Bol. Soc. Paran. Mat. (3s.) **v. 39** 1 (2021): 35–50.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.40499

Simulation Functions and Geraghty Type Results

Sumit Chandok, Ankush Chanda, Lakshmi Kanta Dey, Mirjana Pavlović, Stojan Radenović

ABSTRACT: We concern this manuscript with Geraghty type contraction mappings via simulation functions and pull down some sufficient conditions for the existence and uniqueness of point of coincidence for several classes of mappings involving Geraghty functions in the setting of metric spaces. These findings touch up many of the existing results in the literature. Additionally, we elicit one of our main results by a non-trivial example and pose an interesting open problem for the enthusiastic readers.

Key Words: Geraghty type contraction mapping, Common fixed point, Point of coincidence, Compatible mapping, Simulation function.

Contents

1. Introduction and Preliminaries

The origin of fixed point theory is dated to the last quarter of nineteenth century and rests in the use of successive approximations to establish existence and uniqueness of solutions, particularly of differential equations. It is worth noting that based on the work of S. Banach [4] in 1922, known as the Banach contraction principle, the metric fixed point theory took off. Since then, due to its wide applications, this principle is being investigated at a large in contemporary researches [2,3,7,8,9]. Fixed point theory gains very large impetus due to its wide range of applications in various fields such as engineering, economics, computer science, and many more.

In his research article, Geraghty [10] stated the Cauchy criteria for the convergence of a contractive iteration in a complete metric space and converted this sequential condition to the functional form which eventually gave birth to Geraghty contractions.

On the other hand, in 2015, Khojasteh et al. [12] originated the notion of $\mathcal{Z}$-contractions using a specific family of functions called simulation functions. Subsequently, many researchers generalized this idea in many ways ([5,6,13,14,15,16,17,18,22] and references therein) and proved many interesting results in the arena of fixed point theory.

In this paper, we define new kinds of Geraghty type contraction mappings via simulation functions and inspect for some sufficient conditions for the existence

2010 *Mathematics Subject Classification:* 47H10, 54H25.

Submitted November 17, 2017. Published February 13, 2018

and uniqueness of point of coincidence as well as of common fixed point for such classes of mappings in complete metric spaces. Besides, we raise an open problem for intent researchers and construct an example to substantiate one of the main result.

Now, we recollect some basic definitions, notations and concepts to be used in this sequel.

Definition 1.1. *[12] A mapping $\zeta : [0,\infty)^2 \to \mathbb{R}$ is called a simulation function if it satisfies the following conditions:*

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$;

(ζ_3) if $\{t_n\}, \{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\overline{\lim_{n\to\infty}} \zeta(t_n, s_n) < 0$.

The set of all simulation functions is denoted by $\mathcal{Z}$.

Example 1.2. *[12] The following are some examples of simulation functions.*

- *(i) Let $\zeta : [0,\infty)^2 \to \mathbb{R}$ be defined by $\zeta(t,s) = f(s) - g(t)$ for all $t, s \in [0,\infty)$, where $f, g : [0,\infty) \to [0,\infty)$ are two continuous functions such that $f(t) = g(t) = 0$ if and only if $t = 0$ and $f(t) < t < g(t)$ for all $t > 0$. Then ζ is a simulation function.*

- *(ii) Let $\zeta : [0,\infty)^2 \to \mathbb{R}$ be defined by $\zeta(t,s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t, s \in [0,\infty)$, where $f, g : [0,\infty)^2 \to [0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t, s > 0$. Then ζ is a simulation function.*

- *(iii) Let $\zeta : [0,\infty)^2 \to \mathbb{R}$ be defined by $\zeta(t,s) = s - f(s) - t$ for all $t, s \in [0,\infty)$, where $f : [0,\infty) \to [0,\infty)$ is a continuous function such that $f(t) = 0$ if and only if $t = 0$. Then ζ is a simulation function.*

Definition 1.3. *[12] Let (X,d) be a metric space and $\zeta \in \mathcal{Z}$. A mapping $T : X \to X$ is called a $\mathcal{Z}$-contraction with respect to ζ if*

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0$$

holds for all $x, y \in X$.

Remark 1.4. *Note that a Banach contraction mapping is a $\mathcal{Z}$-contraction mapping which can be obtained by taking $\lambda \in [0,1)$ and $\zeta(t,s) = \lambda s - t$ for all $t, s \geq 0$. Also, we know that every $\mathcal{Z}$-contraction mapping is a contractive mapping and hence it is also continuous.*

Let T and S be two self-maps defined on a non-empty set X. If $w = Tx = Sx$ for some $x \in X$, then x is called a coincidence point of T and S and w is called a point of coincidence of T and S. Moreover w is called a common fixed point of

T and S if $x = w$. A pair (T, S) of self-maps is called weakly compatible if they commute at their coincidence points.

Given two self-mappings $T, S : X \to X$ and a sequence $\{x_n\} \subseteq X$, the sequence $\{x_n\}$ is said to be a Picard-Jungck sequence of the pair (T, S) (based on x_0) if $y_n = Tx_n = Sx_{n+1}$ holds for all $n \in \mathbb{N}_0$.

It is obvious that if $TX \subseteq SX$, then for each $x_0 \in X$ there exists a Picard-Jungck sequence $\{y_n\} = \{Tx_n\} = \{Sx_{n+1}\}$, $n \in \mathbb{N}_0$. In general, the converse is not true.

Now, here we make a note of the following well known result due to Abbas and Jungck [1] which is playing a crucial role in this sequel.

