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Involute-Evolute D-Curves in Minkowski 3-Space E_1^3

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abstract: In this paper, we examine the notion of the involute-evolute curves for the curves lying the surfaces in Minkowski 3-space E_1^3. We call these new associated curves as involute-evolute and by using the Darboux frame of the curves. We give the representation formulae for spacelike curves in Minkowski 3-space E_1^3 and using this formulae we give some characterizations of these curves. Besides, we find the relations between the normal curvatures, the geodesic curvatures and the geodesic torsions of these curves.

Key Words: Involute-evolute curve, darboux frame, normal curvature, geodesic curvature.

Contents

1. Introduction

The theory of curves has been one of the exciting subject because of having many application area from geometry to the different branch of science. In differential gometry, there are many important consequences and properties in the theory of the curves. Hence a lot of researchers follow labours about the curves. C. Huygens, who is also known for his works in optics, discovered involutes while trying to build a more accurate clock, [4]. In [5], the relations Frenet apparatus of involute-evolute curve couple in the space E^3 were given. In [11], A. Turgut and E. Erdoğan examined involute-evolute curve couple in the space E^n. In [3], O. Bektas and S. Yuce examined special involute-evolute curve couple by taking into account the Darboux frames of them and gived some examples in the space E^3.

In this study, we consider the notion of the involute-evolute curves in Minkowski $3-$space E_1^3. We give the representation formulae for spacelike curves in Minkowski $3-$space E_1^3. By using the Darboux frame of the curves we obtain the necessary and sufficient conditions between k_g, k_n, τ_g and k_{g_*}, k_{n_*} and we also give some theorems, results and example.

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2. Preliminaries

The Minkowski 3-space E_1^3 is real vector space $\mathbb{R}^3$ provided with the standart flat metric given as following

$$\langle , \rangle = dy_1^2 + dy_2^2 - dy_3^2,$$

where (y_1, y_2, y_3) is a coordinate system of E_1^3. A vector V on E_1^3 is called spacelike if $\langle V, V\rangle > 0$ or $V = 0$, timelike if $\langle V, V\rangle < 0$ and null if $\langle V, V\rangle = 0$ and $V \neq 0$.

Let M be an oriented surface in 3-dimensional Minkowski space E_1^3, and let consider a non-null curve $y(s)$ lying on M fully. Since the curve $y(s)$ is also in space, there exists Frenet frame $\{T, N, B\}$ at each points of the curve where $\overrightarrow{T}$ is unit tangent vector, $\overrightarrow{N}$ is principal normal vector and $\overrightarrow{B}$ is binormal vector, respectively. Since the curve $y(s)$ lies on the surface M, there exists another frame of the curve $y(s)$ which is called Darboux frame and denoted by $\{T, g, n\}$. In this frame $\overrightarrow{T}$ is the unit tangent of the curve, $\overrightarrow{n}$ is the unit normal of the surface M and $\overrightarrow{g}$ is a unit vector given by $\overrightarrow{g} = \overrightarrow{n} \times \overrightarrow{T}$. Since the unit tangent $\overrightarrow{T}$ is common in both Frenet frame and Darboux frame, the vectors $\overrightarrow{N}$, $\overrightarrow{B}$, $\overrightarrow{g}$ and $\overrightarrow{n}$ lie on the same plane.

Suppose that the surface M is an oriented spacelike surface, then the curve $y(s)$ lying on M is a spacelike curve. So, the relations between the frames can be given as follows:

$$\begin{bmatrix} T \\ g \\ n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

in all cases, φ is the angle between the vectors $\overrightarrow{g}$ and $\overrightarrow{N}$.

Thus, the derivative formulae of the Darboux frame of $y(s)$ is given by

$$\begin{bmatrix} T' \\ g' \\ n' \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ k_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ g \\ n \end{bmatrix} \tag{2.1}$$

$$\langle T, T\rangle = \langle g, g\rangle = 1, \langle n, n\rangle = -1. \tag{2.2}$$

In this formulae, k_g, k_n, τ_g are called the geodesic curvature, the normal curvature and geodesic torsion, respectively. Furthermore, the relations between geodesic curvature, the normal curvature and geodesic torsion and κ, τ are given as follows:

i) $k_g = \kappa\cos\varphi$, $k_n = \kappa\sin\varphi$, $\tau_g = \tau + \frac{d\varphi}{ds}$ if both M and $y(s)$ are timelike or spacelike.

ii) $k_g = \kappa\cosh\varphi$, $k_n = \kappa\sinh\varphi$, $\tau_g = \tau + \frac{d\varphi}{ds}$ if M is timelike and $y(s)$ is spacelike, [9].

