ z \cdot y$. It is easy to see that

$$z \cdot x\ \overline{\overline{\chi}}^*_{\mathcal{A}}\ z \cdot y \text{ and } x \cdot z\ \overline{\overline{\chi}}^*_{\mathcal{A}}\ y \cdot z.$$

□

Theorem 4.3. *The quotient $R/\chi^*_{\mathcal{A}}$ is a commutative ring with generators*

$$\{\chi^*_{\mathcal{A}}(b), \chi^*_{\mathcal{A}}(a_1), \chi^*_{\mathcal{A}}(a_2), \cdots, \chi^*_{\mathcal{A}}(a_m) \mid b \in (R - \mathcal{D}), a_1, \cdots, a_m \in \mathcal{A}\},$$

where $\chi^*_{\mathcal{A}}(a_1), \chi^*_{\mathcal{A}}(a_2), \cdots, \chi^*_{\mathcal{A}}(a_m) \in R/\chi^*_{\mathcal{A}}$ necessarily are not distinct.

Proof. By Lemma 4.2, $\chi^*_{\mathcal{A}}$ is a strongly regular relation, so the quotient structure $R/\chi^*_{\mathcal{A}}$ is a ring with respect to the following operations:

$$\chi^*_{\mathcal{A}}(x) \oplus \omega^*_{\mathcal{A}}(y) = \chi^*_{\mathcal{A}}(z), \text{ for all } z \in x + y,$$

$$\chi^*_{\mathcal{A}}(x) \otimes \chi^*_{\mathcal{A}}(y) = \chi^*_{\mathcal{A}}(t), \text{ for all } t \in x \cdot y.$$

Since $\alpha^* \subseteq \chi^*_{\mathcal{A}}$, we conclude that $\chi^*_{\mathcal{A}}$ is a commutative ring. For all $(x,y) \in (R - \mathcal{D})^2$ since $\{x,y\} \cap \mathcal{D} = \emptyset$, we have $(\{x\},\{y\}) \in \Re^{\mathcal{A}}_{1,1}$ and hence $x\chi^*_{\mathcal{A}}y$. This implies that $\chi^*_{\mathcal{A}}(x) = \chi^*_{\mathcal{A}}(y)$. If $b \in (R - \mathcal{D})$, then for every $x \in (R - \mathcal{D})$ we have $\chi^*_{\mathcal{A}}(x) = \chi^*_{\mathcal{A}}(b)$. Now, suppose that $\chi^*_{\mathcal{A}}(h)$ is given. If $h \in (R - \mathcal{D})$, then $\chi^*_{\mathcal{A}}(h) = \chi^*_{\mathcal{A}}(b)$ and if $h \in \mathcal{D}$, then $\chi^*_{\mathcal{A}}(h) \in \langle \chi^*_{\mathcal{A}}(a_1), \cdots, \chi^*_{\mathcal{A}}(a_m)\rangle$. Thus, $R/\chi^*_{\mathcal{A}} = \{\chi^*_{\mathcal{A}}(b)\} \cup \langle \chi^*_{\mathcal{A}}(a_1), \cdots, \chi^*_{\mathcal{A}}(a_m)\rangle$. □

Now, suppose that $\mathcal{A} = \{a\}$, so $\mathcal{D} = \{t \mid t \in ra + as + na + \sum_{i=1}^{m} r_i a s_i,\ r, s, r_i, s_i \in R, m \in \mathbb{N}, n \in \mathbb{Z}\}$. Put $\psi_a := \chi_{\mathcal{A}}$ and $\psi^*_a := \chi^*_{\mathcal{A}}$. Then, we have the following corollary.

Corollary 4.4. *The quotient R/ψ^*_a is a commutative ring generated by $\psi^*_a(a)$, i.e, $R/\psi^*_a = \langle \psi^*_a(a)\rangle = \{n\psi^*_a(a) \oplus (\psi^*_a(r) \otimes \psi^*_a(a)) \mid n \in \mathbb{Z}, \psi^*_a(r) \in R/\psi^*_a\}$ or $|R/\rho^*_a| \leq 2$.*

Corollary 4.5. *If $R - \mathcal{D} = \emptyset$, which means that R is a hyperring generated by the element a, then* $R/\psi_a^* = \langle \rho_a^*(a) \rangle = \{n\psi_a^*(a) \oplus (\psi_a^*(r) \otimes \psi_a^*(a)) \mid n \in \mathbb{Z}, \psi_a^*(r) \in R/\psi_a^*\}$.