Theorem 1.5. *Let T and S be weakly compatible self-maps defined on a non-empty set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.*

The following lemma is also necessary to obtain our desired results.

Lemma 1.6. *[19] Let (X, d) be a metric space and let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon > 0$ and two sequences $\{n(k)\}$ and $\{m(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following sequences tend to ε when $k \to \infty$:*

$$\{d(x_{m(k)}, x_{n(k)})\}, \{d(x_{m(k)}, x_{n(k)+1})\}, \{d(x_{m(k)-1}, x_{n(k)})\},$$

$$\{d(x_{m(k)-1}, x_{n(k)+1})\}, \{d(x_{m(k)+1}, x_{n(k)+1})\}.$$

Also, here we note down the definition of compatible mappings.

Definition 1.7. *[11] Two self-mappings f and g of a metric space (X, d) are compatible if*

$$\lim_{n\to\infty} d(gf(x_n), fg(x_n)) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$$

for some $t \in X$.

2. Main results

In this section, we investigate for some existence and uniqueness conditions for the point of coincidence for a few kinds of Geraghty type contraction mappings using simulation functions in the framework of metric spaces. Alongside, we discuss an example to authenticate one of our findings. Throughout this paper, $\mathbb{N}_0$ will stand for the set of all non-negative integers and $\mathbb{N}$ will denote the set of all positive integers.

Beforehand, we put down the following definition.

Definition 2.1. *[21] Let (X,d) be a metric space and $T,S : X \to X$ be two self-mappings. A mapping T is called a $(\mathcal{Z},S)$-contraction if there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(d(Tx,Ty),d(Sx,Sy)) \geq 0$$

for all $x,y \in X$ with $Sx \neq Sy$.

Let $\beta : [0,\infty) \to (0,1)$ be such that, for each sequence $\{r_n\}$ in $[0,\infty)$, one of the following conditions holds:

$$\lim_{n\to\infty} \beta(r_n) = 1^- \Rightarrow \lim_{n\to\infty} r_n = 0^+, \tag{G1}$$

or,

$$\overline{\lim_{n\to\infty}} \beta(r_n) = 1^- \Rightarrow \lim_{n\to\infty} r_n = 0^+. \tag{G2}$$

It is clear that $(G1)$ implies $(G2)$. But the converse is not true, in general, as this may be supported by the following example:

Example 2.2. *Let $\beta : [0,\infty) \to (0,1)$ be defined as*

$$\beta(t) = \begin{cases} e^{-t}, & \text{if } t \in \left\{1, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2n-1}, \ldots\right\} \\ \frac{1}{2}, & \text{if } t \in [0,\infty) \setminus \left\{1, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2n-1}, \ldots\right\}. \end{cases}$$

Then $\beta(t_n)$ has no limit as $n \to \infty$, where $t_n = \frac{1}{n}$ but $\overline{\lim\limits_{n\to\infty}} \beta(t_n) = 1$. This means that $(G2)$ does not imply $(G1)$.

Theorem 2.3. *If $\beta : [0,\infty) \to (0,1)$ satisfies $(G2)$, then $\zeta(t,s) = s\beta(s) - t$ is a simulation function.*

Proof: It is clear that $\zeta(0,0) = 0$ as well as

$$\begin{aligned} \zeta(t,s) =& s\beta(s) - t \\ <& s - t \end{aligned}$$

for all $t, s \in (0,\infty)$.

So, (ζ_1) and (ζ_2) are satisfied. Now we check for (ζ_3).

If $\{t_n\}, \{s_n\}$ are two sequences in $(0,\infty)$ such that $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n = l > 0$, then

$$\overline{\lim_{n\to\infty}} \zeta(t_n, s_n) = \overline{\lim_{n\to\infty}} (s_n\beta(s_n) - t_n). \tag{2.1}$$

Since $t_n \to l, s_n \to l$ and $l > 0$, we have

$$t_n = l + \alpha_n, \;\; s_n = l + \sigma_n \tag{2.2}$$

where $\alpha_n, \sigma_n \to 0$ as $n \to \infty$.

From (2.1), we get

$$\begin{aligned}\overline{\lim_{n\to\infty}}\zeta(t_n,s_n) &= \overline{\lim_{n\to\infty}}((l+\sigma_n)\beta(l+\sigma_n)-(l+\alpha_n))\\ &= \overline{\lim_{n\to\infty}}(l(\beta(l+\sigma_n)-1)+\sigma_n\beta(l+\sigma_n)-\alpha_n)\\ &= \overline{\lim_{n\to\infty}}(l(\beta(l+\sigma_n)-1)+\delta_n), \end{aligned}\tag{2.3}$$

where $\delta_n = \sigma_n\beta(l+\sigma_n)-\alpha_n \to 0-0=0$ as $n\to\infty$.

Since $\overline{\lim_{n\to\infty}}(x_n+y_n) \le \overline{\lim_{n\to\infty}}x_n + \overline{\lim_{n\to\infty}}y_n$, from (2.3) it follows that

$$\begin{aligned}\overline{\lim_{n\to\infty}}\zeta(t_n,s_n) &\le \overline{\lim_{n\to\infty}}l(\beta(l+\sigma_n)-1)+\overline{\lim_{n\to\infty}}\delta_n\\ &= l[\overline{\lim_{n\to\infty}}(\beta(l+\sigma_n)-1)]+0\\ &< 0,\end{aligned}$$

because if $\overline{\lim_{n\to\infty}}(\beta(l+\sigma_n)-1)=0$, then by $(G2)$, it follows that $l+\sigma_n\to 0^+$, as $n\to\infty$, which is a contradiction. So (ζ_3) is also verified.

