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On Ternary Left Almost Semigroups

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ABSTRACT: A ternary LA-semigroup is a nonempty set together with a ternary multiplication which is non associative. Analogous to the theory of LA-semigroups, a regularity condition on a ternary LA-semigroup is introduced and the properties of ternary LA-semigroups are studied. Some characterizations of quasi-prime and quasi-ideals were obtained.

Key Words: Ternary LA-semigroup, Quasi-prime ideal, Quasi-ideal, Ternary LA-subsemigroup, Left ideal.

Contents

1. Introduction

The theory of ternary algebraic systems was introduced by Lehmer [3] in 1932. The notion of ternary semigroup was known to Banach [4] who is credited with example of a ternary semigroup which can not reduce to a semigroup. The quasi-ideal theory in ternary semigroups was studied by Sioson [5] in the year 1965. Los [4] studied some properties of ternary semigroup and proved that every ternary semigroup can be embedded in a semigroup. Dixit and Dewan [1,2] studied quasi-ideals and bi-ideals in ternary semigroups. Recently, Bashir and Shabir [4] defined the concepts of weakly pure ideal and purely prime ideal in a ternary semigroup without order.

In this study we followed lines as adopted in [7,8,9] and established the notion of ternary LA-semigroups. Specifically we characterize the quasi-prime and quasi-ideals in ternary LA-semigroups with left identity.

2. Preliminaries

In this section, we give some preliminary results of ternary LA-semigroups which will be required for our later discussions.

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Definition 2.1. *Let S be a nonempty set. Then S is called a ternary left almost semigroup (or simply a ternary LA-semigroup) if there exists a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \mapsto x_1x_2x_3$, such that*

$$((x_1x_2)x_3)(x_4x_5) = ((x_4x_5)x_3)(x_1x_2) = ((x_3x_2)x_1)(x_4x_5)$$

for all $x_1, x_2, x_3, x_4, x_5 \in S$.

Example 2.2. *Let $S = \{0, i, -i\}$. Then by defining $S\times S\times S \to S$, as $(x_1, x_2, x_3) \mapsto x_1x_2x_3$, for all $x_1, x_2, x_3 \in S$. It can be easily verified that S is a ternary LA-semigroup under complex number multiplication while S is not an LA-semigroup.*

Example 2.3. *Let $S = \mathbb{Z}$. Define a mapping $S \times S \times S \to S$ by $(x_1, x_2, x_3) \mapsto -x_1 + x_2 - x_3$, for all $x_1, x_2, x_3 \in S$, where $-$ is a usual subtraction of integers. Then S is a ternary LA-semigroup while S is not a ternary semigroup. Indeed*

$$\begin{aligned}((x_1x_2)x_3)(x_4x_5) &= (-x_1 + x_2 - x_3)(x_4x_5)\\ &= -(-x_1 + x_2 - x_3) + x_4 - x_5\\ &= x_1 - x_2 + x_3 + x_4 - x_5\\ &= -(-x_4 + x_5 - x_3) + x_1 - x_2\\ &= (-x_4 + x_5 - x_3)(x_1x_2)\\ &= ((x_4x_5)x_3)(x_1x_2)\end{aligned}$$

and

$$\begin{aligned}((x_1x_2)x_3)(x_4x_5) &= (-x_1 + x_2 - x_3)(x_4x_5)\\ &= -(-x_1 + x_2 - x_3) + x_4 - x_5\\ &= x_1 - x_2 + x_3 + x_4 - x_5\\ &= -(-x_3 + x_2 - x_1) + x_4 - x_5\\ &= (-x_3 + x_2 - x_1)(x_4x_5)\\ &= ((x_3x_2)x_1)(x_4x_5)\end{aligned}$$

which implies $((x_1x_2)x_3)(x_4x_5) = ((x_4x_5)x_3)(x_1x_2) = ((x_3x_2)x_1)(x_4x_5)$, for all $x_1, x_2, x_3, x_4, x_5 \in S$.

