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Submanifolds of a Conformal Sasakian Manifold

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ABSTRACT: In the present paper, some results on geometry of conformal Sasakian manifolds and their associated submanifolds are provided. Besides these an example of a three-dimensional conformal Sasakian manifold is constructed to illustrate the argument for non-Sasakian manifolds.

Key Words: Conformal Sasakian manifold, Sasakian manifold.

Contents

1. Introduction

A $(2n+1)$-dimensional Riemannian manifold (M, g) said to be a Sasakian manifold if it admits an endomorphism ϕ of its tangent bundle TM, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1,$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$
$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

for all vector fields X, Y on M, where ∇ denotes the Riemannian connection [1]. The close relationship between Kaehler manifolds and Sasakian manifolds naturally leads to the question which objects, methods and theorems can be transfered from one to the other. The locally conformal Kaehler manifold is one of the sixteen classes of almost Hermitian manifolds [7]. Libermann did the first study on locally conformal Kaehler manifolds [3]. Vaisman, put down some geometrical conditions for locally conformal Kaehler manifolds [4], and Tricerri mentioned different examples about the locally conformal Kaehler manifolds [5].

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We introduced conformal Sasakian manifolds by using an idea of conformal Kaehler manifolds in [2].
The paper is organized as follows. In Section 2, we recall som preliminary definitions about conformal Sasakian manifolds. Furthermore, we give some basic results on conformal Sasakian manifolds and their submanifolds. In Section 3, we obtain a necessary and sufficient condition for the invariant submanifolds of a conformal Sasakian manifold to be minimal. In Section 4, we study anti-invariant submanifolds of a conformal Sasakian manifold and obtain the conditions under which these type submanifolds have a flat normal connection. Section 5 considers CR-submanifolds of a conformal Sasakian manifold with distributions D and $D^{\perp}$. we find the conditions under which $D^{\perp}$ is integrable or totally geodesic. In the final section, we give an example of a three-dimensional conformal Sasakian manifold that is not Sasakian.

2. Riemannian geometry of conformal Sasakian manifolds

A differentiable manifold M^{2n+1} is said an almost contact manifold if it admits a vector field ξ, a one-form η and a (1,1)-tensor field φ with the following properties

$$\varphi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta o \varphi = 0. \tag{2.1}$$

Furthermore, if M be a Riemannian manifold with the Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all X, Y on M, then (φ, ξ, η, g) is called an almost contact metric structure on M and M is said an almost contact metric manifold.
A Sasakian manifold is a normal contact metric manifold, that is, an almost contact metric manifold such that $d\varphi = 0$ and $[\varphi, \varphi](X, Y) = -2d\eta(X, Y)\xi$ for all X, Y on M, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ.
An almost contact metric manifold $(M^{2n+1}, \varphi, \xi, \eta, g)$ is said a Sasakian manifold if and only if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.2}$$

for all vector fields X, Y on M, where ∇ denotes the Levi-Civita connection with respect to g [1].
Let $(\acute{M}^m, \acute{g})$ be a Riemannian (sub)manifold into Riemannian manifold (M^n, g), $m < n$, with isometric immersion $\iota : (\acute{M}, \acute{g}) \longrightarrow (M, g)$. Then the Gauss and Weingarten formulas are given by

$$\nabla_X Y = \acute{\nabla}_X Y + h(X, Y),$$
$$\nabla_X N = -A_N X + \nabla^{\perp}_X N,$$

for all X, Y tangent to $\acute{M}$ and normal vector field N on $\acute{M}$, where $\acute{\nabla}$ and ∇ are the Levi-Civita connections of $\acute{M}$ and M, respectively, also h and A_N are the second fundamental form and the shape operator corresponding to N, respectively and $\nabla^{\perp}$ is the normal connection on $T^{\perp}\acute{M}$. Let $\acute{R}$ and R denote the curvature tensors on $\acute{M}$ and M, respectively, then the Gauss and Codazzi equations are given by

$$
\begin{aligned}
g(R(X,Y)Z,W) &= \acute{g}(\acute{R}(X,Y)Z,W) + h(Y,Z)h(X,W) \\
&\quad -h(X,Z)h(Y,W),
\end{aligned} \tag{2.3}
$$

$$
\begin{aligned}
\acute{g}(\acute{R}(X,Y)Z,N_a) &= g((\acute{\nabla}_X A_a)Y - (\acute{\nabla}_Y A_a)X, Z) \\
&\quad + \sum_{b=1}^{p}\{S_{ba}(X)g'(A_bY,Z) - S_{ba}(Y)g'(A_bX,Z)\},
\end{aligned} \tag{2.4}
$$

for all $X,Y,Z,W \in T\acute{M}$ and $N_a \in T^{\perp}\acute{M}$, where A_a is the shape operator with respect to N_a, $a : 1, \cdots, p = n - m$ and the s_{ab} are the coefficients of the third fundamental form of $\acute{M}$ in M. Also, let $R^{\perp}$ be the normal curvature tensor of $\acute{M}$ then we will have the Ricci equation by following

$$
g(R(X,Y)N_1,N_2) = g(R^{\perp}(X,Y)N_1,N_2) - g([A_1,A_2]X,Y), \tag{2.5}
$$

where N_1, N_2 are unit normal vector fields on $\acute{M}$ and A_1, A_2 are the shape operators with respect to N_1, N_2.