This means that $\zeta(t,s)=s\beta(s)-t$ is a simulation function. □

Now we recall the notion of a strong Geraghty function and also the definition of a strong Geraghty contraction.

Definition 2.4. *[10] A function $\beta:[0,\infty)\to(0,1)$ is called a strong Geraghty function if $\{r_n\}\subset[0,\infty)$ and $\overline{\lim_{n\to\infty}}\beta(r_n)=1$ implies $r_n\to 0^+$ as $n\to\infty$, that is, if it satisfies $(G2)$.*

Definition 2.5. *[10] A mapping $T:X\to X$ is called a strong Geraghty contraction if there exists a strong Geraghty function β such that*

$$d(Tx,Ty)\le\beta(d(x,y))d(x,y)\tag{2.4}$$

holds for all $x,y\in X$.

Employing Theorem 2.3, we can easily claim the next result.

Theorem 2.6. *[20] Every strong Geraghty contraction T from a complete metric space (X,d) into itself has a unique fixed point in X and for every $x_0\in X$, the Picard sequence $\{x_n\}$, where $x_n=Tx_{n-1}$ for all $n\in\mathbb{N}$ converges to the fixed point of T.*

Here we present the concepts of a Geraghty function and a Geraghty contraction which were discussed by Geraghty [10] in his research article.

Definition 2.7. *[10] A function $\beta:[0,\infty)\to(0,1)$ is called a Geraghty function if $\{r_n\}\subset[0,\infty)$ and $\lim_{n\to\infty}\beta(r_n)=1^+$ implies $r_n\to 0^+$ as $n\to\infty$, that is, if it satisfies $(G1)$.*

Definition 2.8. *[10] A mapping $T : X \to X$ is called a Geraghty contraction if there exists a Geraghty function β such that*

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \tag{2.5}$$

for all $x, y \in X$.

Now we state the fixed point result due to Geraghty [10].

Theorem 2.9. *Every Geraghty contraction T from a complete metric space (X, d) into itself has a unique fixed point.*

Now, we furnish our first main result which generalizes the Theorem 2.9.

Theorem 2.10. *Let (X, d) be a complete metric space and $T, S : X \to X$ be two self-mappings. Assume there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(d(Tx, Ty), \beta(d(Sx, Sy))d(Sx, Sy)) \geq 0 \tag{2.6}$$

for all $x, y \in X$, where $\beta : [0, \infty) \to (0, 1)$ is a Geraghty function.

Suppose that there exists a Picard-Jungck sequence $\{j_n\}$ of (T, S). Also assume that, at least, one of the following conditions holds:

(i) (TX, d) or (SX, d) is complete;

(ii) (X, d) is complete, S is continuous and T and S are compatible.

Then T and S have a unique point of coincidence.

Proof: First of all, we prove that the point of coincidence of T and S is unique (if it exists). Suppose that w_1 and w_2 are distinct points of coincidence of T and S. From this, it follows that there exist two points s_1 and s_2 ($s_1 \neq s_2$) such that $Ts_1 = Ss_1 = w_1$ and $Ts_2 = Ss_2 = w_2$. Then $d(Ts_1, Ts_2) > 0$ and clearly $\beta(d(Ss_1, Ss_2))d(Ss_1, Ss_2) > 0$. Therefore using (ζ_2), we obtain from (2.6)

$$\begin{aligned} 0 &\leq \zeta(d(Ts_1, Ts_2), \beta(d(Ss_1, Ss_2))d(Ss_1, Ss_2)) \\ &= \zeta(d(w_1, w_2), \beta(d(w_1, w_2))d(w_1, w_2)) \\ &< \beta(d(w_1, w_2))d(w_1, w_2) - d(w_1, w_2) \\ &< d(w_1, w_2) - d(w_1, w_2) \\ &= 0, \end{aligned}$$

which is a contradiction.

In order to prove that T and S have a point of coincidence, suppose that there is a Picard-Jungck sequence $\{j_n\}$ such that $j_n = Tx_n = Sx_{n+1}$, where $n \in \mathbb{N}_0$.

If $j_m = j_{m+1}$ for some $m \in \mathbb{N}_0$, then $Sx_{m+1} = j_m = j_{m+1} = Tx_{m+1}$ and T and S have a coincidence point x_{m+1}. Therefore, we suppose that $j_n \neq j_{n+1}$ for all

$n \in \mathbb{N}_0$. This implies that $d(j_{n+1}, j_{n+2}) > 0$ and also $\beta(d(j_n, j_{n+1}))d(j_n, j_{n+1}) > 0$ for each $n \in \mathbb{N}_0$. Putting $x = x_{n+1},\ y = x_{n+2}$ in (2.6), we obtain that

$$\begin{aligned} 0 &\leq \zeta(d(Tx_{n+1}, Tx_{n+2}), \beta(d(Sx_{n+1}, Sx_{n+2}))d(Sx_{n+1}, Sx_{n+2})) \\ &= \zeta(d(j_{n+1}, j_{n+2}), \beta(d(j_n, j_{n+1}))d(j_n, j_{n+1})) \\ &< \beta(d(j_n, j_{n+1}))d(j_n, j_{n+1}) - d(j_{n+1}, j_{n+2}) \\ &< d(j_n, j_{n+1}) - d(j_{n+1}, j_{n+2}). \end{aligned}$$

Hence, we have that $d(j_{n+1}, j_{n+2}) < d(j_n, j_{n+1})$ for all $n \in \mathbb{N}_0$.