Proposition 2.4. *If S is a ternary LA-semigroup, then $((x_1x_2)x_3)((x_4x_5)x_6) = ((x_1x_2)(x_4x_5))(x_3x_6)$, for all $x_1, x_2, x_3, x_4, x_5, x_6 \in S$.*

Proof: Let $x_1, x_2, x_3, x_4, x_5, x_6 \in S$. Then by Definition of ternary LA-semigroup, we get

$$\begin{aligned}((x_1x_2)x_3)((x_4x_5)x_6) &= (((x_4x_5)x_6)x_3)(x_1x_2)\\ &= ((x_3x_6)(x_4x_5))(x_1x_2)\\ &= ((x_1x_2)(x_4x_5))(x_3x_6).\end{aligned}$$

Hence $((x_1x_2)x_3)((x_4x_5)x_6) = ((x_1x_2)(x_4x_5))(x_3x_6)$. □

Proposition 2.5. *If S is a ternary LA-semigroup, then*

$$[((x_1x_2)x_3)(x_4x_5)]\,[(x_6x_7)((x_8x_9)x_{10})] = [((x_1x_2)x_3)(x_6x_7)]\,[(x_4x_5)((x_8x_9)x_{10})],$$

for all $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \in S$.

Proof: Let $x_1, x_2, x_3, x_4, x_5, x_6 \in S$. Then by Definition of ternary LA-semigroup, we have

$$\begin{aligned}[((x_1x_2)x_3)(x_4x_5)][(x_6x_7)((x_8x_9)x_{10})] &= [[x_6x_7)((x_8x_9)x_{10})](x_4x_5)] \\ &\quad ((x_1x_2)x_3) \\ &= [[(x_4x_5)((x_8x_9)x_{10})](x_6x_7)] \\ &\quad ((x_1x_2)x_3) \\ &= [((x_1x_2)x_3)(x_6x_7)] \\ &\quad [(x_4x_5)((x_8x_9)x_{10})].\end{aligned}$$

□

Definition 2.6. *An element e of a ternary LA-semigroup S is called a left identity if $(ee)x = x$, for all $x \in S$.*

Note 1. *By $(AB)C$ $(A(BC))$ we mean the set*

$$\{(ab)c : a \in A, b \in B, c \in C\} \ (\{a(bc) : a \in A, b \in B, c \in C\})$$

for non empty subsets A, B, C of S. If $A = \{a\}$, then we write $(\{a\}\,B)\,C$ as $(aB)C$ and similarly if $B = \{b\}$ or $C = \{c\}$, we write $(Ab)C$ and $(AB)c$ respectively.

Lemma 2.7. *If S is a ternary LA-semigroup with left identity e, then $(SS)S = S$ and $(ee)S = S$.*

Proof: Let $x \in S$. Then $x = (ee)x \in (SS)S$ so $S \subseteq (SS)S$. Hence $S = (SS)S$. Now as e is left identity in S, it is obvious that $(ee)S = S$. □

Lemma 2.8. *If S is a ternary LA-semigroup with left identity, then $x_1(x_2x_3) = x_2(x_1x_3)$, for all $x_1, x_2, x_3 \in S$.*

Proof: Let $x_1, x_2, x_3 \in S$. Then by Definition of ternary LA-semigroup, we get

$$\begin{aligned}x_1(x_2x_3) &= ((ee)x_1)(x_2x_3) \\ &= ((x_2x_3)x_1)(ee) \\ &= ((x_1x_3)x_2)(ee) \\ &= ((ee)x_2)(x_1x_3) \\ &= x_2(x_1x_3).\end{aligned}$$

Hence $x_1(x_2x_3) = x_2(x_1x_3)$. □

Proposition 2.9. *If S is a ternary LA-semigroup with left identity, then*

$$(x_1x_2)((x_3x_4)x_5) = (x_5(x_3x_4))(x_2x_1),$$

for all $x_1, x_2, x_3, x_4, x_5 \in S$.

Proof: Let $x_1, x_2, x_3, x_4, x_5 \in S$. Then by Definition of ternary LA-semigroup, we get

$$\begin{aligned}(x_1x_2)((x_3x_4)x_5) &= (((ee)x_1)x_2)((x_3x_4)x_5)\\ &= ((x_2x_1)(ee))((x_3x_4)x_5)\\ &= (((x_3x_4)x_5)(ee))(x_2x_1)\\ &= (((ee)x_5)(x_3x_4))(x_2x_1)\\ &= (x_5(x_3x_4))(x_2x_1).\end{aligned}$$

Hence $(x_1x_2)((x_3x_4)x_5) = (x_5(x_3x_4))(x_2x_1)$. □

Lemma 2.10. *If S is a ternary LA-semigroup with left identity, then*

$$(x_1x_2)(x_3x_4) = (x_1x_3)(x_2x_4),$$

for all $x_1, x_2, x_3, x_4 \in S$.

