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# Hausdorff Measure of Noncompactness of Matrix Mappings on Cesàro Spaces *

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ABSTRACT: In this study we establish some identities or estimates for operator norms and the Hausdorff measure of noncompactness of certain operators on the spaces $|C_\alpha|_k$, which have more recently been introduced in [22]. Further, by applying the Hausdorff measure of noncompactness, we determine the necessary and sufficient conditions for such operators to be compact and so the some well known results are generalized.

Key Words: Sequence spaces, Absolute Cesàro summability, Matrix transformations, BK spaces, Compact operator, Hausdorff measure of noncompactness.

## Contents



## 1. Background, notation and preliminaries

Any vector subspace of $w$ is called *a sequence space*, where $w$ is the set of all sequences of complex numbers. For $\ell_k$ $(k \geq 1,\ \ell_1 = \ell) \subset w$, we write the sets of all $k$-absolutely convergent series. Let $X$ and $Y$ be arbitrary subspaces of $w$ and $A = (a_{nv})$ be an arbitrary infinite matrix of complex numbers. If $x = (x_v) \in w$, then we denote $A$-transform of the sequence $x$ as the sequence $A(x) = (A_n(x))$, i.e., $A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v$, provided that the series converges for $v, n \geq 0$. Then, it is called that $A$ defines a matrix transformation from $X$ into $Y$, which is denoted by $A \in (X, Y)$ or $A : X \to Y$ if $Ax = (A_n(x)) \in Y$ for every $x \in X$, and also the sets

$$X_A = \{x \in w : A(x) \in X\} \tag{1.1}$$

is said to be the domain of the matrix $A$ in $X$. Also, $X$ is said to be an $BK$-space if it is a complete normed space with continuous coordinates $p_n : X \to \mathbb{C}$ defined by $p_n(x) = x_n$ for $n \geq 0$. Further, if $X$ and $Y$ are Banach spaces, then we write $\mathcal{B}(X, Y)$ for the set of all bounded linear operators $L : X \to Y$, which is a Banach space with the operator norm given by $\|L\| = \sup_{x \in S_X} \|L(x)\|_Y$ for all $L \in \mathcal{B}(X, Y)$, where $S_X$ denotes the unit sphere in $X$, that is $S_X = \{x \in X : \|x\| \leq 1\}$. A linear operator $L : X \to Y$ is called compact if its domain is all of $X$ and every bounded sequence $(x_n)$ in $X$, the sequence $(L(x_n))$ has a convergent subsequence in $Y$. We denote the class of such operators by $\mathcal{C}(X, Y)$.

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**Absolute Cesàro spaces**

Let $\Sigma x_n$ be an infinite series with partial sum $s_n$. Let $(\sigma_n^\alpha)$ and $(t_n^\alpha)$ be the nth Cesàro mean $(C,\alpha)$ of order $\alpha$ with $\alpha > -1$ of the sequence $(s_n)$ and $(na_n)$ respectively, *e.i.*,

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v,$$

and

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v$$

where

$$A_0^\alpha = 1,\ A_n^\alpha = \binom{\alpha+n}{n},\ A_{-n}^\alpha = 0,\ n \geq 1,$$

$|A_n^\alpha| \leq M(\alpha) n^\alpha$ for all $\alpha$, and $A_n^\alpha \geq m(\alpha) n^\alpha$ for $\alpha > -1$. The series $\Sigma x_n$ is said to be summable $|C,\alpha|_k$ with index $k \geq 1$ if (see [7])

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta \sigma_n^\alpha|^k < \infty, \tag{1.2}$$

where $\Delta\sigma_n^\alpha = \sigma_n^\alpha - \sigma_{n-1}^\alpha$ for $n \geq 0$, $\sigma_{-1}^\alpha = 0$. By using well known identity $t_n^\alpha = n\left(\Delta\sigma_n^\alpha\right)$ [10], condition (1.2) can be stated by

$$\sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty. \tag{1.3}$$

In the special case $k = 1$ and $\alpha = 0$, summability $|C,\alpha|_k$ reduces to summability $|C,\alpha|$ [6] and summability $|C,0|_k$, respectively.

