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A Study on Local Properties of Fourier Series *

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ABSTRACT: In this paper, a general theorem on the local property of $|A, p_n; \delta|_k$ summability of factored Fourier series has been proved. This new theorem also includes several new and known results.

Key Words: Summability factors, absolute matrix summability, Fourier series, local property, infinite series, Hölder inequality, Minkowski inequality.

Contents

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). By (u_n) and (t_n) we denote the n-th $(C,1)$ means of the sequences (s_n) and (na_n), respectively. The series $\sum a_n$ is said to be summable $|C,1|_k$, $k \geq 1$, if (see [16])

$$\sum_{n=1}^{\infty} n^{k-1}|u_n - u_{n-1}|^k = \sum_{n=1}^{\infty} \frac{1}{n}|t_n|^k < \infty. \tag{1.1}$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{1.2}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v \tag{1.3}$$

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defines the sequence (σ_n) of the Riesz mean or simply the $\left(\bar{N},p_n\right)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [17]). The series $\sum a_n$ is said to be summable $\left|\bar{N},p_n\right|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|\Delta\sigma_{n-1}|^k < \infty, \tag{1.4}$$

where

$$\Delta\sigma_{n-1} = -\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v, \quad n \geq 1.$$

In the special case, when $p_n = 1$ for all values of n (resp. $k = 1$), $|\bar{N},p_n|_k$ summability is the same as $\mid C,1 \mid_k$ (resp. $|\bar{N},p_n|$) summability. If we take $k = 1$ and $p_n = 1/(n+1)$, then summability $|\bar{N},p_n|_k$ is equivalent to the summability $|R,logn,1|$.

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n}a_{nv}s_v, \quad n = 0,1,... \tag{1.5}$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [31])

$$\sum_{n=1}^{\infty}n^{k-1}\left|\bar{\Delta}A_n(s)\right|^k < \infty, \tag{1.6}$$

and it is said to be summable $|A,p_n;\delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [26])

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|\bar{\Delta}A_n(s)|^k < \infty, \tag{1.7}$$

where

$$\bar{\Delta}A_n(s) = A_n(s) - A_{n-1}(s).$$

If we take $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$, then $|A,p_n;\delta|_k$ summability reduces to $|\bar{N},p_n|_k$ summability. Also, if we take $\delta = 0$, then $|A,p_n;\delta|_k$ summability reduces to $|A,p_n|_k$ summability (see [30]). In the special case $\delta = 0$ and $p_n = 1$ for all n, $|A,p_n;\delta|_k$ summability is the same as $|A|_k$ summability. Furthermore, if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A,p_n;\delta|_k$ summability is the same as $|\bar{N},p_n;\delta|_k$ summability (see [7]).

A sequence (λ_n) is said to be convex if $\Delta^2\lambda_n \geq 0$ for every positive integer n, where $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$ (see [33]).

Let $f(t)$ be a periodic function with period 2π, and integrable (L) over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)dt = 0 \tag{1.8}$$

and

$$f(t) \sim \sum_{n=1}^{\infty}(a_n cosnt + b_n sinnt) \quad = \quad \sum_{n=1}^{\infty} C_n(t), \tag{1.9}$$

where (a_n) and (b_n) denote the Fourier coefficients. It is well known that the convergence of the Fourier series at $t = x$ is a local property of the generating function $f(t)$ (i.e. it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t = x$ by any regular linear summability method is also a local property of the generating function $f(t)$ (see [32]).

Before stating the main theorem, let us introduce some further notations.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{1.10}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{1.11}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{1.12}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{1.13}$$

2. Known Results

Mohanty [22] has demonstrated that the summability $|R, logn, 1|$ of

$$\sum \frac{C_n(t)}{log(n+1)}, \tag{2.1}$$

at $t = x$, is a local property of the generating function of $\sum C_n(t)$. In [20], Matsumoto has improved this result by replacing the series (2.1) by

$$\sum \frac{C_n(t)}{\{loglog(n+1)\}^{1+\epsilon}}, \quad \epsilon > 0. \tag{2.2}$$

Bhatt [2] has generalized the above result in the following form.

Theorem 2.1. *If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then the summability $|R, logn, 1|$ of the series $\sum C_n(t)\lambda_n logn$ at a point can be ensured by a local property.*

Also, Mishra [21] has proved the following most general theorem.

Theorem 2.2. *Let the sequence (p_n) be such that*

$$P_n = O(np_n), \tag{2.3}$$

$$P_n \Delta p_n = O(p_n p_{n+1}). \tag{2.4}$$

Then the summability $|\bar{N}, p_n|$ of the series

$$\sum \frac{C_n(t)\lambda_n P_n}{np_n} \tag{2.5}$$

at a point can be ensured by local property, where (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent.