Proof: Let $x_1, x_2, x_3, x_4 \in S$. Then by Definition of ternary LA-semigroup, we have

$$\begin{aligned}(x_1x_2)(x_3x_4) &= (((ee)x_1)x_2)(x_3x_4)\\ &= ((x_3x_4)x_2)((ee)x_1)\\ &= ((x_2x_4)x_3)((ee)x_1)\\ &= (((ee)x_1)x_3)(x_2x_4)\\ &= (x_1x_3)(x_2x_4).\end{aligned}$$

Hence $(x_1x_2)(x_3x_4) = (x_1x_3)(x_2x_4)$. □

Lemma 2.11. *If S is a ternary LA-semigroup with left identity, then*

$$(x_1x_2)(x_3x_4) = (x_4x_3)(x_2x_1),$$

for all $x_1, x_2, x_3, x_4 \in S$.

Proof: Let $x_1, x_2, x_3, x_4 \in S$. Then by Definition of ternary LA-semigroup, we have

$$\begin{aligned}(x_1x_2)(x_3x_4) &= (((ee)x_1)x_2)(x_3x_4)\\ &= ((x_2x_1)(ee))(x_3x_4)\\ &= ((x_3x_4)(ee))(x_2x_1)\\ &= (((ee)x_4)x_3)(x_2x_1)\\ &= (x_4x_3)(x_2x_1).\end{aligned}$$

Hence $(x_1x_2)(x_3x_4) = (x_4x_3)(x_2x_1)$. □

Lemma 2.12. *If S is a ternary LA-semigroup with left identity, then*

$$(x_1x_2)x_3 = (x_3x_2)x_1,$$

for all $x_1, x_2, x_3 \in S$.

Proof: Let $x_1, x_2, x_3 \in S$. Then by Definition of ternary LA-semigroup, we have

$$\begin{aligned}(x_1x_2)x_3 &= (x_1x_2)((ee)x_3) \\ &= (x_1(ee))(x_2x_3) \\ &= (x_3x_2)((ee)x_1) \\ &= (x_3x_2)x_1.\end{aligned}$$

Hence $(x_1x_2)x_3 = (x_3x_2)x_1$. □

3. Left ideals of ternary LA-semigroups

The results of the following lemmas seem to play an important role to study ternary LA-semigroups; these facts will be used frequently and normally we shall make no reference to this definition.

Definition 3.1. *A non-empty subset A of a ternary LA-semigroup S is said to be a*

1. *ternary LA-subsemigroup if $(AA)A \subseteq A$;*
2. *left ideal if $(SS)A \subseteq A$.*

Remark 3.2. *It is easy to see that every left ideal is ternary LA-subsemigroup.*

Example 3.3. *Let $S = \{0, -1, -2, -3, -4\}$ the binary operation $\cdot$ be defined on S as follows:*

·	0	-1	-2	-3	-4
0	0	0	0	0	0
-1	0	1	2	3	4
-2	0	2	4	1	3
-3	0	3	1	4	2
-4	0	4	3	2	1

·	0	-1	-2	-3	-4
0	0	0	0	0	0
1	0	-1	-2	-3	-4
2	0	-2	-4	-1	-3
3	0	-3	-1	-4	-2
4	0	-4	-3	-2	-1

·	0	1	2	3	4
0	0	0	0	0	0
-1	0	-1	-2	-3	-4
-2	0	-2	-4	-1	-3
-3	0	-3	-1	-4	-2
-4	0	-4	-3	-2	-1

Define a mapping $S \times S \times S \to S$ by $(x_1, x_2, x_3) \mapsto x_1^{-1} \cdot x_2 \cdot x_3^{-1}$, for all $x_1, x_2, x_3 \in S$ and $x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1 = x_3 \cdot x_3^{-1} = x_3^{-1} \cdot x_3$. Then S is a ternary LA-semigroup. It is easy to see that $\{0, -1\}$ is a ternary LA-subsemigroup of S. But $\{0, -1\}$ is not a left ideal of S, since $(SS)\{0, -1\} = S \not\subseteq \{0, -1\}$.