In a more recent paper Sarıgöl [22] has introduced the space $|C_\alpha|_k$ for the case $\alpha > -1,\ k \geq 1$ as the set of all series summable by the method $|C,\alpha|_k$, and shown that it is also the domain of the matrix $T^{\alpha,k} = (t_{nv}^{\alpha,k})$ in the space $l_k$, the space of all $k$-absolutely convergent series, where $t_{00}^{\alpha,k} = 1$ and

$$t_{nv}^{\alpha,k} = \begin{cases} \dfrac{v A_{n-v}^{\alpha-1}}{n^{1/k} A_n^\alpha}, & 1 \leq v \leq n, \\ 0, & v > n. \end{cases} \tag{1.4}$$

And also some topological structures of the space $|C_\alpha|_k$ have been investigated and some related matrix mappings have been characterized. We refer the reader to [22] for relevant terminology, which also extend some well known results of Flett [7], Orhan & Sarıgöl [17], Bosanquet [3], Mehdi [16], Mazhar [15]. Besides, the problems of absolute summability factors and comparison of these methods is studied by many authors in [3-7, 15-17, 25-30] and the important sequence spaces on the matrix domains have been examined by several authors in [1-2, 8-9, 12, 18-20].

Further, Das [5] defined a matrix $A$ to be absolutely kth power conservative, $k \geq 1$, if $A \in B(\mathcal{A}_k, \mathcal{A}_k)$, where

$$\mathcal{A}_k = \left\{ s = (s_v) : \sum_{v=1}^{\infty} v^{k-1} |\Delta s_v|^k < \infty \right\},$$

and proved every conservative Hausdorff matrix $H \in B(\mathcal{A}_k, \mathcal{A}_k)$. Note that there exists a relation between $\mathcal{A}_k$ and $|C_0|_k$ obtained in the special case $\alpha = 0$ if $A$ lower triangular matrix. In fact, $x \in |C_0|_k$ if and only if $s \in \mathcal{A}_k$, and so $A \in (\mathcal{A}_k, \mathcal{A}_k)$ iff $\widetilde{A} \in (|C_0|_k, |C_0|_k)$, where

$$\widetilde{a}_{nv} = \begin{cases} \sum_{r=v}^{n} (a_{nr} - a_{n-1,r}), & 0 \leq v \leq n \\ 0, & v > n. \end{cases} \tag{1.5}$$

For real number $\alpha$ and nonnegative integers $n$ we use the notation

$$\Delta^{\alpha} x_n = \sum_{v=n}^{\infty} A_{v-n}^{-\alpha-1} x_v,$$

whenever the series convergent and throughout the paper $k^*$ denotes the conjugate of $k > 1$, i.e., $1/k + 1/k^* = 1$, and $1/k^* = 0$ for $k = 1$.

The following known results play important roles in our investigation.

**Lemma 1.1.** [31] Let $1 < k < \infty$. Then, $A \in (\ell_k, \ell)$ if and only if

$$\|A\|_{(\ell_k,\ell)} = \sup_N \left\{ \sum_{v=0}^{\infty} \left| \sum_{n \in N} a_{nv} \right|^{k^*} \right\}^{1/k^*} < \infty,$$

where $N$ is any finite set of positive numbers.

However following lemma is more useful in many cases, which gives equivalent norm.

**Lemma 1.2.** [23] Let $1 < k < \infty$. Then, $A \in (\ell_k, \ell)$ if and only if

$$\|A\|^*_{(\ell_k,\ell)} = \left\{ \sum_{v=0}^{\infty} \left( \sum_{n=0}^{\infty} |a_{nv}| \right)^{k^*} \right\}^{1/k^*} < \infty,$$

and there exists $1 \leq \xi \leq 4$ such that $\|A\|_{(\ell_k,\ell)} = \frac{1}{\xi} \|A\|^*_{(\ell_k,\ell)}$.

The second part of this lemma is easily seen by following the lines in [23] that

$$\|A\|_{(\ell_k,\ell)} \leq \|A\|^*_{(\ell_k,\ell)} \leq 4 \|A\|_{(\ell_k,\ell)}.$$

**Lemma 1.3.** [11] Let $1 \leq k < \infty$. Then, $A \in (\ell, \ell_k)$ if and only if

$$\|A\|_{(\ell,\ell_k)} = \sup_v \left\{ \sum_{n=0}^{\infty} |a_{nv}|^k \right\}^{1/k} < \infty.$$

**Lemma 1.4.** Let $\alpha > -1$ and $1 \leq k < \infty$, then, $|C_\alpha|_k$ is norm isomorphic to the space $l_k$, i.e., $|C_\alpha|_k \cong l_k$.

