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A Multilevel Local Mesh Refinement with the PCD Method

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ABSTRACT: In this paper, we propose a successive local mesh refinement with the PCD method. The multilevel local refinement improves the accuracy and gives better precision, locally and globally, with a lower computational costs particularly if the considered problem has an anomaly. Here we present how this successive local mesh refinement can be handled. We conclude by presenting numerical experiments to show the interest of the multilevel local mesh refinement for the 2D steady diffusion equation.

Key Words: Boundary value problem, Discretization technique, PCD method, Compact schemes, Most sparse stiffness matrix, Multilevel local refinement, Computational costs, $O(h)$-convergence rate.

Contents

1. Introduction

The PCD (piecewise constant distributions) method is a discretization technique of the boundary value problems in which the unknown distribution and its derivatives are represented by piecewise constant distributions but on distinct meshes. It has the advantage of producing the most sparse stiffness matrix resulting from the approximate problem.

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The aim of the PCD method is to produce a best accuracy with lower computational cost. With this method we can introduce a local mesh refinement without the use of the slave nodes that appear in some finite element discretizations with local mesh refinement. In this way, no interpolation is performed between the nodes of the interface boundary (the intersection between the coarse zone and refined zone).

The local refinement gives a better precision, locally and globally, with lower computational costs particularly if the considered problem has an anomaly. see for example Z. Cai et al. [2,3], R. Ewing [4] and P. Vassilevski et al. [10]. Using the PCD method we refer to A. Tahiri [6,7,8].

The PCD method is well adapted for multilevel local refinement. The multilevel local refinement is more attractive especially when we are interested in approximation of the solution just in a given zone of the domain (determined in advance), as for instance to determine the pressure in a very small part of the domain (as in petroleum industry). It is not attractive to use a uniform mesh, for discretizing the problem considered, from the point of view of storage cost, operational cost and the problem size.

The results established for a simple local mesh refinement are still valid for multilevel local refinement, we refer to A. Tahiri [5,6,7,8,9].

2. Preliminaries

2.1. The PCD Discretization

The principle of the PCD method is given in three steps. First, we split the domain of study denoted by Ω into M elements Ω_ℓ such that:

$$\overline{\Omega} = \bigcup_{\ell=1}^{M} \overline{\Omega}_\ell \ , \ \Omega_k \cap \Omega_\ell = \emptyset \text{ if } k \neq \ell \ .$$

Second, we define different sub-meshes on each element Ω_ℓ to represent elements of $H^1(\Omega)$ and their derivatives. Third, we require that the discrete representations of elements of $H^1(\Omega)$ must be continuous across the elements boundaries, i.e., along the normal to the element boundary. We denote the representation of $v \in H^1(\Omega)$ by v_h and the representation of its derivatives $\partial_i v$ $(i = 1, 2)$ by $\partial_{hi} v_h$ $(i = 1, 2)$. The operators ∂_{hi} $(i = 1, 2)$ are the finite difference quotients taken along the element edges, in the case of the rectangular elements. They would need to be appropriately adapted for other elements.

We introduce local mesh refinement by subdividing the elements of the zone to be refined by the ratio 2 in each direction i. e. each coarse element is subdivided by 4 fine elements, since the optimal local mesh refinement ratio is 2, see A. Tahiri [8].

On the intersection between the coarse zone and refined zone (the interface boundary) some irregular nodes appear. Therefore irregular element are created. As

on regular element, we define different sub-meshes on these elements to represent elements of $H^1(\Omega)$ and their derivatives.

we recall that, for a irregular element, we define sub-meshes to represent the derivatives without the use of the slave nodes. In this way, no interpolation is performed between the nodes of the interface boundary. To avoid this interpolation which will create the slave nodes we neglect the existence of the irregular nodes in the direction where we will approach the derivative. For more detail we refer to A. Tahiri [5,6,7,8,9].