Therefore there exists $d^* \geq 0$ such that $\lim_{n\to\infty} d(j_n, j_{n+1}) = d^* \geq 0$. Suppose that $d^* > 0$. In this case we obtain that

$$\frac{d(j_{n+1}, j_{n+2})}{d(j_n, j_{n+1})} \leq \beta(d(j_n, j_{n+1})) < 1,$$

i.e., $\beta(d(j_n, j_{n+1})) \to 1$ as $n \to \infty$, which is a contradiction to the fact that $\lim_{n\to\infty} d(j_n, j_{n+1}) = d^* > 0$.

Hence we obtain $\lim_{n\to\infty} d(j_n, j_{n+1}) = d^* = 0$.

Further, we have to prove that $j_n \neq j_m$, whenever $n \neq m$. Indeed, suppose that $j_n = j_m$ for some $n > m$. Then we can claim that $x_{n+1} = x_{m+1}$. Because if $x_{n+1} \neq x_{m+1}$, then

$$\begin{aligned} Tx_n &\neq Tx_m \\ \Rightarrow j_n &\neq j_m, \end{aligned}$$

which is obviously impossible. Therefore,

$$\begin{aligned} x_{n+1} &= x_{m+1} \\ \Rightarrow Tx_{n+1} &= Tx_{m+1} \\ \Rightarrow j_{n+1} &= j_{m+1}. \end{aligned}$$

Then following the previous arguments, we have

$$d(j_{m+1}, j_m) < d(j_m, j_{m-1}) < \cdots < d(j_{n+1}, j_n) = d(j_{m+1}, j_m),$$

which is a contradiction.

Now, we have to show that $\{j_n\}$ is a Cauchy sequence. Suppose, to the contrary, that it is not a Cauchy sequence. Putting $x = x_{m(k)+1},\ y = x_{n(k)+1}$ in (2.6), we have

$$0 \leq \zeta(d(j_{m(k)+1}, j_{n(k)+1}), \beta(d(j_{m(k)}, j_{n(k)}))d(j_{m(k)}, j_{n(k)})) = \zeta(t_k, s_k),$$

where $0 < t_k = d(j_{m(k)+1}, j_{n(k)+1}),\ 0 < s_k = \beta(d(j_{m(k)}, j_{n(k)}))d(j_{m(k)}, j_{n(k)})$.

Now, since the sequence $\{j_n\}$ is not a Cauchy sequence, then by Lemma 1.6, we have $\{d(j_{m(k)}\, j_{n(k)})\}$ and $\{d(j_{m(k)+1}, j_{n(k)+1})\}$ both the sequences tend to $\varepsilon > 0$, as $k \to \infty$.

Indeed,

$$\begin{aligned} t_k =&d(j_{m(k)+1}, j_{n(k)+1}) \\ \leq&\beta(d(j_{m(k)}, j_{n(k)}))d(j_{m(k)}, j_{n(k)}) \\ =&s_k \\ <&d(j_{m(k)}, j_{n(k)}) \end{aligned}$$

and so by the sandwich theorem, $\{s_k\}$, where $s_k = \beta(d(j_{m(k)}, j_{n(k)}))d(j_{m(k)}, j_{n(k)})$, tends to ε as $k \to \infty$. Therefore, we have $0 < t_k,\ s_k \to \varepsilon$.

Therefore,

$$0 \leq \overline{\lim_{k\to\infty}}\zeta(t_k, s_k) < \overline{\lim_{k\to\infty}}[s_k - t_k] \to \varepsilon - \varepsilon = 0,$$

and we arrive at a contradiction. So, the Picard-Jungck sequence $\{j_n\}$ is a Cauchy sequence.

Suppose that (i) holds, i.e., (SX, d) is complete. Then there exists $z \in X$ such that $j_n = Sx_{n+1} \to Sz$ as $n \to \infty$ which implies

$$\lim_{n\to\infty} d(Sx_{n+1}, Sz) = 0. \tag{2.7}$$

We prove that $Tz = Sz$. Let, on the contrary, $Tz \neq Sz$ and so, $d(Tz, Sz) = \delta > 0$. Again, from (2.7), there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} d(Tx_n, Sz) <&\delta \\ =&d(Tz, Sz) \end{aligned}$$

for all $n \geq n_0$. This leads us to

$$\begin{aligned} Tx_n \neq&Tz \\ \Rightarrow d(Tx_n, Tz) >&0 \end{aligned} \tag{2.8}$$

for all $n \geq n_0$. Now, there does not exist some $n_3 \in \mathbb{N}$ such that for all $n \geq n_3$

$$Sx_{n+1} = Sz.$$

Therefore, there exists a partial subsequence $\{Sx_{p_k}\}$ of $\{Sx_{n+1}\}$ such that

$$Sx_{p_k} \neq z \tag{2.9}$$

for all $k \in \mathbb{N}$. Now, let $n_2 \in \mathbb{N}$ be such that $p_{n_2} \geq n_0$. Hence by (2.8) and (2.9), we have $d(Tx_{p_n}, Tz) > 0$ and $d(Sx_{p_n}, z) > 0$ for all $n \geq n_2$.

So utilizing the previous facts and (ζ_2), we have

$$\begin{aligned} 0 &\leq \zeta(d(Tz, Tx_{p_n}), \beta(d(Sz, Sx_{n+1}))d(Sz, Sx_{n+1})) \\ &< \beta(d(Sz, Sx_{n+1}))d(Sz, Sx_{n+1}) - d(Tz, Tx_{p_n}) \\ &< d(Sz, Sx_{n+1}) - d(Tz, Tx_{p_n}). \end{aligned}$$

Letting $n \to \infty$, we have

$$\begin{aligned} 0 <& d(Sz, Sz) - d(Tz, Sz) \\ =& 0 - d(Tz, Sz). \end{aligned}$$

This implies that $w = Sz = Tz$ and w is the (unique) point of coincidence of T and S.