Example 3.4. *Let $S = \{0, -a, -b, -c\}$ the binary operation $\cdot$ be defined on S as follows:*

$\cdot$	0	-a	-b	-c
0	0	0	0	0
-a	0	a	b	c
-b	0	b	0	b
-c	0	c	b	a

$\cdot$	0	a	b	c
0	0	0	0	0
-a	0	-a	-b	-c
-b	0	-b	0	-b
-c	0	-c	-b	-a

$\cdot$	0	-a	-b	-c
0	0	0	0	0
a	0	-a	-b	-c
b	0	-b	0	-b
c	0	-c	-b	-a

Define a mapping $S \times S \times S \to S$ by

$$(x_1, x_2, x_3) \mapsto \begin{cases} x_1^{-1} \cdot x_2 \cdot x_3^{-1} & ; \exists x_1^{-1}, x_3^{-1} \in S \left[x_1 \cdot x_1^{-1} = a = x_3 \cdot x_3^{-1}\right] \\ 0 & ;otherwise. \end{cases}$$

Then S is a ternary LA-semigroup. It is easy to see that $\{0, -b\}$ is a left ideal of S.

Lemma 3.5. *Let S be a ternary LA-semigroup with left identity. If A is a left ideal of S, then $B(AA)$ is a left ideal of S, where $\emptyset \neq A \subseteq S$.*

Proof: Suppose that S is a ternary LA-semigroup with left identity. Let A be a left ideal of S. Then by Lemma 2.12, we have

$$\begin{aligned} (SS)(B(AA)) &= B((SS)(AA)) \\ &= B(A((SS)A)) \\ &\subseteq B(AA) \end{aligned}$$

By Definition of left ideal, we get $B(AA)$ is a left ideal of S. □

Corollary 3.6. *Let S be a ternary LA-semigroup with left identity. If A is a left ideal of S, then $a(AA)$ is a left ideal of S, where $a \in S$.*

Proof: It is straightforward by Lemma 3.5. □

Definition 3.7. *If A is a left ideal of a ternary LA-semigroup S and $\emptyset \neq B, C \subseteq S$, then*

$$(A : B : C) = \{s \in S : (BC)s \subseteq A\}$$

and it is called the extension of A by B, C. Let $(A : a : b)$ stand for $(A : \{a\} : \{b\})$.

Proposition 3.8. *Let A be a left ideal of a ternary LA-semigroup S and $\emptyset \neq B, C, D, E \subseteq S$. Then the following statements hold.*

1. *$A \subseteq (A : a : b)$, where $a, b \in S$.*
2. *If $D \subseteq B$, then $(A : B : C) \subseteq (A : D : C)$.*
3. *If $E \subseteq C$, then $(A : B : C) \subseteq (A : B : E)$.*
4. *If $A \subseteq D$, then $(A : B : C) \subseteq (D : B : C)$.*
5. *If $B \subseteq A$, then $(A : B : C) = S$.*

6. *If $C \subseteq A$, then $(A : B : C) = S$.*

Proof: It is straightforward by Definition 3.7. $\square$

Proposition 3.9. *Let A be a left ideal of a ternary LA-semigroup with left identity S and $\emptyset \neq B, C \subseteq S$. Then the following statements hold.*

1. *If $B \subseteq A$, then $(A : B : C) = S$.*

2. *If $C \subseteq A$, then $(A : B : C) = S$.*

Proof: It is straightforward by Definition 3.7. $\square$

Lemma 3.10. *Let S be a ternary LA-semigroup with left identity. If A is a left ideal of S, then $(A : a : b)$ is a left ideal of S, where $a, b \in S$.*

Proof: Suppose that S is a ternary LA-semigroup with left identity. Let $r, s \in S$ and $x \in (A : a : b)$. Then $(ab)x \in A$. Then by Lemma 2.12, we have

$$\begin{array}{rcl} (ab)((rs)x) & = & (rs)((ab)x) \\ & \in & (rs)A \\ & \subseteq & A. \end{array}$$

Therefore $(rs)x \in (A : a : b)$ so that $(SS)(A : a : b) \subseteq (A : a : b)$. Hence $(A : a : b)$ is a left ideal in S. $\square$