We denote by H_{h0} (respectively H_{hi} $(i = 1, 2)$) the subspace of $L^2(\Omega)$ of the piecewise constant distributions used to define v_h the approximation of an element v of $H^1(\Omega)$ (respectively $\partial_{hi}\, v_h\,,\ i = 1, 2$ the approximation of its derivative).
The meshes used to define the above subspaces define themselves cells in which these approximations are constants. We denote the cells of these meshes by $\Omega_{\ell i}\,,\ \ell \in J_i\,,\ i = 0, 1, 2$ respectively. The measures of these cells will be denoted by $|\,\Omega_{\ell i}\,|\,,\ i = 0, 1, 2$.
The elements are denoted by $\Omega_\ell,\ \ell \in J = \{\,1, \ldots, M\}$ (M is the number of elements). We similarly denote the cells of the H_{h0}-mesh by $\Omega_{\ell 0}$ with $\ell \in J_0 = \{\,1, \ldots, N_G\}$ where N_G is the number of the grid nodes and N denotes the number of unknowns. It is important to note that each node of the mesh may be uniquely associated with a cell of H_{h0}-mesh. We therefore denote them by $N_\ell,\ \ell \in J_0$. We note that, with the PCD discretization, for any pair of nodes of the mesh we can find a path connecting these nodes (succession of mesh grid segments).

We split the domain Ω into two subdomains Ω_C (the coarse zone) and Ω_R (the refined zone) with $\Omega = \Omega_C \cup \Omega_R$. We denote by $\Gamma = \partial\Omega$ the boundary of the domain Ω.
Ω_I denote the union of all irregular elements, $\Omega_I = \cup_\ell\, \Omega_\ell$ such that $\Omega_\ell \cap \Omega_R = \emptyset$ and $\overline{\Omega}_\ell \cap \partial\Omega_R \neq \emptyset$. The subdomain Ω_I is a strip in Ω with an $O(h)$-width and has the interface boundary as part of its boundary.

We further denote by H_h the space H_{h0} equipped with the inner product:

$$(\,v_h\,,\,w_h\,)_h = (\,v_h\,,\,w_h\,)_\Omega \;+\; (\,\partial_{h1}\, v_h\,,\,\partial_{h1}\, w_h\,)_\Omega \;+\; (\,\partial_{h2}\, v_h\,,\,\partial_{h2}\, w_h\,)_\Omega\,, \qquad (2.1)$$

and its associate norm is denoted $\|\,.\,\|_h\,.$

We denote by h the mesh size defined by $h = \max(h_\ell),\ \ell \in J$, where $h_\ell = \text{diam}\,(\Omega_\ell)\,,\ \ell \in J$ and we denote by $h_{\ell 1}$ and $h_{\ell 2}$, the width and the height of the element Ω_ℓ.

The notation C is used throughout this work to denote a generic positive constant independent of the mesh size.

Before closing this section we note that triangular elements may also be introduced. In this way, the method can accommodate any shape of the domain under investigation through the combined use of a local mesh refinement and triangular elements, see A. Tahiri [9].
We note that the use of rectangular and triangular elements is not a restriction of the PCD discretization. Other elements and other forms of submeshes on such elements can be used, see R. Beauwens [1].

2.2. Properties of the PCD discretization

We investigate in this section general properties of the discrete space H_h and of the possible discrete representations of $v \in H$ in H_h.
The PCD discretization has the following properties which represent a discrete version of the first Friedrichs inequalities, the second Friedrichs inequalities and the trace inequality, for the proof we refer to A. Tahiri [7,9].

Lemma 2.1. *Let Ω be a bounded polygonal domain. Then, there exists a constant $C > 0$, independent of h such that:*

$$\| v_h \|_h \;\leq\; C \left(\| \partial_{h1} v_h \|^2 \;+\; \| \partial_{h2} v_h \|^2 \;+\; \| v_h \|_\Gamma^2 \right)^{1/2} \quad \textit{for all } v_h \in H_h \;. \tag{2.2}$$

Lemma 2.2. *Let Ω be a bounded polygonal domain. Then, there exists a constant $C > 0$, independent of h such that:*

$$\int_\Gamma v_h(x)^2 \, ds \;=\; \| v_h \|_\Gamma^2 \;\leq\; C \, \| v_h \|_h^2 \qquad \textit{for all } v_h \in H_h \;. \tag{2.3}$$

As usual, the interpolation error that can be obtained depends on the regularity of v, an element of $H^1(\Omega)$. Here, we assume that $v \in H^2(\Omega)$. In this case, v is continuous on $\overline{\Omega}$ and we can then define its interpolant v_I in H_h through:

$$v_I(\, N_\ell \,) = \, v(N_\ell) \text{ for all nodes } N_\ell \,, \;\; \ell \in J_0 \,. \tag{2.4}$$

On the other hand, we have shown in A. Tahiri [7] the following theoretical results.