**Proof.** To prove the theorem, we need a linear bijection preserving the norm between $|C_\alpha|_k$ and $l_k$. Now, consider transformation $T^{\alpha,k} : |C_\alpha|_k \to l_k$ defined by

$$T_0^{\alpha,k}(x) = x_0, T_n^{\alpha,k}(x) = \frac{1}{n^{1/k} A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v x_v. \tag{1.6}$$

The linearity of $T^{\alpha,k}$ is obvious. Furthermore, it is trivial that if $T^{\alpha,k}(x) = 0$, then $x = \theta$. So, $T^{\alpha,k}$ is injective. Let $T^{\alpha,k}(x) = y \in l_k$ be given, and take the sequence $x$ as

$$x_0 = y_0, \ x_n = \frac{1}{n} \sum_{v=1}^{n} v^{1/k} A_{n-v}^{-\alpha-1} A_v^\alpha y_v, \ n \geq 1.$$

Then we have $\|x\|_{|C_\alpha|_k} = \|y\|_{l_k} < \infty, 1 \leq k < \infty$, which gives $x \in |C_\alpha|_k$. Thus, $T^{\alpha,k}$ is surjective and norm preserving, which completes the proof.

### Hausdorff measure of noncompactness

If $S$ and $H$ are subsets of a metric space $(X, d)$ and $\varepsilon > 0$ then $S$ is called an $\varepsilon$-net of $H$ , if, for every $h \in H$, there exists an $s \in S$ such that $d(h, s) < \varepsilon$; if $S$ is finite, then the $\varepsilon$-net $S$ of $H$ is called a finite $\varepsilon$-*net* of $H$. If $Q$ is a bounded subset of the metric space $X$, then the Hausdorff measure of noncompactness of $Q$ is defined by

$$\chi(Q) = \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon\text{-net in } X\},$$

and $\chi$ is called the Hausdorff measure of noncompactness.

The following result is an important tool to compute the Hausdorff measure of noncompactness of a bounded subset of the $BK$ space $\ell_k, k \geq 1$.

**Lemma 1.5.** [21] Let $Q$ be a bounded subset of the normed space $X$ where $X = \ell_k$, for $1 \leq k < \infty$ or $X = c_0$.If $P_n : X \to X$ is the operator defined by $P_r(x) = (x_0, x_1, ..., x_r, 0, ...)$ for all $x \in X$, then

$$\chi(Q) = \lim_{r\to\infty} \left( \sup_{x\in Q} \|(I - P_r)(x)\| \right),$$

where $I$ is the identity operator on $X$.

If $X$ and $Y$ be Banach spaces and $\chi_1$ and $\chi_2$ be Hausdorff measures on $X$ and $Y$, then, the linear operator $L : X \to Y$ is said to be $(\chi_1, \chi_2)$-bounded if $L(Q)$ is bounded subset of $Y$ for every bounded subset of $X$ and there exists a positive constant $M$ such that $\chi_2(L(Q)) \leq M \chi_1(Q)$ for every bounded $Q$ of $X$. If an operator $L$ is $(\chi_1, \chi_2)$-bounded then the number $\|L\|_{(\chi_1,\chi_2)} = \inf\{M > 0 : \chi_2(L(Q)) \leq M\chi_1(Q) \text{ for all bounded } Q \subset X\}$ is called the $(\chi_1, \chi_2)$-measure of noncompactness of $L$. In particular, we write $\|L\|_{(\chi,\chi)} = \|L\|_\chi$ for $\chi_1 = \chi_2 = \chi$.

**Lemma 1.6.** [14] Let $X$ and $Y$ be Banach spaces, $L \in B(X,Y)$ and $S_X = \{x \in X : \|x\| \leq 1\}$ denote the unit sphere in $X$. Then,

$$\|L\|_\chi = \chi(L(S_X)),$$

and

$$L \in \mathcal{C}(X,Y) \text{ if and only if } \|L\|_\chi = 0.$$

**Lemma 1.7.** [13] Let $X$ be normed sequence space and $\chi_T$ and $\chi$ denote the Hausdorff measures of noncompactness on $\mathcal{M}_{X_T}$ and $\mathcal{M}_X$, the collections of all bounded sets in $X_T$ and $X$, respectively. Then, $\chi_T(Q) = \chi(T(Q)$ for all $Q \in \mathcal{M}_{X_T}$, where $T = (t_{nv})$ is a triangular infinite matrix.