Similarly, we can prove that $u = Tz = Sz$ is a (unique) point of coincidence of T and S, when (TX, d) is complete.

Finally, suppose that (ii) holds. Since (X, d) is complete, there exists $z \in X$ such that $j_n = Tx_n = Sx_{n+1} \to z$ when $n \to \infty$. As S is continuous, we have

$$\begin{aligned} \lim_{n\to\infty} S(Tx_n) =& Sz \\ \Rightarrow \lim_{n\to\infty} d(S(Tx_n), Sz) =& 0 \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \lim_{n\to\infty} S(Sx_{n+1}) =& Sz \\ \Rightarrow \lim_{n\to\infty} d(S(Sx_{n+1}), Sz) =& 0. \end{aligned} \tag{2.11}$$

Our claim is

$$\lim_{n\to\infty} T(Sx_n) = Tz.$$

If not, then there exists a subsequence $\{T(Sx_{p_k})\}$ of $\{T(Sx_n)\}$ such that

$$d(T(Sx_{p_k}), Tz) > 0 \tag{2.12}$$

for all $k \in \mathbb{N}$. Again, there does not exist some $k_1 \in \mathbb{N}$ such that for all $n \geq k_1$

$$S(Sx_{n+1}) = Sz.$$

Hence, there exists a partial subsequence $\{S(Sx_{p_r})\}$ of $\{S(Sx_{n+1})\}$ such that

$$S(Sx_{p_r}) \neq Sz \tag{2.13}$$

for all $r \in \mathbb{N}$. Therefore, by (2.12) and (2.13), we have $d(T(Sx_{p_k}), Tz) > 0$ and $d(S(Sx_{p_r}), Sz) > 0$ for all $k, r \in \mathbb{N}$.

Now employing (ζ_2) we get

$$\begin{aligned} 0 &\leq \zeta(d(T(Sx_{p_k}), Tz)), \beta(d(S(Sx_{p_r}), Sz))d(S(Sx_{p_r}), Sz)) \\ &< \beta(d(S(Sx_{p_r}), Sz))d(S(Sx_{p_r}), Sz) - d(T(Sx_{p_k}), Tz) \\ &< d(S(Sx_{p_r}), Sz) - d(T(Sx_{p_k}), Tz). \end{aligned}$$

Therefore we have

$$d(T(Sx_{p_k}), Tz) < d(S(Sx_{p_r}), Sz) \to 0$$

as $k \to \infty$, which is a contradiction. It implies that

$$\lim_{n\to\infty} d(T(Sx_n), Tz) = 0. \tag{2.14}$$

Further, as T and S are compatible, we have

$$\lim_{n\to\infty} d(T(Sx_n), S(Tx_n)) = 0. \tag{2.15}$$

Finally we obtain using (2.10), (2.14) and (2.15)

$$\begin{aligned} d(Tz, Sz) \leq & d(Tz, T(Sx_n)) + d(T(Sx_n), S(Tx_n)) + d(S(Tx_n), Sz) \\ \Rightarrow d(Tz, Sz) \leq & 0 \\ \Rightarrow d(Tz, Sz) = & 0. \end{aligned}$$

This implies that $v = Sz = Tz$ and v is the (unique) point of coincidence of T and S.

Hence, the result is proved in both the cases, i.e., the mappings T and S have a unique point of coincidence. □

Here we figure out an additional condition which guarantees the existence and uniqueness of a common fixed point of these two self-maps.

Theorem 2.11. *Let $T, S : X \to X$ be two self-maps defined on a complete metric space (X, d). Assume there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(d(Tx, Ty), \beta(d(Sx, Sy))d(Sx, Sy)) \geq 0 \tag{2.16}$$

for all $x, y \in X$, where $\beta : [0, \infty) \to (0, 1)$ is a Geraghty function.

Suppose that, there exists a Picard-Jungck sequence $\{x_n\}$ of (T, S). Also assume that, (TX, d) or (SX, d) is complete and T and S are weakly compatible. Then T and S have a unique common fixed point in X.

Proof: Using Theorem 2.10, T and S have a unique point of coincidence. Further, since T and S are weakly compatible, then according to Theorem 1.5, they have a unique common fixed point in X. □

The succeeding example endorses our previous result.

Example 2.12. *Let $X = \{0, 2, 3\}$ and $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = |x - y|$. Define $T, S : X \to X$ as*

$$T = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & 2 & 3 \\ 3 & 2 & 0 \end{pmatrix}.$$

Suppose $\zeta(t, s) = \frac{s}{s+1} - t$, $\beta(t) = \frac{1}{1+\frac{t}{9}}$, for $t > 0$ and $\beta(t) = \frac{1}{2}$, for $t = 0$.