Lemma 2.3. *Under the general assumptions and the notations defined above, there exists a positive constant C independent of the mesh size h such that for all v in $H^2(\Omega)$:*

$$\| v - v_I \|_h \leq Ch \|v\|_{2,\Omega} \,, \tag{2.5}$$

where v_I denotes the interpolant of v in H_h.

We recall that:

$$\| v - v_I \|_h^2 = \|v - v_I\|^2 + \|\partial_1 v - (\partial_{h1} v_I)\|^2 + \|\partial_2 v - (\partial_{h2} v_I)\|^2$$

The results given in the previous lemmas are independent of the presence or not of the local mesh refinement (or the multilevel local mesh refinement).

3. Diffusion problem

To keep the presentation of this discretization as simple as possible, we restrict this contribution to the analysis of the 2D diffusion equation on rectangular mesh with a refined subregion. The convergence analysis and technical results of the PCD method can be found in A. Tahiri [7,9].

3.1. Continuous problem

We consider solving the following boundary value problem (BVP) on a rectangular domain Ω:

$$-\nabla \cdot (p(x)\nabla u(x)) + q(x)u(x) = s(x) \ x \ \text{in} \ \Omega, \tag{3.1}$$

$$u(x) = 0 \ x \ \text{on} \ \partial\Omega\,. \tag{3.2}$$

We assume that $p(x)$ is measurable, bounded and strictly positive on $\overline{\Omega}$, $q(x)$ is measurable, bounded and nonnegative on Ω and we have a well posed problem. We note that the extension of the theory to general boundary conditions does not raise any difficulty.

The discrete version of this problem will be based on its variational formulation:

$$\text{find } u \in H \text{ such that } a(\,u,\,v\,) = (s,\,v), \text{ for all } v \in H \tag{3.3}$$

where $H = \{v \in H^1(\Omega), v = 0 \text{ on } \partial\Omega\}$, $(s,\,v)$ denotes the $L^2(\Omega)$ scalar product and

$$a(u,v) = \int_\Omega p(x)\nabla u(x) \cdot \nabla v(x)dx \ + \ \int_\Omega q(x)u(x)(v(x)dx. \tag{3.4}$$

3.2. Discrete Problem

The discrete problem to be solved in H_h is defined by:

$$\text{find } u_h \in H_h \text{ such that } a_h(\,u_h,\,v_h\,) = (s,\,v_h)\ , \text{ for all } \ v_h \in H_h \tag{3.5}$$

$$\text{where } a_h(\,u_h\,,\,v_h\,) = \sum_{i=1}^{2} (\,p(x)\,\partial_{hi}\,u_h\,,\ \partial_{hi}\,v_h\,)_\Omega \ + \ (\,q(x)\,u_h\,,\ v_h\,)_\Omega\ . \tag{3.6}$$

By introducing the basis $(\phi_i)_{\,i\in J_0}$ of the space H_h we get the linear system $\mathcal{A}\,\xi = b$ where $\mathcal{A}$ is the stiffness matrix defined by $\mathcal{A} = (\,a_h(\,\phi_j\,,\,\phi_i\,)\,)_{(i,\,j)\in J_0}$, b is the vector with components defined by $b_i = (\,s\,,\,\phi_i\,)_\Omega$ and ξ the unknown vector.
The basis $(\phi_i)_{\,i\in J_0}$ of H_h is defined as usual through the conditions:

$$\phi_i \in H_h \quad \text{and} \quad \phi_i(N_j) = \delta_{i\,j} \quad \text{for all nodes } N_j\,,\ j \in J_0\,.$$

We note that this basis is reduced to the characteristic function of the cells $\Omega_{\ell 0}, \ell \in J_0$.
It should be stated that the presented method has the advantage of producing the most compact discrete schemes and the most sparse stiffness matrix resulting from the approximate problem independently of the presence or not of the local mesh

refinement (or the multilevel local mesh refinement).

Under the assumptions previously mentioned and the regularity of u the exact solution of (3.1), we can give the following Theorems, for their proofs we refer to A. Tahiri [7].