Corollary 3.11. *Let S be a ternary LA-semigroup with left identity. If A is a left ideal of S, then $(A : B : C)$ is a left ideal of S, where $\emptyset \neq B, C \subseteq S$.*

Proof: It is straightforward by Definition 3.10. $\square$

Lemma 3.12. *Let S be a ternary LA-semigroup with left identity. If A is a left ideal of S, then $(SS)a$ is a left ideal of S, where $a \in S$.*

Proof: Suppose that S is a ternary LA-semigroup with left identity. Then by Definition of ternary LA-semigroup, we have

$$\begin{array}{rcl} (SS)((SS)a) & = & (((ee)S)S)((SS)a) \\ & = & ((SS)(ee))((SS)a) \\ & = & (((SS)a)(ee))(SS) \\ & = & (((ee)a)(SS))(SS) \\ & = & ((SS)(SS))((ee)a) \\ & = & (S((SS)S))((ee)a) \\ & = & (SS)a \end{array}$$

where $a \in S$. By Definition of left ideal, we get $(SS)a$ is a left ideal of S. $\square$

Definition 3.13. *Let S be a ternary LA-semigroup. A left ideal P is called quasi-prime if $(AB)C \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$ for all left ideals A, B and C in S.*

Example 3.14. *By Example 3.15, $S = \{0, -a, -b, -c\}$ is a ternary LA-semigroup with left identity a. Then $\{0, -b\}$ is a quasi-prime ideal of S.*

Theorem 3.15. *Let S be a ternary LA-semigroup with left identity. Then a left ideal P of S is quasi-prime if and only if $(ab)c \in P$ implies that $a \in P$ or $b \in P$ or $c \in P$, where $a, b, c \in S$.*

Proof: Suppose that S is a ternary LA-semigroup with left identity. Let P be a left ideal of S. Then by Definition of left ideal, we get

$$\begin{aligned}
(((SS)a)((SS)b))((SS)c) &= ((((SS)b)a)(SS))((SS)c) \\
&= (((ab)(SS))(SS))((SS)c) \\
&= (((SS)(SS))(ab))((SS)c) \\
&= (((SS)c)(ab))((SS)(SS)) \\
&= (((ab)c)(SS))((SS)(SS)) \\
&= (((SS)(SS))(SS))((ab)c) \\
&= ((((SS)S)S)(SS))((ab)c) \\
&= (((SS)S)S)((ab)c) \\
&= (SS)((ab)c) \\
&\subseteq (SS)P \\
&\subseteq P.
\end{aligned}$$

By Lemma 3.12, we get $(SS)a, (SS)b, (SS)c$ are left deals S so that $(ee)a \in (SS)a \subseteq P$ or $(ee)b \in (SS)b \subseteq P$ or $(ee)c \in (SS)c \subseteq P$. Conversely, assume that if $(ab)c \in P$, then $a \in P$ or $b \in P$ or $c \in P$, where $a, b, c \in S$. Suppose that $(AB)C \subseteq P$, where A, B and C are left ideals of S such that $A \not\subseteq P$ and $B \not\subseteq P$. Then there exist $a \in A, b \in B$ such that $a, b \in P$. Now consider $(ab)c \in (AB)C \subseteq P$, for all $c \in C$. So by hypothesis, $c \in P$ for all $c \in C$ implies that $C \subseteq P$. Hence P is a quasi-prime ideal in S. □

Example 3.16. *In the ternary LA-semigroup $\mathbb{Z}^-$ of all negative integers, the ideal $P = \{2x : x \in \mathbb{Z}^-\}$ is a quasi-prime ideal of $\mathbb{Z}^-$. But the ideal $Q = \{70x : x \in \mathbb{Z}^-\}$ is not a quasi-prime ideal of $\mathbb{Z}^-$, since $(-2)(-5)(-7) = -70 \in Q$ but $-2 \notin Q, -5 \notin Q$ and $-7 \notin Q$.*

Corollary 3.17. *Let S be a ternary LA-semigroup with left identity. Then a left ideal P of S is quasi-prime if and only if $a, bc \notin P$, implies that $(ab)c \notin P$, where $a, b, c \in S$.*