Case-1 *For $x = 0, y = 2$. From (2.6), we have*

$$\begin{aligned} \zeta(d(T0, T2), \beta(d(S0, S2))d(S0, S2)) = & \zeta(0, \beta(1)1) \\ = & \frac{\beta(1)}{\beta(1) + 1} \\ \geq & 0. \end{aligned}$$

Case-2 *For $x = 0, y = 3$. From (2.6), we obtain*

$$\begin{aligned}\zeta(d(T0,T3),\beta(d(S0,S3))d(S0,S3)) =& \zeta(0,\beta(3)3)\\ =& \frac{\beta(3)}{\beta(3)+1}\\ \geq & 0.\end{aligned}$$

Case-3 *For $x = 2, y = 3$. From (2.6), we get*

$$\begin{aligned}\zeta(d(T2,T3),\beta(d(S2,S3))d(S2,S3)) =& \zeta(0,\beta(2)2)\\ =& \frac{\beta(2)}{\beta(2)+1}\\ \geq & 0.\end{aligned}$$

Hence all the assumptions of Theorem 2.11 are satisfied and by the conclusion of it, T and S have a unique point of coincidence $x = 2$ and also it is their unique common fixed point.

Using the similar reasoning as in Theorems 2.10-2.11, we have the following new results and also these results generalize several ones from the existing literature.

Theorem 2.13. *Let (X, d) be a complete metric space and $T, S : X \to X$ be two given self-maps. Assume that there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(d(Tx,Ty),\beta(M(x,y))M(x,y)) \geq 0$$

for all $x, y \in X$ with $Sx \neq Sy$, where $\beta : [0,\infty) \to (0,1)$ is a Geraghty function and

$$M(x,y) = \max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{d(Sx,Ty)+d(Sy,Tx)}{2}\right\}.$$

Suppose that, there exists a Picard-Jungck sequence $\{x_n\}$ of (T, S). Also assume that, at least, one of the following conditions holds:

(*i*) *(TX, d) or (SX, d) is complete;*

(*ii*) *(X, d) is complete, S is continuous and T and S are compatible.*

Then T and S have a unique point of coincidence. Moreover if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Theorem 2.14. *Let (X, d) be a complete metric space and $T, S : X \to X$ be two given mappings. Assume that there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(d(Tx,Ty),\beta(M(x,y))M(x,y)) \geq 0$$

for all $x, y \in X$ with $Sx \neq Sy$, where $\beta : [0,\infty) \to (0,1)$ is a Geraghty function and

$$M(x,y) = \max\left\{d(Sx,Sy), \frac{d(Sx,Tx)+d(Sy,Ty)}{2}, \frac{d(Sx,Ty)+d(Sy,Tx)}{2}\right\}.$$

Also assume that, at least, one of the following conditions holds:

(i) (TX, d) *or* (SX, d) *is complete;*

(ii) (X, d) *is complete,* S *is continuous and* T *and* S *are compatible.*

Then T *and* S *have a unique point of coincidence. Moreover if* T *and* S *are weakly compatible, then* T *and* S *have a unique common fixed point in* X*.*

Now we secure another main result of this manuscript.

Theorem 2.15. *Let* (X, d) *be a metric space and* $T, S : X \to X$ *be two given mappings. Assume that there exists* $\zeta \in \mathcal{Z}$ *such that*

$$\zeta(d(Tx, Ty), \beta(E(x, y))E(x, y)) \geq 0 \tag{2.17}$$

for all $x, y \in X$*, where*

$$E(x, y) = d(Sx, Sy) + |d(Sx, Tx) - d(Sy, Ty)|$$

and $\beta : [0, \infty) \to (0, 1)$ *is a Geraghty function. If* $TX \subseteq SX$ *and* TX *or* SX *is a complete subset of* X*, then* T *and* S *have a unique point of coincidence in* X*. Moreover, if* T *and* S *are weakly compatible, then* T *and* S *have a unique common fixed point in* X*.*

Proof: Firstly, we prove that T and S have a unique point of coincidence, if it exists. Suppose that ω_1 and ω_2 are distinct points of coincidence of T and S. This means that there exist two points u_1 and u_2 ($u_1 \neq u_2$) such that $Tu_1 = Su_1 = \omega_1$ and $Tu_2 = Su_2 = \omega_2$. Then we have

$$\begin{aligned} E(u_1, u_2) =& d(Su_1, Su_2) + |d(Su_1, Tu_1) - d(Su_2, Tu_2)| \\ =& d(\omega_1, \omega_2) + |d(\omega_1, \omega_1) - d(\omega_2, \omega_2)| \\ =& d(\omega_1, \omega_2). \end{aligned}$$

Since $d(\omega_1, \omega_2) > 0$, we can conclude that $E(u_1, u_2)$ and $\beta(d(\omega_1, \omega_2))d(\omega_1, \omega_2)$ are positive. Hence, using (2.17) and (ζ_2), we get

$$\begin{aligned} 0 \leq& \zeta(d(Tu_1, Tu_2), \beta(E(u_1, u_2))E(u_1, u_2)) \\ =& \zeta(d(\omega_1, \omega_2), \beta(E(u_1, u_2))E(u_1, u_2)) \\ =& \zeta(d(\omega_1, \omega_2), \beta(d(\omega_1, \omega_2))d(\omega_1, \omega_2)) \\ <& \beta(d(\omega_1, \omega_2))d(\omega_1, \omega_2) - d(\omega_1, \omega_2) \\ <& d(\omega_1, \omega_2) - d(\omega_1, \omega_2) \\ =& 0, \end{aligned}$$

which is a contradiction.

Since $TX \subseteq SX$, there is at least one Picard-Jungck sequence $p_n = Tx_n = Sx_{n+1}$, where $n \in \mathbb{N}_0$ and $x_0 \in X$ is an arbitrary element. Also, as in the proof of Theorem 2.10, without loss of generality, we can suppose that $p_n \neq p_{n+1}$ for all $n \in \mathbb{N}_0$.