Corollary 4.6. *If the hyperring R has an identity element, then* $R/\psi_a^* = \langle \rho_a^*(a) \rangle = \{\psi_a^*(a) \otimes \psi_a^*(r) \mid \psi_a^*(r) \in R/\psi_a^*\}$ *or* $|R/\psi_a^*| \leq 2$.

Example 4.7. *Suppose that*

$$R = M_2(\mathbb{Z}_4) = \left\{ \begin{pmatrix} c & 0 \\ b & 0 \end{pmatrix} \mid c, b \in \mathbb{Z}_4 \right\}, \ \mathcal{A} = \left\{ \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} \right\}$$

It is easy to see that R/ψ_a^* is a commutative ring such that $R/\psi_a^* \cong \mathbb{Z}_2$ and $R/\alpha^* \cong \mathbb{Z}_4$ and so, $\psi_a^* \neq \alpha^*$. If $\mathcal{A} = \left\{ \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \right\}$ *then* $R/\psi_a^* = R/\alpha^* \cong \mathbb{Z}_4$.

Example 4.8. *Let $R = \{0, 1, 2, 3, 4, 5, 6\}$ be a set with the hyperoperations $+$ and $\cdot$ defined as follow:*

$+$	0	1	2	3	4	5	6
0	$\{0,5\}$	1	$\{2,6\}$	$\{3,4\}$	$\{3,4\}$	$\{0,5\}$	$\{2,6\}$
1	1	$\{0,5\}$	$\{3,4\}$	$\{2,6\}$	$\{2,6\}$	1	$\{3,4\}$
2	$\{2,6\}$	$\{3,4\}$	$\{0,5\}$	1	1	$\{2,6\}$	$\{0,5\}$
3	$\{3,4\}$	$\{2,6\}$	1	$\{0,5\}$	$\{0,5\}$	$\{3,4\}$	1
4	$\{3,4\}$	$\{2,6\}$	1	$\{0,5\}$	$\{0,5\}$	$\{3,4\}$	1
5	$\{0,5\}$	1	$\{2,6\}$	$\{3,4\}$	$\{3,4\}$	$\{0,5\}$	$\{2,6\}$
6	$\{2,6\}$	$\{3,4\}$	$\{0,5\}$	1	1	$\{2,6\}$	$\{0,5\}$

$\cdot$	0	1	2	3	4	5	6
0	$\{0,5\}$	$\{0,5\}$	$\{0,5\}$	$\{0,5\}$	$\{0,5\}$	$\{0,5\}$	$\{0,5\}$
1	$\{0,5\}$	1	$\{0,5\}$	1	1	$\{0,5\}$	$\{0,5\}$
2	$\{0,5\}$	$\{2,6\}$	$\{0,5\}$	$\{2,6\}$	$\{2,6\}$	$\{0,5\}$	$\{0,5\}$
3	$\{0,5\}$	$\{3,4\}$	$\{0,5\}$	$\{3,4\}$	$\{3,4\}$	$\{0,5\}$	$\{0,5\}$
4	$\{0,5\}$	$\{3,4\}$	$\{0,5\}$	$\{3,4\}$	$\{3,4\}$	$\{0,5\}$	$\{0,5\}$
5	$\{0,5\}$	$\{0,5\}$	$\{0,5\}$	$\{0,5\}$	$\{0,5\}$	$\{0,5\}$	$\{0,5\}$
6	$\{0,5\}$	$\{2,6\}$	$\{0,5\}$	$\{2,6\}$	$\{2,6\}$	$\{0,5\}$	$\{0,5\}$

Then $(R, +, \cdot)$ is a non-commutative hyperring such that is not a ring. Set $\mathcal{A} = \{2\}$. Then, $\mathcal{D} = \{0, 2, 5, 6\}$ and $R - \mathcal{D} = \{1, 3, 4\}$. Since $1 \cdot 3 + 1 \cdot 3 = \{0, 5\}$ and $1 \cdot 3 + 3 \cdot 1 = \{2, 6\}$ so $\psi_a^*(0) = \{0, 2, 5, 6\}$ and $\psi_a^*(1) = \{1, 3, 4\}$. Therefore $R/\psi_a^* \cong \mathbb{Z}_2$. But $R/\gamma^* = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$ when $\bar{0} = \{0, 5\}$, $\bar{1} = \{1\}$, $\bar{2} = \{2, 6\}$ and $\bar{3} = \{3, 4\}$, is a non-commutative ring with the following table:

$+$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{1}$	$\bar{1}$	$\bar{0}$	$\bar{3}$	$\bar{2}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{2}$	$\bar{1}$	$\bar{0}$

$\cdot$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
$\bar{1}$	$\bar{0}$	$\bar{1}$	$\bar{0}$	$\bar{1}$
$\bar{2}$	$\bar{0}$	$\bar{2}$	$\bar{0}$	$\bar{2}$
$\bar{3}$	$\bar{0}$	$\bar{3}$	$\bar{0}$	$\bar{3}$

In this Example we have $\gamma^* \neq \psi_a^*$.