Many works dealing with Fourier series have been done (see [1]- [2], [5]- [14], [18]- [29]). Few of them are given above. Furthermore, Bor [4] has proved the following theorem.

Theorem 2.3. *Let $k \geq 1$ and (p_n) be a sequence such that the conditions (2.3) and (2.4) of Theorem 2.2 are satisfied. Then the summability $|\bar{N}, p_n|_k$ of the series (2.5) at a point can be ensured by local property, where (λ_n) is as in Theorem 2.2.*

3. Main Result

The aim of this paper is to generalize Theorem 2.3 for $|A, p_n; \delta|_k$ summability. Now, we shall prove the following theorem.

Theorem 3.1. *Let $k \geq 1$ and $0 \leq \delta < 1/k$. Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \;\; n = 0, 1, ..., \tag{3.1}$$

$$a_{n-1,v} \geq a_{nv}, \;\; for \; n \geq v + 1, \tag{3.2}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{3.3}$$

$$|\hat{a}_{n,v+1}| = O\left(v\left|\Delta_v \hat{a}_{nv}\right|\right). \tag{3.4}$$

If all the conditions of Theorem 2.3 and the conditions

$$\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{v}(\lambda_v)^k = O(1) \quad as \quad m\to\infty, \tag{3.5}$$

$$\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\Delta\lambda_v = O(1) \quad as \quad m\to\infty, \tag{3.6}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left|\Delta_v \hat{a}_{nv}\right| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k-1}\right\} \quad as \quad m\to\infty, \tag{3.7}$$

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}| \quad = \quad O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\right\} \quad as \quad m\to\infty \tag{3.8}$$

are satisfied, then the summability $|A,p_n;\delta|_k$ *of the series* $\sum \frac{C_n(t)\lambda_n P_n}{np_n}$ *at a point can be ensured by local property.*

It should be noted that if we take $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 2.3. In this case, the conditions (3.1)-(3.6) are obvious and the conditions (3.7) and (3.8) reduce to

$$\sum_{n=v+1}^{m+1}\left|\Delta_v \hat{a}_{nv}\right| = O\left(\frac{p_v}{P_v}\right) \quad as \quad m\to\infty, \tag{3.9}$$

and

$$\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}| = O(1) \quad as \quad m\to\infty, \tag{3.10}$$

which always hold.
We need the following lemmas for the proof of Theorem 3.1.

Lemma 3.2. *[21]. If the sequence* (p_n) *is such that the conditions (2.3) and (2.4) of Theorem 2.2 are satisfied, then*

$$\Delta\left(\frac{P_n}{np_n}\right) = O\left(\frac{1}{n}\right). \tag{3.11}$$

Lemma 3.3. *[15]. If (λ_n) is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then (λ_n) is non-negative and decreasing, and $n\Delta\lambda_n \to 0 \quad as \quad n \to \infty$.*

Lemma 3.4. *Let $k \geq 1$ and $0 \leq \delta < 1/k$. If (s_n) is bounded and all conditions of Theorem 3.1 are satisfied, then the series*

$$\sum_{n=1}^{\infty} \frac{a_n \lambda_n P_n}{n p_n}, \tag{3.12}$$

is summable $|A, p_n; \delta|_k$, where (λ_n) is as in Theorem 2.2.

Remark 3.5. *Since (λ_n) is a convex sequence, therefore $(\lambda_n)^k$ is also convex sequence and*

$$\sum \frac{1}{n}(\lambda_n)^k < \infty. \tag{3.13}$$

4. Proof of Lemma 3.4

Let (I_n) denotes the A-transform of the series $\sum \frac{a_n \lambda_n P_n}{n p_n}$. Then, by (1.12) and (1.13), we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v \lambda_v P_v}{v p_v}.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} \lambda_v P_v}{v p_v} \right) \sum_{r=1}^{v} a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n p_n} \sum_{v=1}^{n} a_v \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} \lambda_v P_v}{v p_v} \right) s_v + \frac{a_{nn} P_n \lambda_n}{n p_n} s_n \\
&= \sum_{v=1}^{n-1} \frac{P_v \lambda_v \Delta_v(\hat{a}_{nv})}{v p_v} s_v + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} \Delta\lambda_v P_v}{v p_v} s_v \\
&+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta \left(\frac{P_v}{v p_v} \right) s_v + \frac{a_{nn} P_n \lambda_n}{n p_n} s_n \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Lemma 3.4, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid I_{n,r}\mid^k< \infty, \quad for \quad r=1,2,3,4. \tag{4.1}$$