## 2. Main results and their applications

In this study, we establish some identities or estimates for operator norms and the Hausdorff measure of noncompactness of certain matrix operators on $|C_\alpha|_k$ and also give the necessary and sufficient conditions for such operators to be compact by applying the Hausdorff measure of noncompactness, and so the some well known results are generalized.

**Theorem 2.1.** Let $\alpha > -1, \delta > -1$, $1 \leq k < \infty$ and $D = (d_{nv})$ be given by

$$d_{nv} = \begin{cases} \dfrac{1}{n^{1/k} A_n^\delta} \sum_{j=1}^n j A_{n-j}^{\delta-1} a_{j0}, & n \geq 1, v = 0 \\ \dfrac{v A_v^\alpha}{n^{1/k} A_n^\delta} \sum_{j=1}^n j A_{n-j}^{\delta-1} \Delta^\alpha \left(\frac{a_{jv}}{v}\right), & n \geq 1, v \geq 1. \end{cases} \tag{2.1}$$

If $A \in (|C_\alpha|, |C_\delta|_k)$, then

$$\|A\|_{(|C_\alpha|,|C_\delta|_k)} = \sup_v \left\{ \sum_{n=0}^\infty |d_{nv}|^k \right\}^{1/k} \tag{2.2}$$

and

$$\|A\|_\chi = \lim_{r\to\infty} \sup_v \left\{ \sum_{n=r+1}^\infty |d_{nv}|^k \right\}^{1/k}. \tag{2.3}$$

**Proof.** Consider the map $T^{\alpha,1} : |C_\alpha| \to l$ and $T^{\delta,k} : |C_\delta|_k \to l_k$ defined by (1.4) for $k = 1$ and

$$T_0^{\delta,k}(x) = x_0, T_n^{\delta,k}(x) = \frac{1}{n^{1/k} A_n^\delta} \sum_{v=1}^n A_{n-v}^{\delta-1} v x_v \tag{2.4}$$

respectively. Then, by Lemma 1.4, $x \in |C_\alpha|$ iff $y = T^{\alpha,1}(x) \in l$, and so $\|x\|_{|C_\alpha|} = \|y\|_l$. Hence, since $A = \left(T^{\delta,k}\right)^{-1} oDoT^{\alpha,1}$, it is clear that for all $x \in |C_\alpha|$ and $y \in l$,

$$\begin{aligned} \|A\|_{(|C_\alpha|,|C_\delta|_k)} &= \sup_{x\neq\theta} \frac{\left\|(DoT^{\alpha,1})(x)\right\|_{l_k}}{\|x\|_{|C_\alpha|}} \\ &= \sup_{y\neq\theta} \frac{\|D(y)\|_{l_k}}{\|y\|_l} = \|D\|_{(\ell,\ell_k)} \end{aligned}$$

which completes the asserted by Lemma 1.3.

Finally, let $S=\{x\in|C_\alpha|:\|x\|\leq 1\}$. Then, by Lemma 1.5-Lemma 1.7, it follows that

$$\begin{aligned}\|A\|_\chi &= \chi(AS)=\chi\left(T^{\delta,k}AS\right)=\chi\left(DT^{\alpha,1}S\right)\\ &= \lim_{r\to\infty}\sup_{y\in T^{\alpha,1}S}\|(I-P_r)D(y)\|_{l_k}=\lim_{r\to\infty}\sup_v\left\{\sum_{n=r+1}^{\infty}|d_{nv}|^k\right\}^{1/k}\end{aligned}$$

where $P_r:l_k\to l_k$ is defined by $P_r(y)=(y_0,y_1,...,y_r,0,...)$. This proves the theorem together with Lemma 1.3.

Now, by combining this theorem with Lemma 1.6, we can characterize the compact operators in the class $(|C_\alpha|,|C_\delta|_k)$.