A smooth manifold M^{2n+1} with an almost contact metric structure (φ,η,ξ,g) is called a conformal Sasakian manifold if there is a positive smooth function $f : M \longrightarrow \mathbb{R}$ such that

$$
\widetilde{g} = exp(f)g, \quad \widetilde{\varphi} = \varphi, \quad \widetilde{\eta} = exp(f)^{\frac{1}{2}}\eta, \quad \widetilde{\xi} = exp(-f)^{\frac{1}{2}}\xi \tag{2.6}
$$

is a Sasakian structure on M [6].

Let $\widetilde{\nabla}$ and ∇ denote connections of M related to metrics $\widetilde{g}$ and g, respectively. Using Koszul formula, we derive the following relation between the connections $\widetilde{\nabla}$ and ∇

$$
\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)\omega^{\sharp}\}, \tag{2.7}
$$

for all vector fields X, Y on M, so that $\omega(X) = X(f)$ and $\omega^{\sharp}$ is vector field of metrically equavalente to one form of ω, that is, $g(\omega^{\sharp}, X) = \omega(X)$. Vector field $\omega^{\sharp} = gradf$ is called the Lee vector field of conformal Sasakian manifold M.

Then with a straightforward computation we will have

$$
\begin{aligned}
\exp(-f)\widetilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) \\
&+ \frac{1}{2}\{B(X,Z)g(Y,W) - B(Y,Z)g(X,W) \\
&+ B(Y,W)g(X,Z) - B(X,W)g(Y,Z)\} \\
&+ \frac{1}{4}\|\omega^{\sharp}\|^2\{g(X,Z)g(Y,W) - g(Y,Z)g(X,W)\},
\end{aligned} \tag{2.8}
$$

for all vector fields X, Y, Z, W on M, where $B := \nabla\omega - \frac{1}{2}\omega \otimes \omega$ and $R, \widetilde{R}$ are the curvature tensors of M related to connections of ∇ and $\widetilde{\nabla}$, respectively. Also, from

(2.2) and (2.6) we have

$$(\nabla_X\varphi)Y = (\exp(f))^{\frac{1}{2}}\{g(X,Y)\xi - \eta(Y)X\} \\ -\frac{1}{2}\{\omega(\varphi Y)X - \omega(Y)\varphi X + g(X,Y)\varphi\omega^\sharp - g(X,\varphi Y)\omega^\sharp\}, \tag{2.9}$$

$$\nabla_X\xi = (\exp(-f))^{\frac{1}{2}}\varphi X + \frac{1}{2}\{\eta(X)\omega^\sharp - \omega(\xi)X\}, \tag{2.10}$$

for all vector fields X, Y on M. Let $(\acute{M}^m, \acute{g})$ be a Riemannian submanifold in conformal Sasakian manifold M^{2n+1} with isometric immersion $\iota : (\acute{M}^m, \acute{g}) \longrightarrow (M, g)$. Suppose $\acute{\nabla}$ and $\acute{R}$ are the Levi-Civita connection and curvature tensor on $\acute{M}^m$, respectively. We set

$$PX = tan(\varphi X) \quad , \quad FX = nor(\varphi X),$$
$$tN = tan(\varphi N) \quad , \quad fN = nor(\varphi N),$$

for each $X \in TM'$ and $N \in TM'^{\perp}$. Then from (2.9) we get

$$\nabla_X(\varphi Y) = \varphi\nabla_X Y + (\exp(f))^{\frac{1}{2}}\{g(X,Y)\xi - \eta(Y)X\} \\ -\frac{1}{2}\{\omega(\varphi Y)X - \omega(Y)\varphi X + g(X,Y)\varphi\omega^\sharp - g(X,\varphi Y)\omega^\sharp\}, \tag{2.11}$$

for all vector field X, Y on $\acute{M}$. Separating the tangential and normal parts from the above equation we will have

$$(\acute{\nabla}_X P)Y = (\exp(f))^{\frac{1}{2}}\{g(X,Y)\xi^\top - \eta(Y)X\} + A_{FY}X + th(X,Y) \\ -\frac{1}{2}\{\omega(\varphi Y)X - \omega(Y)PX + g(X,Y)(\varphi\omega^\sharp)^\top \\ -g(X,\varphi Y)\omega^{\sharp^\top}\}, \tag{2.12}$$