Theorem 4.9. *The relation ψ_a^* is the smallest strongly regular relation such that the quotient R/ψ_a^* is a commutative ring generated by $\rho_a^*(a)$ or $|R/\psi_a^*| \leq 2$, where the equivalence classes of all elements of $R - \mathfrak{D}$ are equal.*

Proof. Let θ be a strongly regular equivalence such that quotient R/θ is a commutative ring generated by $\theta^*(a)$ or $|R/\theta^*| \leq 2$, and the equivalence classes of θ of all elements of $(R - \mathfrak{D})$ are equal. Suppose that $\phi : R \Longrightarrow R/\theta$ is the canonical projection. ϕ is a good homomorphism. We show that $\psi_a^* \subseteq \theta$. Let $x\ \psi_a^*\ y$. So there exists $(A, B) \in \Re^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$ such that $x \in A$ and $y \in B$. We have three cases:

Case 1. If $(A, B) \in \alpha^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$, then $A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ and there exist $\sigma \in \mathbb{S}_n$ and $\sigma_i \in \mathbb{S}_{k_i}$ such that $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{k_i} y_{ij} = \sum\limits_{i=1}^{n} A_{\sigma(i)}$, where $A_i = \prod\limits_{j=1}^{k_i} x_{i\sigma_i(j)}$. Therefore,

$$\phi(x) = \oplus \sum_{i=1}^{n}(\otimes \prod_{j=1}^{k_i} x_{ij}) \ \text{ and } \phi(y) = \oplus \sum_{i=1}^{n}(\otimes \prod_{j=1}^{k_i} (y_{ij})).$$

By the commutativity of R/θ, it follows that $\phi(x) = \phi(y)$. Thus $x\ \theta\ y$.

Case 2. If $(A, B) \in \wp^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$, then $A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathfrak{D} = \{y_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathfrak{D} \neq \varnothing$ and there exists $\sigma \in \mathbb{S}_n$ and $\sigma_i \in \mathbb{S}_{k_i}$ such that for all $1 \leqslant t \leqslant n, 1 \leqslant s \leqslant k_n$ if $y_{ts} \in \mathfrak{D}$, then $y_{ts} = x_{\sigma(t)\sigma_{\sigma(t)}(s)}$. Renumber of the elements of the sets $\{x_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\}$ and $\{y_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\}$ such that $\{y_{ij}|1 \leq i \leq n', 1 \leq j \leq k'_i\} \subseteq \mathfrak{D}$, where $1 \leqslant n' \leqslant n$ and $1 \leqslant k'_i \leqslant k_i$ and $x_{t's'}, y_{t's'} \notin \mathfrak{D}$ for all $n'+1 \leqslant t' \leqslant n$ and $k'_i + 1 \leqslant s' \leqslant k_i$. So, $\phi(x) = (\oplus \sum\limits_{i=1}^{n'}(\otimes \prod\limits_{j=1}^{k'_i} \theta(x_{ij}))) \bigoplus (\oplus \sum\limits_{t'=n'+1}^{n}(\otimes \prod\limits_{s'=k'_i+1}^{k_i} \theta(x_{ij})))$ and $\phi(y) = (\oplus \sum\limits_{i=1}^{n'}(\otimes \prod\limits_{j=1}^{k'_i} \theta(x_{\sigma(i)\sigma_{\sigma(i)}(j)}))) \bigoplus (\oplus \sum\limits_{t'=n'+1}^{n}(\otimes \prod\limits_{s'=k'_i+1}^{k_i} \theta(y_{ij})))$. For all $n'+1 \leqslant l, l' \leqslant n$ and $k'_i + 1 \leqslant d, d' \leqslant k_i$ we have $\phi(x_{ld}) = \phi(y_{l'd'})$ and since R/θ is a commutative ring so, $\phi(x) = \phi(y)$ and $x\ \theta\ y$.

Case 3. If $(A, B) \in \jmath^{\mathcal{A}}_{n,k_1,k_2,\cdots,k_n}$, then $A = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} x_{ij})$ and $B = \sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ such that $\{x_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathfrak{D} = \{y_{ij}|1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathfrak{D} = \varnothing$. Therefore, for all $1 \leq i \leq n$ and $1 \leq j \leq k_i$ we have $\phi(x_{ij}) = \phi(y_{ij})$. Thus, $\phi(x) = \phi(y)$ and $x\ \theta\ y$.