**Corollary 2.2.** Under hypotheses of Theorem 2.1,

$$A\in\mathcal{C}\left(|C_\alpha|,|C_\delta|_k\right)\text{ if and only if }\lim_{r\to\infty}\sup_v\sum_{n=r+1}^{\infty}|d_{nv}|^k=0.$$

If $A$ is chosen as a diagonal matrix, i.e., $a_{nn}=\varepsilon_n$, zero otherwise, then $A\in(|C_\alpha|,|C_\delta|_k)$ states the form of summability factors that $\Sigma\varepsilon_v x_v$ is summable $|C_\delta|_k$ when $\Sigma x_v$ is summable $|C_\alpha|$, and also $I\in(|C_\alpha|,|C_\delta|_k)$ means the comparisons of these methods, i.e., $|C_\alpha|\subset|C_\delta|_k$, where $I$ is identity matrix. For the case $\alpha,\delta>-1$ and $A=I$, Theorem 2.1 reduces to the following result, which includes the norm of operators characterizing the class of Flett [7] and shows that this operators are not compact.

**Corollary 2.3.** *If $\alpha>-1$, $\delta>\alpha+1/k^*$ and $k\geq 1$, then $|C_\alpha|\subset|C_\delta|_k$, i.e., $I\in(|C_\alpha|,|C_\delta|_k)$,*

$$\|I\|_{(|C_\alpha|,|C_\delta|_k)}=\sup_v vA_v^\alpha\left\{\sum_{n=v}^{\infty}\left(\frac{A_{n-v}^{\delta-\alpha-1}}{n^{1/k}A_n^\delta}\right)^k\right\}^{1/k}$$

*and*

$$I\notin\mathcal{C}\left(|C_\alpha|,|C_\delta|_k\right).$$

**Proof.** Let $r$ be given and $\sigma(v,r)=vA_v^\alpha\left\{\sum_{n=r+1}^{\infty}\left(\frac{A_{n-v}^{\delta-\alpha-1}}{n^{1/k}A_n^\delta}\right)^k\right\}^{1/k}$ for $v\geq 1$. Then, by Theorem 2.1, $\|L_I\|_\chi=\lim_{r\to\infty}\sup_v\sigma(v,r)$. Now, take into account $A_n^\alpha\leq M(\alpha)n^\alpha$ for all $\alpha$ and $A_n^\alpha\geq m(\alpha)n^\alpha$ for $\alpha>-1,n\geq 1$, and $\delta>\alpha+1/k^*$, we get, for $v=r+1$,

$$\sigma(v,v-1)\geq m_1\sum_{n=v}^{\infty}\frac{A_{n-v}^{(\delta-\alpha-1)k}}{nA_n^{(\delta k)}}$$

$$= m_1 \int_0^1 (1-x)^{\delta k} x^{v-1} \sum_{n=v}^{\infty} A_{n-v}^{(\delta-\alpha-1)k} x^{n-v} dx \tag{2.5}$$

$$= m_1 \int_0^1 (1-x)^{\delta k} x^{v-1} (1-x)^{-\delta k+\alpha k+k-1} dx \geq \frac{m_1}{v^{\alpha k+k}}$$

which gives $\sigma(v,r) \geq m_1$, where $m_1$ is a positive constant not always the same in (2.5) different occasion. The term-by-term integration is legitimate, since everything is positive. Thus, we obtain $\|L_I\|_\chi \neq 0$, which proves the result by Lemma 1.6.

If it is chosen that $\delta \geq 0$ and $\alpha$ is nonnegative integer, $k \geq 1$ and $A$ is a diagonal matrix with $a_{nn} = \varepsilon_n$, zero otherwise, in Theorem 2.1, then we can obtain following result, in which matrix transformation also characterized by Bosanquet [4] and Mehdi [16] for $k = 1$ and $k \geq 1$, respectively.

**Corollary 2.4.** [16] If $A \in (|C_\alpha|, |C_\delta|_k)$ for $k \geq 1$, $\delta \geq 0$ and nonnegative integer $\alpha$ , then

$$\|L_A\|_{(|C_\alpha|,|C_\delta|_k)} = \sup_v v A_v^\alpha \left\{ \sum_{n=v}^{\infty} \left| \sum_{j=v}^{n} \frac{A_{n-j}^{\delta-1} A_{j-v}^{-\alpha-1} \varepsilon_j}{n^{1/k} A_n^\delta} \right|^k \right\}^{1/k}.$$