$$(\acute{\nabla}_X F)Y = fh(X,Y) - h(X,PY) \\ +\frac{1}{2}\{\omega(Y)FX - g(X,Y)(\varphi\omega^\sharp)^\perp + g(X,\varphi Y)\omega^{\sharp^\perp}\}, \tag{2.13}$$

$$(\acute{\nabla}_X t)N = A_{fN}X - PA_NX - \frac{1}{2}\{-\omega(N)PX + \omega(\varphi N)X \\ -g(X,\varphi N)\omega^{\sharp^\top}\}, \tag{2.14}$$

$$(\acute{\nabla}_X f)N = -h(X,tN) - FA_NX + \frac{1}{2}\{\omega(N)FX + g(X,\varphi N)\omega^{\sharp^\perp}\}, \tag{2.15}$$

for all $X, Y \in TM'$ and $N \in TM'^{\perp}$.
We need the equations of Gauss, Codazzi and Ricci between manifolds $\acute{M}^m$ and M^{2n+1} conformal Sasakian manifold M, thus from (2.3), (2.4), (2.5) and (2.8) we

get

$$\begin{aligned}
exp(-f)\widetilde{R}(X,Y,Z,W) &= \acute{R}(X,Y,Z,W) - \frac{1}{2}\{B(X,Z)\acute{g}(Y,W) \\
&- B(Y,Z)\acute{g}(X,W) + B(Y,W)\acute{g}(X,Z) \\
&- B(X,W)\acute{g}(Y,Z)\} \\
&- \frac{1}{4}\|\omega^\sharp\|^2\{\acute{g}(X,Z)\acute{g}(Y,W) \\
&- \acute{g}(Y,Z)\acute{g}(X,W)\} \\
&+ \sum_{a=1}^{p}\{\acute{g}(A_aY,Z)\acute{g}(A_aX,W) \\
&- \acute{g}(A_aX,Z)\acute{g}(A_aY,W)\}, \qquad (2.16) \\
exp(-f)\widetilde{R}(X,Y,Z,N_a) &= \acute{g}((\acute{\nabla}_XA_a)Y - (\acute{\nabla}_YA_a)X,Z) \\
&+ \sum_{b=1}^{p}\{S_{ba}(X)\acute{g}(A_bY,Z) - S_{ba}(Y)\acute{g}(A_bX,Z)\} \\
&+ \frac{1}{2}\{\acute{g}(X,Z)g(\nabla_Y\omega^\sharp,N_a) \\
&- \acute{g}(Y,Z)g(\nabla_X\omega^\sharp,N_a)\}, \qquad (2.17) \\
exp(-f)\widetilde{R}(X,Y,N_a,N_b) &= \acute{g}([A_2,A_1]X,Y) + g(R^\perp(X,Y)N_1,N_2),
\end{aligned}$$

for all $X,Y,Z \in T\acute{M}$ and $N_a \in T^\perp\acute{M}$, where A_a is the shape operator with respect to N_a, $a : 1,\cdots,p = 2n-m+1$.

3. Invariant submanifolds

A submanifold $\acute{M}$ of a conformal Sasakian manifold M is called an invariant submanifold of M if $\varphi T\acute{M} \subset T\acute{M}$. Hence, $\varphi N \in T^\perp\acute{M}$ for each $N \in T^\perp\acute{M}$, that is, $tN \equiv 0$.

Theorem 3.1. *Let $\acute{M}^m$ be an invariant submanifold of a conformal Sasakian manifold M^{2n+1} tangent to ξ. Then $\acute{M}$ is minimal if $\omega^\sharp$ is tangent to $\acute{M}$.*

Proof: *By relation (2.9) and the Gauss formula we have*

$$\begin{aligned}
h(X,\varphi Y) &= \varphi h(X,Y) - (\acute{\nabla}_X\varphi)Y + exp(f)^{\frac{1}{2}}\{g(X,Y)\xi - \eta(Y)X\} \\
&- \frac{1}{2}\{\omega(\varphi Y)X - \omega(Y)\varphi X - g(X,\varphi Y)\omega^\sharp + g(X,Y)\varphi\omega^\sharp\}, \quad (3.1)
\end{aligned}$$

for all $X,Y \in T\acute{M}$. Since $\acute{M}$ is invariant, comparing tangential and normal parts we get

$$h(X,\varphi Y) = \varphi h(X,Y) - \frac{1}{2}\{g(X,Y)(\varphi\omega^\sharp)^\perp - g(X,\varphi Y){\omega^\sharp}^\perp\}. \qquad (3.2)$$