Proof: It is straightforward by Theorem 3.15. □

Theorem 3.18. *Let S be a ternary LA-semigroup with left identity. If A is a left ideal of S and P is a quasi-prime ideal of S, then $A \cap P$ is a quasi-prime ideal of A.*

Proof: Suppose that S is a ternary LA-semigroup with left identity. Clearly $A \cap P$ is a left ideal of A. Let $(ab)c \in A \cap P$. Then $(ab)c \in P$, since $A \cap P \subseteq P$. Since S is a quasi-prime ideal of S, we have $a \in P$ or $b \in P$ or $c \in P$. Therefore $a \in A \cap P$ or $b \in A \cap P$ or $c \in A \cap P$. Consequently, by Theorem 3.15, we get $A \cap P$ is a quasi-prime ideal in S. □

Theorem 3.19. *Let S be a ternary LA-semigroup with left identity. If P is a quasi-prime ideal of S, then $(P : a : b)$ is a quasi-prime ideal of S, where $a, b \in S$.*

Proof: Assume that P is a quasi-prime ideal of S. By Lemma 3.10, we have $(P : a : b)$ is a left ideal in S. Let $(xy)z \in (P : a : b)$. Then $(xy)((ab)z) = (ab)((xy)z) \in P$. By Theorem 3.15, we get $(ab)x \in (ab)P \subseteq P$ or $(ab)y \in (ab)P \subseteq P$ or $(ab)z \in P$. Therefore $x \in (P : a : b)$ or $y \in (P : a : b)$ or $z \in (P : a : b)$. Hence $(P : a : b)$ is a quasi-prime ideal of S. □

Corollary 3.20. *Let S be a ternary LA-semigroup with left identity. If P is a quasi-prime ideal of S, then $(P : a : b)$ is a quasi-prime ideal of S, where $a, b \in S$.*

Proof: It is straightforward by Definition 3.19. □

Theorem 3.21. *Let S be a ternary LA-semigroup with left identity. A left ideal P of S is a quasi-prime ideal if and only if $S - P$ is either ternary LA- subsemigroup of S or empty.*

Proof: Suppose that P is a quasi-prime ideal of S and $S - P \neq \emptyset$. Let $a, b, c \in S - P$. Then $a, b, c \notin P$. By Corollary 3.17, we get $(ab)c \notin P$ so $(ab)c \in S - P$. Hence $S - P$ is a ternary LA- subsemigroup of S. Conversely suppose that $S - P$ is either ternary LA- subsemigroup of S or empty. If $S - P$ is empty, then $S = P$ and hence P is a quasi-prime ideal of S. Assume that $S - P$ is a ternary LA-subsemigroup of S. Let $(ab)c \in P$. Then $(ab)c \notin S - P$. Since $S - P$ is a ternary LA- subsemigroup, we get $a \notin S - P$ or $b \notin S - P$ or $c \notin S - P$. Therefore $a \in P$ or $b \in P$ or $c \in P$. By Theorem 3.15, we have P of S is a quasi-prime ideal of S. □

4. Quasi-ideals of ternary LA-semigroups

The results of the following theorems seem to play an important role to study quasi–ideals in ternary LA-semigroups; these facts will be used frequently and normally we shall make no reference to this definition.

Definition 4.1. *A non-empty subset Q of a ternary LA-semigroup S is called a quasi-ideal of S if*

1. $(QS)S \cap (SQ)S \cap (SS)Q \subseteq Q$
2. $(QS)S \cap ((SS)Q)(SS) \cap (SS)Q \subseteq Q$.

Remark 4.2. *Let S is a ternary LA-semigroup.*

1. *Each quasi-ideal Q of S is a ternary LA-subsemigroup. In fact,* $(QQ)Q \subseteq (QS)S \cap (SQ)S \cap (SS)Q \subseteq Q$.
2. *Every left ideal of S is a quasi-ideal of S.*

Proposition 4.3. *Let S be a ternary LA-semigroup. If Q_i is a quasi-ideal of S, then $\bigcap_{i\in I} Q_i$ is a quasi-ideal of S.*