Now, from a different point of view, let $(p_n)$ and $(q_n)$ be two positive sequence with $P_n = p_0 + p_1 + \cdots + p_n \to \infty$ and $Q_n = q_0 + q_1 + \cdots + q_n \to \infty$ as $n \to \infty$. By following lines in [17] , it is easy to see that $|R_p| \subset |R_q|_k$ if and only if $A \in (|C_0|, |C_0|_k)$ , where $A$ defined by

$$a_{nv} = \begin{cases} \frac{q_n}{Q_n Q_{n-1}} \left( Q_v - \frac{q_v P_v}{p_v} \right), & 1 \leq v \leq n-1 \\ \frac{q_v P_v}{Q_v p_v}, & v = n \\ 0, & n > v. \end{cases} \tag{2.6}$$

and $|R_q|_k$ is the set of series summable absolute weighted mean, i.e.,

$$|R_p|_k = \left\{ x = (a_v) : \sum_{n=1}^{\infty} n^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v \right|^k < \infty \right\}, k \geq 1$$

So, by Theorem 2.1, we determine exactly or estimate the norms and Hausdorff measure of noncompactness of bounded matrix operators charaterized by Orhan and Sarıgöl [17] , which includes that of Bosanquet [3] and Sunouchi [32] for the case $k = 1$.

**Corollary 2.5.** If $1 \leq k < \infty$ and $I \in (|R_p|, |R_q|_k)$ , *i.e.* $|R_p| \subset |R_q|_k$ , then,

$$\|L_I\|_{(|R_p|,|R_q|_k)} = \sup_v \left\{ \sum_{n=v}^{\infty} \left| n^{1/k^*} a_{nv} \right|^k \right\}^{1/k}, \tag{2.7}$$

$$\|L_I\|_\chi = \lim_{r\to\infty}\sup_v \left\{\sum_{n=r+1}^{\infty}\left|n^{1/k^*}a_{nv}\right|^k\right\}^{1/k}. \tag{2.8}$$

**Proof.** Take $\alpha=\delta=0$ in Theorem 2.1. If $I\in\left(|R_p|,|R_q|_k\right)$, then $A\in(|C_0|,|C_0|_k)$, where $A$ is defined by (2.6), and so, (2.2) and (2.3) reduce to (2.7) and (2.8), respectively.

**Corollary 2.6.** [24] If A is a triangular infinite matrix, then $A\in(\mathcal{A}_1,\mathcal{A}_k)$, $k\geq 1$, if and only if

$$\|L_A\|_{(\mathcal{A}_1,\mathcal{A}_k)} = \sup_v\left\{\sum_{n=v}^{\infty} n^{k-1}\left|\hat{a}_{nv}\right|^k\right\}^{1/k} < \infty, \tag{2.9}$$

and

$$\left\|L_{\widehat{A}}\right\|_\chi = \lim_{r\to\infty}\sup_v\sum_{n=r+1}^{\infty} n^{k-1}\left|\hat{a}_{nv}\right|^k.$$

**Proof.** In Theorem 2.1, take $\alpha=\delta=0$. If $\widetilde{A}$ is defined by (1.5), then, $A\in(\mathcal{A}_1,\mathcal{A}_k)$ iff $\widetilde{A}\in(|C_0|,|C_0|_k)$. On the other hand, it is obvious that the condition (2.2) are reduced to (2.9), which completes the proof.

**Theorem 2.7.** Let $\alpha,\delta>-1, 1<k<\infty$ and $\widetilde{D}=\left(\widetilde{d}_{nv}\right)$ be given by

$$\widetilde{d}_{nv} = \begin{cases} \dfrac{1}{nA_n^\delta}\sum\limits_{i=1}^{n}A_{n-i}^{\delta-1}ia_{i0}, & n\geq 1, v=0\\ \dfrac{v^{1/k}A_v^\alpha}{nA_n^\delta}\sum\limits_{i=1}^{n}A_{n-i}^{\delta-1}i\Delta^\alpha\left(\dfrac{a_{iv}}{v}\right), & n\geq 1, v\geq 1.\end{cases} \tag{2.10}$$

If $A\in(|C_\alpha|_k,|C_\delta|)$, then there exists $1\leq\xi\leq 4$ such that

$$\|A\|_{\left(|C_\alpha|_k,|C_\delta|\right)} = \frac{1}{\xi}\left\{\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}\left|\widetilde{d}_{nv}\right|\right)^{k^*}\right\}^{1/k^*}, \tag{2.11}$$

$$\|A\|_\chi = \frac{1}{\xi}\lim_{r\to\infty}\left\{\sum_{v=0}^{\infty}\left(\sum_{n=r+1}^{\infty}\left|\widetilde{d}_{nv}\right|\right)^{k^*}\right\}^{1/k^*}. \tag{2.12}$$