Putting $x = x_n,\ y = x_{n+1}$ in (2.17), we obtain that

$$\begin{aligned}0 \leq &\zeta(d(Tx_n, Tx_{n+1}), \beta(E(x_n, x_{n+1}))E(x_n, x_{n+1}))\\ =&\zeta(d(p_n, p_{n+1}), \beta(d(p_{n-1}, p_n) + |d(p_{n-1}, p_n) - d(p_n, p_{n+1})|)\\ &(d(p_{n-1}, p_n) + |d(p_{n-1}, p_n) - d(p_n, p_{n+1})|)).\end{aligned}$$

Let there exists $n \in \mathbb{N}$ such that $d(p_{n-1}, p_n) \leq d(p_n, p_{n+1})$. In this case, we have

$$E(x_n, x_{n+1}) = d(p_{n-1}, p_n) + d(p_n, p_{n+1}) - d(p_{n-1}, p_n) = d(p_n, p_{n+1}).$$

From this, (ζ_2) and the fact that $d(p_n, p_{n+1}) > 0$, we obtain

$$\begin{aligned}0 \leq &\zeta(d(p_n, p_{n+1}), \beta(d(p_n, p_{n+1}))d(p_n, p_{n+1}))\\ <&\beta(d(p_n, p_{n+1}))d(p_n, p_{n+1}) - d(p_n, p_{n+1})\\ <&d(p_n, p_{n+1}) - d(p_n, p_{n+1})\\ =&0,\end{aligned}$$

which is a contradiction.

Hence,

$$d(p_n, p_{n+1}) < d(p_{n-1}, p_n)$$

for all $n \in \mathbb{N}_0$. From this, it follows that there exists

$$\lim_{n\to\infty} d(p_n, p_{n+1}) = p^* \geq 0.$$

Let us suppose, $p^* > 0$. Then, choosing two sequences $\{t_n\}$ and $\{s_n\}$, where $t_n = d(p_n, p_{n+1})$ and $s_n = \beta(2d(p_{n-1}, p_n) - d(p_n, p_{n+1}))(2d(p_{n-1}, p_n) - d(p_n, p_{n+1}))$, with same positive limit, we obtain that

$$\begin{aligned}0 \leq &\overline{\lim_{n\to\infty}} \zeta(d(p_n, p_{n+1}), \beta(2d(p_{n-1}, p_n) - d(p_n, p_{n+1}))(2d(p_{n-1}, p_n) - d(p_n, p_{n+1})))\\ <&\overline{\lim_{n\to\infty}} [\beta(2d(p_{n-1}, p_n) - d(p_n, p_{n+1}))(2d(p_{n-1}, p_n) - d(p_n, p_{n+1})) - d(p_n, p_{n+1})]\\ <&\overline{\lim_{n\to\infty}} [(2d(p_{n-1}, p_n) - d(p_n, p_{n+1})) - d(p_n, p_{n+1})]\\ =&\overline{\lim_{n\to\infty}} [2(d(p_{n-1}, p_n) - d(p_n, p_{n+1}))] \to 2(p^* - p^*) = 0.\end{aligned}$$

But this is a contradiction with (ζ_3). Hence we have, $\lim_{n\to\infty} d(p_n, p_{n+1}) = 0$.

Using Lemma 1.6 and the arguments as Theorem 2.10, we can prove that the sequence $\{p_n\}$ is a Cauchy sequence.

Suppose that (SX, d) is a complete metric space. Therefore,

$$\begin{aligned}\lim_{n\to\infty} p_n &= Sp\\ \lim_{n\to\infty} d(p_n, Sp) &= 0 \qquad (2.18)\end{aligned}$$

for some $p \in X$. We prove that $Tp = Sp$. On the contrary, let $Tp \neq Sp$. In this case, put $x = x_n,\ y = p$ in (2.17), we get

$$0 \leq \zeta(d(Tx_n, Tp), \beta(E(x_n, p))E(x_n, p)),$$

where $E(x_n, p) = d(Sx_n, Sp) + |d(Sx_n, Tx_n) - d(Sp, Tp)| \to d(Sp, Tp)$ as $n \to \infty$.

Now since $Sp \neq Tp$, we have

$$d(Tp, Sp) = \delta > 0.$$

Again from (2.18), we can find $k \in \mathbb{N}$ such that

$$\begin{aligned} d(p_n, Sp) =& d(Tx_n, Sp) \\ <& \delta \\ =& d(Tp, Sp) \end{aligned}$$

for all $n \geq k$. This implies $Tx_n \neq Tp$ for all $n \geq k$ and hence

$$d(Tx_n, Tp) > 0.$$

Now considering two sequences $\{t_n\}$ and $\{s_n\}$ with same positive limit $d(Tp, Sp) > 0$, where $t_n = d(Tx_n, Tp),\ s_n = \beta(E(x_n, p))E(x_n, p)$, we obtain

$$\begin{aligned} 0 \leq& \overline{\lim_{n\to\infty}} \zeta(d(Tx_n, Tp), \beta(E(x_n, p))E(x_n, p)) \\ <& \overline{\lim_{n\to\infty}} [\beta(E(x_n, p))E(x_n, p) - d(Tx_n, Tp)] \\ <& \overline{\lim_{n\to\infty}} [E(x_n, p) - d(Tx_n, Tp)] \to 0, \end{aligned}$$

which contradicts with (ζ_3). This implies that $w = Sp = Tp$ is a (unique) point of coincidence of T and S.

Using similar arguments, we can easily prove that $u = Sp = Tp$ is a (unique) point of coincidence of T and S, when TX is a complete subset of X.

The rest of the result follows from Theorem 1.5. □

Finally, we have the following open question.