Theorem 3.1. *Let Ω be a rectangular bounded open set. Assume that the unique variational solution u of (3.1) belongs to $H^2(\Omega)$. Then, there exists a constant $C > 0$ independent of h, such that:*

$$\left(\| u - u_h \|^2 + \| \partial_1 u - (\partial_{h1} u_h) \|^2 + \| \partial_2 u - (\partial_{h2} u_h) \|^2 \right)^{\frac{1}{2}} \leq C h \| u \|_{2,\Omega},$$

where u_h is the solution of the problem (3.5) without local refinement.

Theorem 3.2. *Let Ω be a rectangular bounded open set. Assume that the unique variational solution u of (3.1) belongs to $H^2(\Omega)$. Then, there exists a constant $C > 0$ independent of h, such that:*

$$\left(\| u - u_h \|^2 + \| \partial_1 u - (\partial_{h1} u_h) \|^2 + \| \partial_2 u - (\partial_{h2} u_h) \|^2 \right)^{\frac{1}{2}}$$

$$\leq C h \| u \|_{2,\Omega} + C h^{\frac{1}{2}} \| u \|_{2,\Omega_I},$$

where u_h is the solution of the problem (3.5) with a local mesh refinement and Ω_I is the strip in Ω.

If the solution u is only in $H^1(\Omega)$, we can still prove the convergence of u_h to u under some assumptions on u on the strip Ω_I which contains the interface boundary, namely u must be in $H^2(\Omega_I)$.

3.3. Multilevel local mesh refinement

The multilevel local refinement is more attractive, from the point of view of storage cost, operational cost, the problem size and gives a better precision locally and globally. Particularly, when we are interested in approximation of the solution just in a given zone of the domain (determined in advance). We refer to A. Tahiri [6,8] to show the interest of a local mesh refinement and its best strategy using the PCD method.
Another case where the multilevel local refinement is highly solicited, when the needed accuracy is not reached in a given part of the domain. In this case we proceed by multilevel local refinement until the required accuracy is reached, in order to have a very small mesh size in the desired part of the domain, and the obtained linear system still with a reasonable size.
The PCD method is well adapted to multilevel local refinement. Also, even if the resulting linear system is very large, its resolution by iterative methods is not difficult because we use a suitable preconditioning technique for this discretization.

We select a succession of subregions containing the needed zone. Successive local mesh refinement can be handled in the similar way than a simple one. The results previously established are still valid.

Since the best refinement rate is 2, we proceed by subdividing the elements of the succession of the subregions to be refined into four equal fine elements. The theoretical error is given in Theorem 3.2 where the term $C\,h^{0.5}\,\|\,u\,\|_{2,\Omega_I}$ summarizes all the interface boundaries contribution.

We process by steps, choosing a sequence of subdomains Ω_μ $(\mu = 0, .., l)$ with $\Omega_0 = \Omega$, at level μ $(\mu = 1, .., l)$ we refine the zone Ω_μ by introducing a mesh of size $h_\mu = h_{\mu-1}/2$. We illustrate on Fig. 1 an example of mesh with multilevel local refinement.