Furthermore, the geodesic curvature k_g and geodesic torsion τ_g of the curve $y(s)$ can be calculated as follows:

$$k_g = \langle \frac{dy}{ds}, \frac{d^2y}{ds^2} \times n\rangle, \tau_g = \langle \frac{dy}{ds}, n \times \frac{dn}{ds}\rangle.$$

In the differential geometry of surfaces, for a curve $y(s)$ lying on a surface M, the followings are well-known

i) $y(s)$ is a geodesic curve $k_g = 0$,
ii) $y(s)$ is an asymptotic line $k_n = 0$,
iii) $y(s)$ is a principal line $\tau_g = 0$, [9].

3. Representation Formulae of Spacelike Curves in Minkowski 3-Space E_1^3

Theorem 3.1. *Let $y(s)$ be a spacelike curve in Minkowski 3−space E_1^3 with arc length parameter s. Then $y = (y_1, y_2, y_3)$ can be given as follows*

$$y(s) = (\frac{1}{2}\mu(s)(g - g^{-1}) + ag^{-1}, a - \mu(s), ag^{-1} - \frac{1}{2}\mu(s)(g + g^{-1})) \tag{3.1}$$

$$\mu(s) = g^{(\frac{1}{2a}-1)}\left(-\frac{1}{2a}\int \frac{g^{2-\frac{1}{2a}}}{g_s}ds + k\right), \tag{3.2}$$

where $a, k \in \mathbb{R}_0$ and $g(s) \neq cons\tan t$.

Proof. Let $y(s)$ be a spacelike curve in Minkowski 3−space E_1^3 with arc length parameter s. We write $y = (y_1, y_2, y_3)$ and have

$$y_1^2 + y_2^2 - y_3^2 = a^2$$

where $a \in \mathbb{R}_0$. From $y_1^2 - y_3^2 = a^2 - y_2^2$, we can write

$$\frac{y_1 - y_3}{a - y_2} = \frac{a + y_2}{y_1 + y_3}. \tag{3.3}$$

Furthermore, for a curve y with $y(s) = (y_1, y_2, y_3)$, we can suppose that

$$\frac{y_1 - y_3}{a - y_2} = \frac{a + y_2}{y_1 + y_3} = g(s), \tag{3.4}$$

and

$$y_2 = a - \mu(s). \tag{3.5}$$

By considering 3.4 and 3.5, we get

$$y_1 = \frac{1}{2}\mu(s)(g - g^{-1}) + g^{-1}.a,$$

$$y_3 = g^{-1}.a - \frac{1}{2}\mu(s)(g + g^{-1}). \tag{3.6}$$

That is, we can write the curve y as following

$$y(s) = (\frac{1}{2}\mu(s)(g - g^{-1}) + ag^{-1}, a - \mu(s), ag^{-1} - \frac{1}{2}\mu(s)(g + g^{-1})).$$

If s is the arc length parameter of the curve $y(s)$, we have

$$\mu_s + \mu(\frac{2a-1}{2a})g_s g^{-1} + \frac{1}{2a} g g_s^{-1} = 0. \tag{3.7}$$

Therefore, we obtain

$$\mu = g^{(\frac{1}{2a}-1)} \left(-\frac{1}{2a} \int \frac{g^{2-\frac{1}{2a}}}{g_s} ds + k \right),$$

for non constant function $g(s)$ and where $k \in \mathbb{R}_0^+$. □

4. Involute-Evolute D-Curves in Minkowski 3-Space E_1^3

In this section, we define involute-evolute spacelike curves and give some characterizations of these curves by considering the Darboux frame.

Definition 4.1. *Let M and M_* be oriented surfaces in Minkowski 3-space E_1^3 and let consider the arc length parameter spacelike curves $y(s)$ and $y_*(s_*)$ lying fully on M and M_*, respectively. Denote the Darboux frames of $y(s)$ and $y_*(s_*)$ by $\{T, g, n\}$ and $\{T_*, g_*, n_*\}$, respectively. If there exists a corresponding relationship between the curves y and y_* such that at the corresponding points of the curves, the Darboux frame element T of y coincides with the Darboux frame element g_* of $y_*(s_*)$. Then $y_*(s_*)$ is called the involute D-curve of $y(s)$ (y is called the evolute D-curve of $y_*(s_*)$) if $\langle T, T_* \rangle = 0$, then $\{y(s), y_*(s_*)\}$ is said to be a involute-evolute D-pair.*