Problem: Let (X, d) be a complete metric space and $T, S : X \to X$ be two given mappings. Assume that there exists $\zeta \in \mathcal{Z}$ such that

$$\zeta\left(d\left(Tx, Ty\right), d\left(Sx, Ty\right) + d\left(Sy, Tx\right)\right) \geq 0$$

for all $x, y \in X$. If $TX \subseteq SX$ and TX or SX is a complete subset of X, then T and S have a unique point of coincidence in X. Moreover if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Acknowledgments

The authors are thankful to the referees for their precious remarks which have made the presentation of the paper better. The second named author would like to convey his cordial thanks to DST-INSPIRE, New Delhi, India for their financial support under INSPIRE fellowship scheme.

References

1. M. Abbas and G. Jungck. Common fixed point results for noncommuting mappings without continuity in cone metric spaces. *J. Math. Anal. Appl.*, 341(1):416–420, 2008.
2. M. Abbas, I. Zulfaqar, and S. Radenović. Common fixed point of (ψ, β)-generalized contractive mappings in partially ordered metric spaces. *Chin. J. Math. (N.Y.)*, 2014, 2014. Article ID 379049.
3. A.H. Ansari, M. Berzig, and S. Chandok. Some fixed point theorems for (CAB)-contractive mappings and related results. *Math. Morav.*, 19(2):97–112, 2015.
4. S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fund. Math.*, 3:133–181, 1922.
5. A. Chanda, B. Damjanović, and L.K. Dey. Fixed point results on θ-metric spaces via simulation functions. *Filomat*, 31(11):3365–3375, 2017.
6. A. Chanda, L.K. Dey, and S. Radenović. Simulation functions: a survey of recent results. Preprint.
7. A. Chanda, S. Mondal, L.K. Dey, and S. Karmakar. C^*-algebra-valued contractive mappings with its application to integral equations. *Indian J. Math.*, 59(1):107–124, 2017.
8. S. Chandok. Some fixed point theorems for (α, β)-admissible Geraghty type contractive mappings and related results. *Math. Sci.*, 9(3):127–135, 2015.
9. A. Fulga and A.M. Proca. Fixed points for φ_E-Geraghty contractions. *J. Nonlinear Sci. Appl.*, 10(9):5125–5131, 2017.
10. M. Geraghty. On contractive mappings. *Proc. Amer. Math. Soc.*, 40(2):604–608, 1973.
11. G. Jungck. Compatible mappings and common fixed points. *Int. J. Math. Math. Sci.*, 9(4):771–779, 1986.
12. F. Khojasteh, S. Shukla, and S. Radenović. A new approach to the study of fixed point theory for simulation functions. *Filomat*, 29(6):1189–1194, 2015.
13. S. Komal, P. Kumam, and D. Gopal. Best proximity point for $\mathcal{Z}$-contraction and Suzuki type $\mathcal{Z}$-contraction mappings with an application to fractional calculus. *Appl. Gen. Topol.*, 17(2):185–198, 2016.
14. P. Kumam, D. Gopal, and L. Budhiya. A new fixed point theorem under Suzuki type $\mathcal{Z}$-contraction mappings. *J. Math. Anal.*, 8(1):113–119, 2017.
15. C. Mongkolkeha, Y.J. Cho, and P. Kumam. Fixed point theorems for simulation functions in b-metric spaces via the wt-distance. *Appl. Gen. Topol.*, 18(1):91–105, 2017.
16. A. Nastasi and P. Vetro. Fixed point results on metric and partial metric spaces via simulation functions. *J. Nonlinear Sci. Appl.*, 8(6):1059–1069, 2015.
17. A. Nastasi, P. Vetro, and S. Radenović. Some fixed point results via $\mathcal{R}$-functions. *Fixed Point Theory Appl.*, 2016:81, 2016.
18. M. Olgun, Ö. Biçer, and T. Alyildiz. A new aspect to Picard operators with simulation functions. *Turkish J. Math.*, 40(4):832–837, 2016.
19. S. Radenović and S. Chandok. Simulation type functions and coincidence points. *Filomat*, 32(1):141-147, 2018.
20. S. Radenović, F. Vetro, and J. Vujaković. An alternative and easy approach to fixed point results via simulation functions. *Demonstr. Math.*, 50(1):223–230, 2017.
21. A.F. Roldán-López-de-Hierro, E. Karapınar, C. Roldán-López-de-Hierro, and J. Martínez-Moreno. Coincidence point theorems on metric spaces via simulation functions. *J. Comput. Appl. Math.*, 275:345–355, 2015.
22. A.F. Roldán-López-de-Hierro and B. Samet. φ-admissibility results via extended simulation functions. *J. Fixed Point Theory Appl.*, 19(3):1997–2015, 2017.

Sumit Chandok,
School of Mathematics,
Thapar University, Patiala,
India.
E-mail address: sumit.chandok@thapar.edu

and

Ankush Chanda,
Department of Mathematics,
National Institute of Technology Durgapur,
India.
E-mail address: ankushchanda8@gmail.com

and

Lakshmi Kanta Dey,
Department of Mathematics,
National Institute of Technology Durgapur,
India.
E-mail address: lakshmikdey@yahoo.co.in

and

Mirjana Pavlović,
Faculty of Sciences,
Department of Mathematics,
University of Kragujevac, Kragujevac,
Serbia.
E-mail address: mpavlovic@kg.ac.rs

and

Stojan Radenović,
Faculty of Mechanical Engineering,
University of Belgrade,
Serbia.
E-mail address: radens@beotel.rs