Since $\xi \in T\acute{M}$, taking $X = \varphi X$ in (3.2) we obtain

$$h(\varphi X,\varphi Y) + h(X,Y) = \{\acute{g}(X,Y) - \frac{1}{2}\eta(X)\eta(Y)\}{\omega^\sharp}^\perp,$$

for all X, Y *on* $\acute{M}$. *Again, since* $\xi \in T\acute{M}$, *we put* $X = Y = \xi$ *in (3.2) then we find*

$$h(\xi, \xi) = \frac{1}{2}\omega^{\sharp\perp}.$$

Let $\{E_\alpha, \varphi E_\alpha, \xi | \alpha = 1, ..., n = \frac{m-1}{2}\}$ *be an orthonormal frame on* $\acute{M}$ *and suppose* H *is the mean curvature vector. Then from the above relation we have*

$$\begin{aligned} H &= \frac{1}{m}\sum_{\alpha=1}^{n}\{h(\xi, \xi) + h(E_\alpha, E_\alpha) + h(\varphi E_\alpha, \varphi E_\alpha)\} \\ &= \frac{1}{m}\{\frac{1}{2} + \sum_{\alpha=1}^{n} g(E_\alpha, E_\alpha)\}\omega^{\sharp\perp}. \end{aligned}$$

Thus the theorem is proved. □

4. Anti-invariant submanifolds

A submanifold $\acute{M}^m$ of a conformal Sasakian manifold M is called an anti-invariant of M if $\varphi T\acute{M} \subset T^\perp \acute{M}$. Then $\varphi X \in T^\perp \acute{M}$, for each $X \in T\acute{M}$, that is, $P \equiv 0$.

Lemma 4.1. *Let* $\acute{M}^m$ *be an* m*-dimensional anti-invariant submanifold of a conformal Sasakian manifold* M^{2n+1}. *Then*

$$\begin{aligned} A_{\varphi Y}X = -\varphi h(X, Y) - (\exp(f))^{\frac{1}{2}}\{g(X, Y)\xi^\top - \eta(Y)X\} \\ +\frac{1}{2}\{\omega(\varphi Y)X + g(X, Y)\varphi\omega^{\sharp\perp}\}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\acute{g}([A_{\varphi Z}, A_{\varphi W}]X, Y) = g(h(X, W), h(Y, Z)) - g(h(Y, W), h(X, Z)) \\ &-\frac{1}{2}\{\acute{g}(Y, Z)\omega(h(X, W)) - \acute{g}(Y, W)\omega(h(X, Z)) + \acute{g}(X, W)\omega(h(Y, Z)) \\ &-\acute{g}(X, Z)\omega(h(Y, W)) + \omega(\varphi Z)\Phi(Y, h(X, W)) - \omega(\varphi W)\Phi(Y, h(X, Z)) \\ &+\omega(\varphi W)\Phi(X, h(Y, W)) - \omega(\varphi Z)\Phi(X, h(Y, W))\} \\ &-\frac{1}{4}\{\omega(\varphi W)\omega(\varphi X)\acute{g}(Y, Z) - \omega(\varphi Z)\omega(\varphi X)\acute{g}(Y, W) \\ &+\omega(\varphi Z)\omega(\varphi Y)\acute{g}(X, W) - \omega(\varphi W)\omega(\varphi Y)\acute{g}(X, Z) \end{aligned}$$

$$
\begin{aligned}
&+\|\omega^\sharp\|^2\{\acute{g}(X,Z)\acute{g}(Y,W)-\acute{g}(X,W)\acute{g}(Y,Z)\}\}\\
&-\frac{1}{2}(\exp(f))^{\frac{1}{2}}\{2\eta(Z)\Phi(Y,h(X,W))-2\eta(W)\Phi(Y,h(X,Z))\\
&+2\eta(W)\Phi(X,h(Y,Z))-2\eta(Z)\Phi(X,h(Y,W))\\
&+\omega(\varphi Z)\eta(Y)\acute{g}(X,W)-\omega(\varphi W)\eta(Y)g'(X,Z)\\
&+\omega(\varphi X)\eta(W)\acute{g}(Y,Z)-\omega(\varphi X)\eta(Z)\acute{g}(Y,W)\\
&+\omega(\varphi W)\eta(X)\acute{g}(Y,Z)-\omega(\varphi Z)\eta(X)\acute{g}(Y,W)+\omega(\varphi Y)\eta(Z)\acute{g}(X,W)\\
&-\omega(\varphi Y)\eta(W)\acute{g}(X,Z)\}\\
&+\exp(f)\{\acute{g}(Y,Z)\acute{g}(X,W)-\acute{g}(X,Z)\acute{g}(Y,W)+\acute{g}(X,Z)\eta(Y)\eta(W)\\
&-\acute{g}(X,W)\eta(Y)\eta(Z)+\acute{g}(Y,W)\eta(X)\eta(Z)-\acute{g}(Y,Z)\eta(X)\eta(W)\}, \qquad (4.2)
\end{aligned}
$$

for all $X,Y,Z,W \in T\acute{M}$, where $\Phi(X,Y)=g(X,\varphi Y)$.