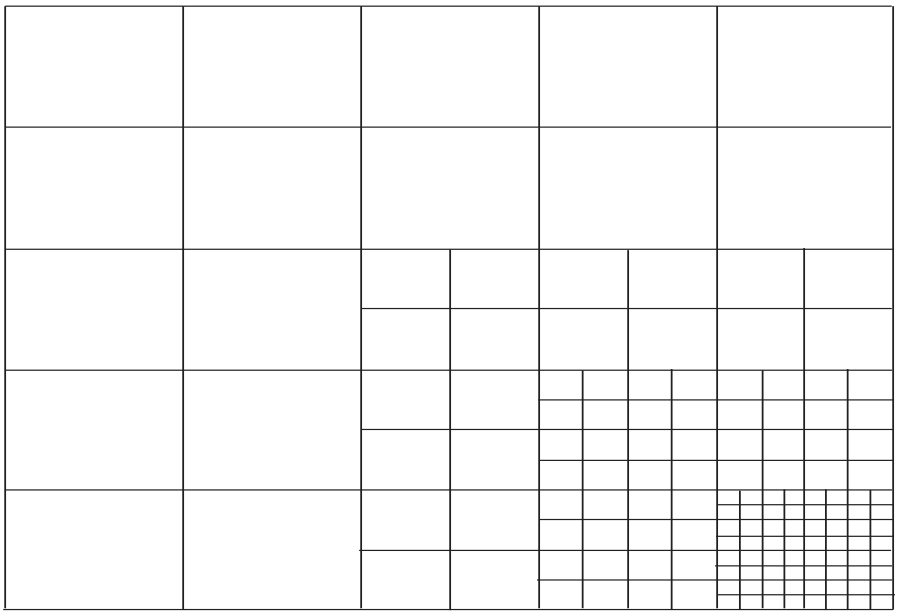

Figure 1: Mesh having a multilevel local refinement

4. Numerical experiments

In this section we present experimental results concerning the multilevel local refinement with the PCD discretization.
Sometimes we solve very large problems (very large linear system) just with the purpose of determining an approximation of the solution in a given part of the domain. To this end, we introduce a local error estimator and we try to determine its behavior. We will show the contribution of multilevel local mesh refinement to get better accuracy for this error estimator with a lower computational cost.

We consider the following error estimators: ε_0^r the relative L^2-error estimator and ε_1^r the relative H^1-error estimator defined by:

$$\varepsilon_0^r = \frac{\| u - u_h \|_0}{\| u \|_0} \text{ and } \varepsilon_1^r = \frac{| (u_I - u_h)^t \mathcal{A} (u_I - u_h) |^{\frac{1}{2}}}{\| u_I \|_h} = \frac{\| u_I - u_h \|_h}{\| u_I \|_h},$$

where u is the exact solution of (3.3) and u_I its interpolant in H_h.

In each example the approximate mean flux in the subregion Ω_F is computed and we try to give the behavior of the error between the exact flux and the computed one. We denote by $|\Omega_F|$ the area of Ω_F, F the exact mean flux in Ω_F and F_h the approximate mean flux in Ω_F. We denote by η the relative local error estimator defined by:

$$\eta = \frac{| F - F_h |}{F}, \text{ where } F = \frac{1}{|\Omega_F|} \int_{\Omega_F} u(x)dx \text{ and } F_h = \frac{1}{|\Omega_F|} \int_{\Omega_F} u_h(x)dx.$$

Finally, we denote by N the number of unknowns and by h_c the size of the coarse mesh.

4.1. Example 1

We consider the problem:

$$\begin{cases} -\Delta u(x,y) = s(x,y) \text{ in } \Omega \\ u(x,y) = 0 \text{ on } \partial\Omega \end{cases}$$

where the domain Ω is the unit square $]0, 1[\times]0, 1[$.

In this example we choose the source term $s(x,y)$ such that the exact solution is: $u(x,y) = x(1-x)y(1-y)\beta(x,y)$, where $\beta(x,y) = \exp(-100\{(x-0.5)^2+(y-0.5)^2\})$. We have smooth solution with a sharp peak at the point (0.5, 0.5). This solution varies much more rapidly in $\Omega_1 = [1/4, 3/4] \times [1/4, 3/4]$ than in the remaining part of Ω. We have an exponential variation of $u(x,y)$ which starts from the boundary of the subregion Ω_1.

The multilevel local refinement is applied in a sequence of subdomains: Ω_1, $\Omega_2 = [3/8, 5/8]\times[3/8, 5/8]$, $\Omega_3 = [7/16, 9/16]\times[7/16, 9/16]$, $\Omega_4 = [15/32, 17/32]\times[15/32, 17/32]$ and $\Omega_5 = \Omega_F = [31/64, 33/64] \times [31/64, 33/64]$.

h_c	N	ε_0^r	ε_1^r	η
1/8	49	1.005	0.9599	0.6950
1/16	225	0.4231	0.5668	0.3918
1/32	961	0.2219	0.3094	0.2412
1/64	3969	0.1122	0.1582	0.1036
1/128	16129	5.631×10^{-2}	7.954×10^{-2}	5.713×10^{-2}
1/256	65025	2.817×10^{-2}	3.982×10^{-2}	3.092×10^{-2}