Theorem 4.2. *Let $y(s)$ and $y_*(s_*)$ be two spacelike curves in the Minkowski 3-space E_1^3. If the $\{y(s), y_*(s_*)\}$ is a involute-evolute D-pair, then*

$$y_*(s_*) = y(s(s_*)) + (-s + c)T(s(s_*)), c \in \mathbb{R}_0. \tag{4.1}$$

Proof. Suppose that the pair $\{y, y_*\}$ is an involute-evolute D-pair. Denote the Darboux frames of $y(s)$ and $y_*(s)$ by $\{T, g, n\}$ and $\{T_*, g_*, n_*\}$, respectively. From the definition, we can assume that

$$y_*(s_*) = y(s(s_*)) + \lambda(s(s_*))T(s(s_*)) \tag{4.2}$$

for function $\lambda(s(s_*))$. By taking the derivative of 4.2 with respect to s_* and applying the Darboux formulaes, we have

$$\begin{aligned} \frac{dy_*}{ds_*} &= \frac{dy}{ds}\frac{ds}{ds_*} + \lambda'(s)T(s_*) + \lambda \frac{dT}{ds}\frac{ds}{ds_*}, \\ T_*(s_*) &= T(s)\frac{ds}{ds_*} + \lambda' T(s) + \lambda(k_g \overrightarrow{g} + k_n \overrightarrow{n})\frac{ds}{ds_*}, \end{aligned}$$

$$T_*(s_*) = (\lambda'(s_*) + \frac{ds}{ds_*})T(s_*) + \lambda k_g \frac{ds}{ds_*}\overrightarrow{g} + \lambda k_n \frac{ds}{ds_*}\overrightarrow{n}. \tag{4.3}$$

Since the curves y and y_* are involute-envolute curves, we get

$$\langle T, T_* \rangle = 0.$$

Furthermore, exposing the inner product T to the both sides of 4.3, we obtain

$$\lambda'(s_*) + \frac{ds}{ds_*} = 0,$$

$$\lambda(s) = -s + c, c \in \mathbb{R}_0^+. \tag{4.4}$$

This means that λ is a nonzero function. □

Theorem 4.3. *Let $y(s)$ and $y_*(s_*)$ be two spacelike curves in the Minkowski 3-space E_1^3. If the $\{y(s), y_*(s_*)\}$ is an involute-evolute D-pair, then the curve y_* is given as*

$$y_*(s_*) = \begin{pmatrix} \frac{\mu+\lambda\mu_s}{2}(g - g^{-1}) + ag^{-1} + \lambda g_s(\frac{\mu}{2}(1+g^{-2}) + ag^{-2}), \\ a - \mu(s) + \lambda\mu_s, \\ ag^{-1} - \frac{\mu+\lambda\mu_s}{2}(g+g^{-1}) - \lambda g_s(\frac{\mu}{2}(1-g^{-2}) + ag^{-2}) \end{pmatrix}, a \in \mathbb{R}_0.$$

Proof. It is obvious from 3.1 and 4.2. □

Conclusion 4.1. *Let $y(s)$ and $y_*(s_*)$ be two spacelike D-curves in the Minkowski 3−space E_1^3. If the $\{y_*(s_*), y(s)\}$ is an involute-evolute D-pair, then the between the curves $y(s)$ and $y_*(s_*)$ is $\lambda(s) = |-s+c|$, $c \in \mathbb{R}_0^+$.*

Theorem 4.4. *Let M and M_* be oriented surfaces in Minkowski 3-space E_1^3 and the arc length spacelike curves $y(s)$ and $y_*(s_*)$ lying fully on M and M_*, respectively. $y_*(s_*)$ is involute D-curve of $y(s)$ if and if only satisfy the following equations*

i) *The geodesic curvature k_g and the normal curvature k_n of $y(s)$ satisfy the following equations*

$$\begin{aligned} k_g &= -\frac{\cosh\theta}{\lambda\lambda'} \\ k_n &= -\frac{\sinh\theta}{\lambda\lambda'} \\ k_g^2 - k_n^2 &= \frac{1}{(\lambda\lambda')^2} \\ \left(\frac{k_n}{k_g}\right)'\left(\frac{k_n}{k_g}\right) &= \frac{d\theta}{ds}. \end{aligned}$$

ii) *The geodesic curvature k_{g_*} and the geodesic normal k_{n_*} of $y_*(s)$ satisfy the following equations*

$$\begin{aligned} k_{g_*} &= -\frac{1}{\lambda} = -\frac{1}{-s+c}; c \in \mathbb{R}_0 \\ k_{n_*} &= \left(\left(\frac{k_n}{k_g}\right)'\left(\frac{k_n}{k_g}\right) - \lambda'\tau_g\right), \end{aligned}$$

where θ is the angle between the vectors T_* and g at the corresponding points of $y(s)$ and $y_*(s_*)$.