Proof: *Since $P \equiv 0$ then (4.1) follows from (2.12), easily. Also, substituting (4.1) in $\acute{g}([A_{\varphi Z},A_{\varphi W}]X,Y)=\acute{g}(A_{\varphi W}X,A_{\varphi Z}Y)-\acute{g}(A_{\varphi Z}X,A_{\varphi W}Y)$, we get (4.2).* □

Proposition 4.2. *Let $\acute{M}^m$ be an anti-invariant submanifold of a conformal Sasakian manifold M^{2n+1} tangent to ξ. Then $\acute{M}$ has a flat normal connection if and only if*

$$
\begin{aligned}
\acute{R}(X,Y)Z &= \eta(R(X,Y)Z)\xi\\
&+ \frac{1}{2}\{\acute{B}(Y,Z)X-\acute{B}(X,Z)Y+\acute{g}(Y,Z)\acute{B}(X,.)^\sharp-\acute{g}(X,Z)\acute{B}(Y,.)^\sharp\\
&+ B(X,Z)\eta(Y)\xi-B(Y,Z)\eta(X)\xi\\
&+ B(Y,\xi)\acute{g}(X,Z)\xi-B(X,\xi)\acute{g}(Y,Z)\xi\}\\
&+ \frac{1}{4}\|\acute{\omega}^\sharp\|^2\{\acute{g}(Y,Z)X-\acute{g}(X,Z)Y\}\\
&+ \frac{1}{4}\|\omega^\sharp\|^2\{\acute{g}(X,Z)\eta(Y)\xi-\acute{g}(Y,Z)\eta(X)\xi\}\\
&+ \{\acute{g}(Y,Z)X-\acute{g}(X,Z)Y+\acute{g}(X,Z)\eta(Y)\xi-\acute{g}(Y,Z)\eta(X)\xi\}\\
&- \exp(f)\{\acute{g}(Y,Z)X-\acute{g}(X,Z)Y\}, \qquad (4.3)
\end{aligned}
$$

for all $X,Y,Z \in T\acute{M}$, where $\acute{\omega}^\sharp=\omega^{\sharp^\top}$ and $\acute{B}=B+\omega o h$.

Proof: *Since $(\tilde{\nabla}_X\varphi)Y=\widetilde{g}(X,Y)\xi-\eta(Y)X$ from that [1] we have*

$$
\begin{aligned}
\widetilde{R}(X,Y)\varphi Z = \varphi\widetilde{R}(X,Y)Z-\widetilde{g}(Y,Z)\varphi X+\widetilde{g}(X,Z)\varphi Y&\\
-\widetilde{g}(\varphi Y,Z)X+\widetilde{g}(\varphi X,Z)Y,& \qquad (4.4)
\end{aligned}
$$

for all $X, Y, Z \in T\acute{M}$. Replacing (2.8) in (4.4) we can write

$$\begin{aligned} R(X,Y)\varphi Z &= \varphi R(X,Y)Z \\ &- \frac{1}{2}\{B(X,\varphi Z)Y - B(Y,\varphi Z)X + B(Y,Z)\varphi X \\ &- B(X,Z)\varphi Y + B(Y,.)^\sharp g(X,\varphi Z) \\ &- B(X,.)^\sharp g(Y,\varphi Z) - \varphi B(Y,.)^\sharp g(X,Z) + \varphi B(X,.)^\sharp g(Y,Z)\} \\ &- (\frac{1}{4}\|\omega^\sharp\|^2 + 1)\{g(Y,Z)\varphi X - g(X,Z)\varphi Y + g(X,\varphi Z)X \\ &- g(Y,\varphi Z)X\}, \end{aligned} \tag{4.5}$$

for all $X, Y, Z, W \in TM'$, where $B(X,Y) = g(B(X,.)^\sharp, Y)$. Taking the inner product from (4.5) with φW and using the Ricci and Gauss equations, we obtain

$$\begin{aligned} & g(R^\perp(X,Y)\varphi Z, \varphi W) - \acute{g}([A_{\varphi Z}, A_{\varphi W}]X, Y) \\ =\ & \acute{g}(\acute{R}(X,Y)Z,W) - g(h(X,W),h(Y,Z)) \\ +\ & g(h(Y,W),h(X,Z)) - \eta(R(X,Y)Z)\eta(W) \\ -\ & \frac{1}{2}\{B(Y,Z)\acute{g}(X,W) - B(X,Z)\acute{g}(Y,W) \\ +\ & B(X,W)\acute{g}(Y,Z) - B(Y,W)\acute{g}(X,Z) + B(X,Z)\eta(Y)\eta(W) \\ -\ & B(Y,Z)\eta(X)\eta(W) + B(Y,\xi)g(X,Z)\eta(W) - B(X,\xi)g(Y,Z)\eta(W)\} \\ -\ & (\frac{1}{4}\|\omega^\sharp\|^2 + 1)\{\acute{g}(Y,Z)\acute{g}(X,W) - \acute{g}(X,Z)\acute{g}(Y,W) \\ +\ & \acute{g}(X,Z)\eta(Y)\eta(W) - \acute{g}(Y,Z)\eta(X)\eta(W)\}, \end{aligned} \tag{4.6}$$