Table 1: Example 1, without refinement

h_c	N	ε_0^r	ε_1^r	η
1/8	105	0.3702	0.5127	6.817×10^{-2}
1/16	433	0.1831	0.2505	2.500×10^{-2}
1/32	1761	9.352×10^{-2}	0.1271	1.173×10^{-2}
1/64	7105	4.776×10^{-2}	6.393×10^{-2}	3.243×10^{-3}
1/128	28545	2.434×10^{-2}	3.205×10^{-2}	7.547×10^{-4}
1/256	114433	1.233×10^{-2}	1.605×10^{-2}	3.529×10^{-4}

Table 2: Example 1, with local refinement in Ω_1

h_c	N	ε_0^r	ε_1^r	η
1/8	329	0.3255	0.3225	1.273×10^{-2}
1/16	1265	0.1506	0.1932	5.905×10^{-3}
1/32	4961	7.291×10^{-2}	0.1212	2.839×10^{-3}
1/64	19649	3.595×10^{-2}	7.979×10^{-2}	1.391×10^{-3}
1/128	78209	1.787×10^{-2}	5.421×10^{-2}	7.157×10^{-4}
1/256	312065	8.909×10^{-3}	3.754×10^{-2}	3.624×10^{-4}

Table 3: Example 1, with multilevel local refinement in Ω_μ $\mu = 1, .., 5$

From Tables 1–3 we observe a monotonic improvement of the accuracy in both error estimators, they decrease when the mesh size h decreases (N the number of unknowns increases). That proves the convergence of the presented method independently of the presence or not of local mesh refinement or multilevel local refinement.

We observe from Table 3 that ε_0^r is reduced by a factor of 2 when the mesh size is reduced by a factor of 2. Also, we observe that ε_1^r is reduced by a factor of 2 when the mesh size is reduced by a factor of 4. That proves an $O(h)$-convergence rate for ε_0^r and an $O(h^{0.5})$-convergence rate for ε_1^r.

Comparing the results in the Tables 1–3, we obtain better results with a local refinement and a multilevel local refinement than the results reported in Table 1 using uniform mesh. The local refinement (simple or multilevel) improves the accuracy of the approximate solution with a lower computational costs.

We note that the local estimator η has nearly the same behavior as the L^2-error estimator ε_0^r (Tables 1–3). Our numerical results are in agreement with the theoretical results given in Theorem 3.2. Moreover, we can improve the estimation of the convergence rate for the L^2-error estimate by $\varepsilon_0^r \leq C\,h$.

To show the interest of multilevel local mesh refinement we shall compare the problem size (N number of unknowns) and the obtained accuracy in all cases.
More explicitly, in Table 1 we must have a uniform mesh with 65025 nodes to get around the point (0.5, 0.5) the local L^2-error $\eta = 3.092 \times 10^{-2}$. With simple local mesh refinement, Table 2, we get $\eta = 1.173 \times 10^{-2}$ only with the use of 1761 nodes. Using multilevel local mesh refinement, Table 3, this error is reached only with the use of 329 nodes, and with 1265 nodes we get better accuracy than a simple local refinement.
From the point of view of storage cost, operational cost and the problem size, multilevel local mesh refinement is more attractive than the use of a uniform mesh or simple local mesh refinement.

4.2. Example 2

We consider a problem with piecewise constant coefficient $p(x, y)$.

$$\begin{cases} -\Delta(p(x,y)\,u(x,y)) = s(x,y) \text{ in } \Omega \\ u = 0 \text{ on } \Gamma_D \\ \partial u/\partial n \,+\, 4\,u = 0 \text{ on } \Gamma_N \end{cases}$$

where the domain Ω is the unit square $]0,\,1[\times]0,\,1[$, $\Gamma_D = \{(x,\,0),\ 0 \leq x \leq 1\,,\ (0,\,y),\ 0 \leq y \leq 1\}$ and

$$p(x,y) = \begin{cases} 100\,, & x > 0.75\ ,\ y > 0.75 \\ 1\,, & \text{otherwise} \end{cases}$$

The exact solution is:

$$u(x,y) = \frac{1}{p(x,y)}(x - 0.75)\,(y - 0.75)\sin\left(\frac{\pi\,x}{2}\right)\,\sin\left(\frac{\pi\,y}{2}\right).$$