Proof: Suppose that A_i is a quasi-ideal of S. Then $(Q_iS)S \cap (SQ_i)S \cap (SS)Q_i \subseteq Q_i$ and $(Q_iS)S \cap ((SS)Q_i)(SS) \cap (SS)Q_i \subseteq Q_i$. Then by Definition of quasi-ideal, we get

$$\begin{aligned}((\bigcap_{i\in I} Q_i)S)S \cap (S(\bigcap_{i\in I} Q_i))S \cap (SS)\left(\bigcap_{i\in I} Q_i\right) &= \bigcap_{i\in I}((Q_iS)S) \cap \bigcap_{i\in I}((SQ_i)S) \\ &\quad \cap \bigcap_{i\in I}((SS)Q_i) \\ &\subseteq (Q_iS)S \cap (SQ_i)S \cap \\ &\quad (SS)Q_i \\ &\subseteq Q_i\end{aligned}$$

and

$$\begin{aligned}((\bigcap_{i\in I} Q_i)S)S \cap ((SS)\bigcap_{i\in I} Q_i)(SS) \cap (SS)(\bigcap_{i\in I} Q_i) &= \bigcap_{i\in I}((Q_iS)S) \cap \\ &\quad \bigcap_{i\in I}(((SS)Q_i)(SS)) \cap \\ &\quad \bigcap_{i\in I}((SS)Q_i) \\ &\subseteq (Q_iS)S \cap ((SS)Q_i)(SS) \\ &\quad \cap (SS)Q_i \\ &\subseteq Q_i.\end{aligned}$$

Therefore $((\bigcap_{i\in I} Q_i)S)S \cap (S(\bigcap_{i\in I} Q_i))S \cap (SS)(\bigcap_{i\in I} Q_i) \subseteq \bigcap_{i\in I} Q_i$ and

$$((\bigcap_{i\in I} Q_i)S)S \cap ((SS)\bigcap_{i\in I} Q_i)(SS) \cap (SS)(\bigcap_{i\in I} Q_i) \subseteq \bigcap_{i\in I} Q_i.$$

Hence $\bigcap_{i\in I} Q_i$ is a quasi-ideal of S. □

Theorem 4.4. *Let S be a ternary LA-semigroup with left identity. Then $S^2a \cap aS^2$ is a quasi-ideals of S, for every $a \in S$.*

Proof: Let S be a ternary LA-semigroup with left identity. Then by Definition of quasi-ideal, we get

$$\begin{aligned}(((SS)a \cap a(SS))S)S &= (((SS)a)S)S \cap ((a(SS))S)S \\ &= ((Sa)S^2)S \cap ((SS^2)a)S \\ &= (SS^2)(Sa) \cap (Sa)(SS^2) \\ &= (aS)(S^2S) \cap (S^2S)(aS) \\ &= (aS)S \cap S(aS) \\ &= (SS)a \cap a(SS).\end{aligned}$$

Therefore $((S^2a \cap aS^2)S)S \cap (S(S^2a \cap aS^2))S \cap S(S(S^2a \cap aS^2)) \subseteq S^2a \cap aS^2$ and $((S^2a \cap aS^2)S)S \cap (S^2(S^2a \cap aS^2))S^2 \cap S(S(S^2a \cap aS^2)) \subseteq S^2a \cap aS^2$. Hence $S^2a \cap aS^2$ is a quasi-ideals of S. $\square$

Theorem 4.5. *Let S be a ternary LA-semigroup with left identity. Then $A \cup S^2A$ is a quasi-ideals of S, for every $\emptyset \neq A \subseteq S$.*

Proof: Let S be a ternary LA-semigroup with left identity. Then

$$\begin{aligned}(((SS)a \cap a(SS))S)S &= (((SS)a)S)S \cap ((a(SS))S)S \\ &= ((Sa)S^2)S \cap ((SS^2)a)S \\ &= (SS^2)(Sa) \cap (Sa)(SS^2) \\ &= (aS)(S^2S) \cap (S^2S)(aS) \\ &= (aS)S \cap S(aS) \\ &= (SS)a \cap a(SS).\end{aligned}$$

Therefore $((S^2a \cap aS^2)S)S \cap (S(S^2a \cap aS^2))S \cap S(S(S^2a \cap aS^2)) \subseteq S^2a \cap aS^2$ and $((S^2a \cap aS^2)S)S \cap (S^2(S^2a \cap aS^2))S^2 \cap S(S(S^2a \cap aS^2)) \subseteq S^2a \cap aS^2$. Hence $S^2a \cap aS^2$ is a quasi-ideals of S. $\square$

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