for all $X, Y, Z, W \in T\acute{M}$. From (4.1) we get

$$\begin{aligned} \Phi(Y, h(X,Z)) = \Phi(Z, h(X,Y)) &- (\exp(f))^{\frac{1}{2}}\{\acute{g}(X,Z)\eta(Y) - \acute{g}(X,Y)\eta(Z)\} \\ &+ \frac{1}{2}\{\omega(\varphi Z)\acute{g}(X,Y) - \omega(\varphi Y)\acute{g}(X,Z)\}, \end{aligned} \tag{4.7}$$

for all $X, Y, Z, W \in T\acute{M}$. Putting (4.2) into (4.6) and using (4.7), we find

$$\begin{aligned} -\varphi R^\perp(X,Y)\varphi Z = & \acute{R}(X,Y)Z - \eta(R(X,Y)Z)\xi \\ -\ & \frac{1}{2}\{\acute{B}(Y,Z)X - \acute{B}(X,Z)Y + \acute{g}(Y,Z)\acute{B}(X,.)^\sharp \\ -\ & \acute{g}(X,Z)\acute{B}(Y,.)^\sharp + B(X,Z)\eta(Y)\xi - B(Y,Z)\eta(X)\xi \\ +\ & B(Y,\xi)\acute{g}(X,Z)\xi - B(X,\xi)\acute{g}(Y,Z)\xi\} \\ -\ & \frac{1}{4}\|\acute{\omega}^\sharp\|^2\{\acute{g}(Y,Z)X - \acute{g}(X,Z)Y\} \\ -\ & \frac{1}{4}\|\omega^\sharp\|^2\{\acute{g}(X,Z)\eta(Y)\xi - \acute{g}(Y,Z)\eta(X)\xi\} \\ -\ & \{\acute{g}(Y,Z)X - \acute{g}(X,Z)Y + \acute{g}(X,Z)\eta(Y)\xi - \acute{g}(Y,Z)\eta(X)\xi\} \\ +\ & \exp(f)\{\acute{g}(Y,Z)X - \acute{g}(X,Z)Y\}, \end{aligned} \tag{4.8}$$

for all $X, Y, Z, W \in T\acute{M}$. Thus $R^{\perp} = 0$ if and only if (4.3) holds. □

Let $\acute{M}^m$ be an anti-invariant submanifold of a conformal Sasakian manifold M^{2n+1}. The normal curvature tensor $R^{\perp}$ of $\acute{M}$ is called reccurent if

$$R^{\perp}(X, Y)N = \theta(X, Y)N, \tag{4.9}$$

for all $X, Y \in T\acute{M}$and $N \in T\acute{M}^{\perp}$ holds on $\acute{M}$, where θ is a 2-form on $\acute{M}$.

Theorem 4.3. *Let $\acute{M}^m$ be an anti-invariant submanifold of a conformal Sasakian manifold M^{2n+1} normal to ξ with reccurent nomal curvature tensor. Then $\acute{M}$ has a flat normal connection.*

Proof: *Since $R^{\perp}$ is reccurent, by (4.9) and using (4.3) in Proposition* 4.2 *we obtain*

$$\begin{aligned}\acute{R}(X,Y)Z = {} & \theta(X,Y)Z - \theta(X,Y)\eta(Z)\xi + \eta(R(X,Y)Z)\xi \\ & + \frac{1}{2}\{\acute{B}(Y,Z)X - \acute{B}(X,Z)Y + \acute{g}(Y,Z)\acute{B}(X,.)^{\sharp} \\ & - \acute{g}(X,Z)\acute{B}(Y,.)^{\sharp} + B(X,Z)\eta(Y)\xi - B(Y,Z)\eta(X)\xi \\ & + B(Y,\xi)\acute{g}(X,Z)\xi - B(X,\xi)\acute{g}(Y,Z)\xi\} \\ & + \frac{1}{4}\|\acute{\omega}^{\sharp}\|^2\{\acute{g}(Y,Z)X - \acute{g}(X,Z)Y\} \\ & + \frac{1}{4}\|\omega^{\sharp}\|^2\{\acute{g}(X,Z)\eta(Y)\xi - \acute{g}(Y,Z)\eta(X)\xi\} \\ & + \{\acute{g}(Y,Z)X - \acute{g}(X,Z)Y + \acute{g}(X,Z)\eta(Y)\xi - \acute{g}(Y,Z)\eta(X)\xi\} \\ & - \exp(f)\{\acute{g}(Y,Z)X - \acute{g}(X,Z)Y\},\end{aligned} \tag{4.10}$$

for all $X, Y, Z \in T\acute{M}$. Since $\xi \in T^{\perp}\acute{M}$, taking the inner product from the above equation with each vector field $W \in T\acute{M}$ and Contracting it over Z and W we get

$$m\theta(X, Y) = 0, \tag{4.11}$$

for all X, Y on $\acute{M}$. Then (4.9) results $R^{\perp} = 0$. Thus, $\acute{M}$ has a flat normal connection. □