The sequence of subdomains used for multilevel local refinement is:
$\Omega_1 = [\,5/8\,,\,1\,] \times [\,5/8\,,\,1\,]$, $\Omega_2 = [\,3/4\,,\,1\,] \times [\,3/4\,,\,1\,]$ and
$\Omega_3 = \Omega_F = [\,14/16\,,\,15/16\,] \times [\,13/16\,,\,14/16\,]$

h_c	N	ε_0^r	ε_1^r	η
1/8	64	9.941×10^{-2}	9.856×10^{-2}	0.4139
1/16	256	5.104×10^{-2}	7.046×10^{-2}	0.3515
1/32	1024	2.582×10^{-2}	4.434×10^{-2}	0.1727
1/64	4096	1.303×10^{-2}	2.554×10^{-2}	8.251×10^{-2}
1/128	16384	6.571×10^{-3}	1.393×10^{-2}	4.002×10^{-2}
1/256	65536	3.301×10^{-3}	7.304×10^{-3}	1.967×10^{-2}

Table 4: Example 2, without local refinement

h_c	N	ε_0^r	ε_1^r	η
1/8	80	7.772×10^{-2}	8.002×10^{-2}	0.3789
1/16	312	3.245×10^{-2}	5.059×10^{-2}	0.2121
1/32	1232	1.527×10^{-2}	3.661×10^{-2}	9.723×10^{-2}
1/64	4896	7.533×10^{-3}	2.042×10^{-2}	4.637×10^{-2}
1/128	19520	3.768×10^{-3}	9.687×10^{-3}	2.264×10^{-2}
1/256	77952	1.889×10^{-3}	5.436×10^{-3}	1.118×10^{-2}

Table 5: Example 2, with local refinement in Ω_2

h_c	N	ε_0^r	ε_1^r	η
1/8	162	5.588×10^{-2}	9.198×10^{-2}	0.1010
1/16	628	1.958×10^{-2}	5.438×10^{-2}	4.789×10^{-2}
1/32	2464	8.369×10^{-3}	3.051×10^{-2}	2.331×10^{-2}
1/64	9760	3.959×10^{-3}	1.632×10^{-2}	1.150×10^{-2}
1/128	38848	1.943×10^{-3}	8.481×10^{-3}	4.807×10^{-3}
1/256	155008	9.658×10^{-4}	4.633×10^{-3}	2.213×10^{-3}

Table 6: Example 2, with multilevel local refinement in $\Omega_\mu,\ \mu = 1,..,3$

Also in this example, we observe a monotonic improvement of the accuracy in both error estimators, they decrease when the mesh size h decreases (N the number of unknowns increases), see Tables 4–6. That proves the convergence of the method independently of the presence or not of local mesh refinement or multilevel local refinement. Also we observe that the local estimator η has the same behavior

as the L^2-error estimator ε_0^r.

We observe from Table 5–6 that ε_0^r and ε_1^r are reduced by a factor of 2 when the mesh size is reduced by a factor of 2. In this example we have a nearly $O(h)$-convergence rate for both error estimators with simple local refinement and with multilevel local refinement.

Also we observe, globally the obtained accuracy is better with the use of multilevel local refinement than that obtained with the use of simple local refinement, see Tables 5–6.

Moreover, we must have a uniform mesh with 65536 nodes to get the local L^2-error $\eta = 1.967 \times 10^{-2}$, see Table 4. With a simple local mesh refinement, we get $\eta = 2.264 \times 10^{-2}$ only with the use of 19520 nodes, see Table 5. Using multilevel local mesh refinement, this error is reached only with the use of 2464 nodes, see Table 6. Better accuracy is obtained with the use of multilevel local refinement. Also here, the multilevel local refinement improves the accuracy of the approximate solution with a lower computational costs.

5. Concluding remarks

The main issue of the present work is the presentation of the interest of multilevel local mesh refinement and the way in which we introduce such refinement using the PCD method.
Our results have shown that the multilevel local refinement is more attractive than the use of a uniform mesh or a simple local mesh refinement. Comparing the storage cost, operational cost, the problem size and the intended accuracy, multilevel local mesh refinement is more efficient than a uniform mesh or a simple local mesh refinement.
The numerical examples presented here are in agreement with the theoretical results.
Our conclusion is that, the multilevel local refinement gives better precision, locally and globally, with lower computational costs.

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