**Proof.** Consider the maps $T^{\alpha,k}:|C_\alpha|_k\to l_k$ and $T^{\delta,1}:|C_\delta|\to l$ defined by (1.4) and (2.4) for $k=1$, respectively. Then, $x\in|C_\alpha|_k$ iff $y=T^{\alpha,k}(x)\in l_k$, and so $\|x\|_{|C_\alpha|_k}=\|y\|_{l_k}$. Hence, since $A=\left(T^{\delta,1}\right)^{-1}o\widetilde{D}oT^{\alpha,k}$, it is clear from Lemma 1.4 for all $x\in|C_\alpha|_k$ and $y\in l_k$,

$$\begin{aligned}\|A\|_{\left(|C_\alpha|_k,|C_\delta|\right)} &= \sup_{x\neq\theta}\frac{\left\|\left(T^{\delta,1}\right)^{-1}o\widetilde{D}oT^{\alpha,k}(x)\right\|_{|C_\delta|}}{\left\|T^{\alpha,k}(x)\right\|_{l_k}}\\ &= \left\|\widetilde{D}\right\|_{(l_k,l)} = \frac{1}{\xi}\left\|\widetilde{D}\right\|^*_{(\ell_k,\ell)} = \frac{1}{\xi}\left\{\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}\left|\widetilde{d}_{nv}\right|\right)^{k^*}\right\}^{1/k^*}\end{aligned}$$

which is desired by Lemma 1.2.

Finally, $S = \{x \in |C_\alpha|_k : \|x\| \leq 1\}$. Then, by considering Lemma 1.5-Lemma 1.7, and Lemma 1.2, we get that there exists $1 \leq \xi \leq 4$ such that

$$\begin{aligned} \|A\|_\chi &= \chi\left(T^{\delta,1}AS\right) = \chi\left(\widetilde{D}T^{\alpha,k}S\right) \\ &= \lim_{r\to\infty} \sup_{y\in T^{\alpha,k}S} \left\|\left(I-P_r\right)\widetilde{D}(y)\right\|_l \\ &= \lim_{r\to\infty} \left\|\widetilde{D}^{(r)}\right\|_{(l_k,l)} = \frac{1}{\xi}\lim_{r\to\infty}\left\|\widetilde{D}^{(r)}\right\|^*_{(l_k,l)} \\ &= \frac{1}{\xi}\lim_{r\to\infty}\left\{\sum_{v=0}^{\infty}\left(\sum_{n=r+1}^{\infty}\left|\widetilde{d}_{nv}\right|\right)^{k^*}\right\}^{1/k^*} \end{aligned}$$

where $P_r : l \to l$ is defined by $P_r(y) = (y_0, y_1, ..., y_r, 0, ...)$ and $\widetilde{D}^{(r)} = \left(\widetilde{d}_{nv}^{(r)}\right)$ is defined by

$$\widetilde{d}_{nv}^{(r)} = \begin{cases} 0, & 0 \leq n \leq r \\ \widetilde{d}_{nv}, & n > r \end{cases},$$

which proves the theorem together with Lemma 1.2. From Theorem 2.7 we have

**Corollary 2.8.** Under hypotheses of Theorem 2.7, $A \in \mathcal{C}\left(|C_\alpha|_k, |C_\delta|\right)$ if and only if

$$\|A\|_\chi = \lim_{r\to\infty}\left\{\sum_{v=0}^{\infty}\left(\sum_{n=r+1}^{\infty}\left|\widetilde{d}_{nv}\right|\right)^{k^*}\right\}^{1/k^*} = 0.$$

By Theorem 2.7, we determine exactly or estimate the norms and Hausdorff measure of noncompactness of bounded matrix operators charaterized by Mazhar [15].

**Corollary 2.9.** Let $\alpha \geq 0,\ k > 1$. If $A \in \left(|C_\alpha|_k, |C_1|\right)$, then there exists $1 \leq \xi \leq 4$ such that

$$\|L_W\|_{\left(|C_\alpha|_k,|C_1|\right)} = \frac{1}{\xi}\left\{\sum_{j=1}^{\infty}\left(\sum_{n=j}^{\infty}\left|\sum_{v=j}^{n}\frac{j^{1/k}A_j^\alpha A_{v-j}^{-\alpha-1}\varepsilon_v}{n(n+1)}\right|\right)^{k^*}\right\}^{1/k^*} < \infty,$$

where $A$ is a diagonal matrix.

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