5. Distribution on submanifolds

Let M^{2n+1} be a conformal Sasakian manifold. Then $\acute{M}^m$ is said a CR-submanifold in M if there exist two orthogonal complementray distributions D and $D^{\perp}$ of $T\acute{M}$ such that $\xi \in T\acute{M}$ and
(1) D is invariant by φ, i.e. $\varphi(D_p) \subset D_p, \forall p \in \acute{M}$.
(2) $D^{\perp}$ is anti-invariant by φ, i.e. $\varphi(D_p^{\perp}) \subset T_p^{\perp}\acute{M} \forall p \in \acute{M}$.

Theorem 5.1. *Let $(\acute{M}^m, D)$ be a CR-submanifold of a conformal Sasakian manifold M^{2n+1}. Then the anti-invariant distribution $D^{\perp}$ of M' is integrable.*

Proof: *Since* $\Phi(X,Y) = g(X,\varphi Y)$ *for all* $X,Y \in T\acute{M}$*, we get* $\Phi(X,Y) = 0$ *and* $\Phi(Z,W) = 0$ *for all* $X \in D$ *and* $Y,Z \in D^{\perp}$*. Since* $(M^{2n+1},\varphi,\widetilde{\xi},\widetilde{\eta},\widetilde{g})$ *is a Sasakian manifold, we have* $d\widetilde{\Phi} = 0$*, where* $\widetilde{\Phi}(X,Y) = \widetilde{g}(X,\varphi Y)$*. Thus, we find*

$$\begin{aligned} 0 &= d\widetilde{\Phi} \\ &= d(\exp(f)) \wedge \Phi + \exp(f) d\Phi \\ &= \exp(f)(\omega \wedge \Phi + d\Phi). \end{aligned}$$

Using $(\Phi \wedge \omega)(X,Y,Z) = 0$ *for all* $X \in D$ *and* $Y,Z \in D^{\perp}$ *in the above equation we can write*

$$\begin{aligned} 0 &= 3(d\Phi)(X,Y,Z) \\ &= X(\Phi(Y,Z)) + Z(\Phi(Z,X)) + W(\Phi(X,Y)) \\ &\quad -\Phi([X,Y],Z) - \Phi([Z,X],Y) - \Phi([Y,Z],X) \\ &= -g([Y,Z],\varphi X), \end{aligned}$$

hence, $[Y,Z] \in D^{\perp}$ *for all* $Y,Z \in D^{\perp}$*.* □

Let $(\acute{M}^m, D)$ be a CR-submanifold of a conformal Sasakian manifold M^{2n+1}. Then M' is said to be mixed totally geodesic if $th(X,Y) = 0$ for each $X \in D$ and $Y \in D^{\perp}$ [2].

Theorem 5.2. *Let* $(\acute{M}^m, D)$ *be a* CR*-submanifold of a conformal Sasakian manifold* M^{2n+1} *normal to* $\omega^{\sharp}$*. Then* $\acute{M}$ *is mixed totally geodesic if and only if each leaf of the anti-invariant distribution* $D^{\perp}$ *is a totally geodesic submanifold of* $\acute{M}$

Proof: *Let* S *be a leaf of* $D^{\perp}$*. Making use of the Gauss formula we have*

$$h_S(Y,Z) = (\acute{\nabla}_Y Z)_D, \tag{5.1}$$

for all $Y,Z \in T(S) = D^{\perp}$ *and* $X \in D$*, where* h_S *is the second fundamental form of* S *in* $\acute{M}$*. Hence we have*

$$-\acute{\nabla}((\acute{\nabla}_Y Z)_D, \varphi X) = \acute{g}(th(X,Y),Z) - \frac{1}{2}\acute{g}(Y,Z)\omega(\varphi X). \tag{5.2}$$

Since $\omega^{\sharp}$ *is normal to* $\acute{M}$*, in view of (5.2) we get*

$$g'((\nabla'_Y Z)_D, \varphi X) \quad = \quad -g'(th(X,Y),Z), \tag{5.3}$$

for all $Y,Z \in T(S) = D^{\perp}$ *and* $X \in D$*. So (5.1) and (5.3) complete the proof of the theorem.* □

6. Example

In this section we construct an example of a conformal Sasakian manifold that is not Sasakian. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vetore fields

$$e_1 = 2(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \qquad e_2 = 2\frac{\partial}{\partial y}, \qquad e_3 = 2exp(z)^{\frac{1}{2}}\frac{\partial}{\partial z},$$

are linearity independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_2) = g(e_1, e_3) = g(e_2, e_3) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = exp(-z), \qquad g(e_3, e_3) = 1.$$