In the all cases we have $x\ \theta\ y$ and hence $x\ \psi_a\ y$ implies that $x\ \theta\ y$. Thus, $x\ \psi_a^*\ y$ implies that $x\ \theta\ y$ by transitivity of R. Therefore, $\psi_a^* \subseteq \theta$. □

Definition 4.10. *We say that M is a $\chi^*_{\mathcal{A}}$-part of R if the following conditions hold:*

(F_1) *For all* $\sigma \in \mathbb{S}_n$ *and* $\sigma_i \in \mathbb{S}_{k_i}$, $\sum\limits_{i=1}^{n} A_{\sigma(i)} \subseteq M$, *where* $A_i = \prod\limits_{j=1}^{k_i} x_{i\sigma_i(j)}$

(F_2) *If* $\{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} \neq \varnothing$, *then for all* $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ *such that* $\{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D}$ *there exist* $\sigma \in \mathbb{S}_n$ *and* $\sigma_i \in \mathbb{S}_{k_i}$ *for which for all* $y_{ij} \in \{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D}$, $y_{ij} = x_{\sigma(i)\sigma_{\sigma(i)}(j)}$ *we have* $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij}) \subseteq M$;

(F_3) *If* $\{x_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \varnothing$, *then for all* $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij})$ *such that* $\{y_{ij} | 1 \leq i \leq n, 1 \leq j \leq k_i\} \cap \mathcal{D} = \varnothing$, *we have* $\sum\limits_{i=1}^{n}(\prod\limits_{j=1}^{k_i} y_{ij}) \subseteq M$.

Example 4.11. *In Example 4.8,* $\chi^*_{\mathcal{A}}$*-parts of* R *are exactly* $\psi^*_a(0) = \{0, 2, 5, 6\}$, $\psi^*_a(1) = \{1, 3, 4\}$ *and* R*. Also, complete parts(*γ^**-parts) of* R *are exactly* $\{0, 5\}$, $\{1\}$, $\{2, 6\}$ *and* $\{3, 4\}$ *or unions of them. It is clear that* $\psi^*_a(0) = \{0, 2, 5, 6\}$ *and* $\psi^*_a(1) = \{1, 3, 4\}$ *are complete parts(*γ^**-parts) of* R*.*

Let $\phi : R \longrightarrow R/\chi^*_{\mathcal{A}}$ be the canonical projection and let $D(R)$ be the *kernel* of ϕ. If we denote by $\overline{0}$ the zero element of $R/\chi^*_{\mathcal{A}}$, then $D(R) = \phi^{-1}(\overline{0})$

Theorem 4.12. *For every non empty subset* B *of hyperring* R*, we have*

1) $\phi^{-1}(\phi(B)) = D(R) + B = B + D(R)$.

2) *If* B *is a* $\chi^*_{\mathcal{A}}$*-part of* R*, then* $\phi^{-1}(\phi(B)) = B$.

Proof. 1) For every $x \in D(R) + B$, there exists a pair $(b, a) \in B \times D(R)$ such that $x \in a + b$, so $\phi(x) = \phi(a) \oplus \phi(b) = \overline{0} \oplus \phi(b) = \phi(b)$. Therefore, $x \in \phi^{-1}(\phi(b)) \subseteq \phi^{-1}(\phi(B))$. Conversely, for every $x \in \phi^{-1}(\phi(B))$, an element $b \in B$ exists such that $\phi(x) = \phi(b)$. By the reproducibility $a \in R$ exists such that $x \in b + a$, so $\phi(b) = \phi(x) = \phi(b) \oplus \phi(a)$, hence $\phi(a) = \overline{0}$ and $a \in \phi^{-1}(\overline{0}) = D(R)$. Therefore, $x \in b + a \subseteq B + D(R)$. This prove that $\phi^{-1}(\phi(B)) = D(R) + B$. In the same way, it is possible to prove that $\phi^{-1}(\phi(B)) = B + D(R)$.

2) It is obvious that $B \subseteq \phi^{-1}(\phi(B))$. Moreover, if $x \in \phi^{-1}(\phi(B))$, then there exists an element $b \in B$ such that $\phi(x) = \phi(b)$. Since B is a $\chi^*_{\mathcal{A}}$-part, it follows that $x \in \phi(x) = \phi(b) \subseteq B$ and therefore $\phi^{-1}(\phi(B)) \subseteq B$. □

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