First, by applying Hölder's inequality, we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |I_{n,1}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left|\sum_{v=1}^{n-1}\frac{P_v\lambda_v\Delta_v(\hat{a}_{nv})}{vp_v}s_v\right|^k \\
&\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)|\Delta_v(\hat{a}_{nv})|\,(\lambda_v)|s_v|\right\}^k \\
&\leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k|\Delta_v(\hat{a}_{nv})|(\lambda_v)^k|s_v|^k\right\} \\
&\times \left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1}.
\end{aligned}$$

By (1.10) and (1.11), we have that

$$\Delta_v(\hat{a}_{nv})=\hat{a}_{nv}-\hat{a}_{n,v+1}=\bar{a}_{nv}-\bar{a}_{n-1,v}-\bar{a}_{n,v+1}+\bar{a}_{n-1,v+1}=a_{nv}-a_{n-1,v}.$$

Thus using (1.10), (3.1) and (3.2)

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|=\sum_{v=1}^{n-1}(a_{n-1,v}-a_{nv})\leq a_{nn}.$$

Hence,

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |I_{n,1}|^k = O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \\
\times\, a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}|\Delta_v(\hat{a}_{nv})|(\lambda_v)^k\right\} \\
=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}(\lambda_v)^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})| \\
=O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k\frac{1}{v^k}(\lambda_v)^k\left(\frac{P_v}{p_v}\right)^{\delta k-1}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1}\frac{1}{v^k}(\lambda_v)^k\left(\frac{P_v}{p_v}\right)^{\delta k}\\
&= O(1)\sum_{v=1}^{m} v^{k-1}\frac{1}{v^k}(\lambda_v)^k\left(\frac{P_v}{p_v}\right)^{\delta k}\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{v}(\lambda_v)^k = O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by virtue of the hypotheses of Lemma 3.4.

Now, by using (2.3), and Hölder's inequality we have

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,2}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}\Delta\lambda_v P_v}{vp_v}s_v\right|^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v|s_v|\right\}^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v|s_v|^k\right\}\\
&\times \left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\right\}^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\\
&\times \left\{\sum_{v=1}^{n-1}\Delta\lambda_v\right\}^{k-1}
\end{aligned}
$$

by using (3.3), we get

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\Delta\lambda_v\right\}\\
&= O(1)\sum_{v=1}^{m}\Delta\lambda_v\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\Delta\lambda_v\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}
$$

by using (3.6), (3.8) and hypotheses of Lemma 3.3. Now, since $\Delta\left(\frac{P_v}{vp_v}\right)=O\left(\frac{1}{v}\right)$ by Lemma 3.2, we have

$$
\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,3}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\Delta\left(\frac{P_v}{vp_v}\right)s_v\right|^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left\{\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})|s_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \\
&\quad\times\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})^k|s_v|^k\left\{\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right\}^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}a_{nn}^{k-1}\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}|(\lambda_{v+1})^k \\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}(\lambda_{v+1})^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\hat{a}_{n,v+1}| \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{v}(\lambda_{v+1})^k = O(1)\quad as\quad m\to\infty,
\end{aligned}
$$

by using (3.3), (3.4), (3.5) and (3.8). Finally, we have

$$
\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,4}|^k &= \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left|\frac{a_{nn}P_n\lambda_n}{np_n}s_n\right|^k \\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\frac{p_n}{P_n}\right)^k \\
&\quad\times\frac{1}{n^k}\left(\frac{P_n}{p_n}\right)^k(\lambda_n)^k|s_n|^k \\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k}\frac{1}{n}(\lambda_n)^k = O(1)\quad as\quad m\to\infty,
\end{aligned}
$$

by using (2.3), (3.3) and (3.5).
This completes the proof of Lemma 3.4.

5. Proof of Theorem 3.1

The convergence of the Fourier series at $t=x$ is a local property of f (i.e., it depends only on the behaviour of f in an arbitrarily small neighbourhood of x), and hence the summability of the Fourier series at $t=x$ by any regular linear summability method is also a local property of f. Since the behaviour of the Fourier series, as far as convergence is concerned, for a particular value of x depends on the behaviour of the function in the immediate neighbourhood of this point only, hence the truth of Theorem 3.1 is a consequence of Lemma 3.4.

If we take $\delta = 0$, then we obtain a theorem on $|A, p_n|_k$ summability method. If we take $\delta = 0$ and $p_n = 1$ for all n, then we obtain a result for $|A|_k$ summability method.

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