Let η be the 1-form defined by

$$\eta(e_3) = 1, \qquad \eta(e_2) = 0, \qquad \eta(e_1) = 0.$$

We define the (1,1) tensor field φ as $\varphi e_1 = e_2$, $\varphi e_2 = -e_1$ and $\varphi e_3 = 0$. Then using the linearly of φ and g we have

$$\varphi^2 X = -X + \eta(X)e_3, \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for all X, Y on M. Thus for $e_3 = \xi$, (φ, ξ, η, g) defines an almost contact metric structure on M.
Let ∇ be the Levi-civita connection with respect to g. Then we have

$$[e_1, e_2] = -2exp(-z)^{\frac{1}{2}}e_3, \qquad [e_1, e_3] = ye_3, \qquad [e_2, e_3] = 0.$$

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &- g(X, [Y, Z]) + g(Y, [X, Z]) + g(Z, [X, Y]), \end{aligned}$$

which is known as Koszul formula. By using the Koszul formula, we obtain

$$\begin{aligned} &\nabla_{e_1} e_2 = -ye_2 - exp(-z)^{\frac{1}{2}}e_3, \quad &&\nabla_{e_1} e_3 = -exp(z)^{\frac{1}{2}}(e_1 - e_2), \\ &\nabla_{e_2} e_3 = -exp(z)^{\frac{1}{2}}(e_1 + e_2), \quad &&\nabla_{e_2} e_2 = ye_1 + exp(-z)^{\frac{1}{2}}e_3, \\ &\nabla_{e_3} e_3 = yexp(z)e_1, \quad &&\nabla_{e_3} e_2 = -exp(z)^{\frac{1}{2}}(e_1 + e_2), \\ &\nabla_{e_1} e_1 = -ye_1 + exp(-z)^{\frac{1}{2}}e_3, \quad &&\nabla_{e_3} e_1 = -exp(z)^{\frac{1}{2}}(e_1 - e_2) - ye_3, \\ &\nabla_{e_2} e_1 = -ye_2 + exp(-z)^{\frac{1}{2}}e_3. \end{aligned}$$

By a contact transformation

$$\tilde{g} = exp(x)g, \qquad \tilde{\xi} = exp(-x)^{\frac{1}{2}}\xi, \qquad \tilde{\eta} = exp(x)^{\frac{1}{2}}\eta, \qquad \tilde{\varphi} = \varphi,$$

$(M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is a Sasakian manifold [1]. So M is a conformal Sasakian manifold but is not Sasakian, Since we have

$$\nabla_X \xi \neq -\varphi X,$$

for each vector field X on M (for instance, $\nabla_{e_3} e_3 \neq 0$). By using above results, we can easily obtain the following :

$$R(e_1, e_2)e_2 = -4e_1 + yexp(-z)^{\frac{1}{2}} e_3,$$
$$R(e_1, e_3)e_3 = -e_1 + 3e_2 + yexp(z)^{\frac{1}{2}} e_3,$$
$$R(e_2, e_3)e_3 = exp(z)e_1 + exp(z)(1 - y^2)e_2.$$

In view of above relations, we get the following results:

$$K(e_1, e_2) = -4exp(z), \quad K(e_1, e_3) = -1, \quad K(e_2, e_3) = exp(z)(1 - y).$$

Note that the sectional curvature of manifold M with almost contact metric structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is [1]

$$K(e_1, e_2) = -3, \quad K(e_1, e_3) = -1, \quad K(e_2, e_3) = -1.$$

References

1. Blair, D. E., *Riemannian geometry of contact and symplectic manifolds*, Birkhauser, Boston-Basel-Berlin, 2002.
2. Dragomir, S., Ornea, L., *Locally conformal Kaehler Geometry*, Prog. Math. V 155, (1998).
3. Libermann, P., *Sur les structures infinitesimales regulieres*, Bull Soc. Math. France, V. 83, 195-224, (1955).
4. Vaisman, I., *A geometric condition for a locally conformal kaehler to be kaehler*. Geom. Dedicata., 10, 129-136, (1981).
5. Tricerri, F., *Some examples of locally conformal Kaehler manifolds*. Rend. Mat. Univ. Politecn. Torino., 40, 81-92, (1982).
6. Abedi, E., Ziabari, R. B., Tripathi, M. M., *Ricci and scalar curvatures of submanifolds of a conformal Sasakian space form*, Archivum Mathematicum (Brno) Tomus, 52, 113-130, (2016).
7. Gray, A., Hervella, L. M., *Sixteen classes of almost Hermitian manifold and their linear invariants*, Ann. Math. Pure Appl., 123 (3), 35-58, (1980).

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