Proof. **i)** Suppose that the pair $\{y, y_*\}$ is an involute-evolute D-pair. Denote the Darboux frames of $y(s)$ and $y_*(s)$ by $\{T,g,n\}$ and $\{T_*, g_*, n_*\}$, respectively. From 4.3, we can write

$$\overrightarrow{T}_*(s_*) = \lambda k_g \frac{ds}{ds_*}\overrightarrow{g} + \lambda k_n \frac{ds}{ds_*}\overrightarrow{n}. \tag{4.5}$$

Furthermore, from 4.5 we say that $\overrightarrow{T}_* \in Sp\{\overrightarrow{g}, \overrightarrow{n}\}$ and using this representation we say that

$$\overrightarrow{T_*}(s_*) = \cosh\theta\overrightarrow{g} + \sinh\theta\overrightarrow{n} \tag{4.6}$$

where θ is the angle between the vectors T_* and $\overrightarrow{g}$ at the corresponding points of y and y_*. By differentiating 4.6 with respect to s_*, we get

$$k_{g_*}\overrightarrow{g_*} + k_{n_*}\overrightarrow{n_*} = (-k_g\cosh\theta + k_n\sinh\theta)\frac{ds}{ds_*}\overrightarrow{T}$$

$$+(\frac{d\theta}{ds} + \tau_g)\frac{ds}{ds_*}\sinh\theta\overrightarrow{g} + (\frac{d\theta}{ds} + \tau_g)\frac{ds}{ds_*}\cosh\theta\overrightarrow{n}. \tag{4.7}$$

Furthermore, from equation 4.7 and the fact that $\overrightarrow{T} \perp (\overrightarrow{n} \times \overrightarrow{g})$ we can say that $\overrightarrow{T} \in Sp\{\overrightarrow{n_*}, \overrightarrow{g_*}\}$ and $\overrightarrow{n}_* \in Sp\{\overrightarrow{g}, \overrightarrow{n}\}$, we write

$$\overrightarrow{n_*} = \sinh\theta\overrightarrow{g} + \cosh\theta\overrightarrow{n}, \tag{4.8}$$

from 4.7 and 4.8, we have

$$k_{g_*}\overrightarrow{g_*} + k_{n_*}\sinh\theta\overrightarrow{g} + k_{n_*}\cosh\theta\overrightarrow{n} = (-k_g\cosh\theta + k_n\sinh\theta)\frac{ds}{ds_*}\overrightarrow{T}$$

$$+(\frac{d\theta}{ds} + \tau_g)\frac{ds}{ds_*}\sinh\theta\overrightarrow{g} + (\frac{d\theta}{ds} + \tau_g)\frac{ds}{ds_*}\cosh\theta\overrightarrow{n}. \tag{4.9}$$

By considering 4.5 and 4.6, we get

$$1 = \frac{\lambda k_g}{\cosh\theta}\frac{ds}{ds_*} = \frac{\lambda k_n}{\sinh\theta}\frac{ds}{ds_*}. \tag{4.10}$$

By solving the previous equation, we obtain

$$\frac{k_g}{k_n} = \coth\theta \text{ or } \frac{k_n}{k_g} = \tanh\theta.$$

Furthermore, from 4.9 we have

$$k_{g_*} = (-k_g\cosh\theta + k_n\sinh\theta)\frac{ds}{ds_*} \tag{4.11}$$

$$k_{n_*}\sinh\theta = (\frac{d\theta}{ds} + \tau_g)\frac{ds}{ds_*}\sinh\theta \tag{4.12}$$

$$k_{n_*}\cosh\theta = (\frac{d\theta}{ds} + \tau_g)\frac{ds}{ds_*}\cosh\theta, \tag{4.13}$$

or since $\frac{ds}{ds_*} = -\lambda'$, we can write

$$k_{g_*} = (-k_g\cosh\theta + k_n\sinh\theta)\left(-\lambda'\right) \tag{4.14}$$

$$k_{n_*} = (\frac{d\theta}{ds} + \tau_g)\left(-\lambda'\right). \tag{4.15}$$

By using 4.10, 4.11, 4.12, 4.13 and $\frac{ds}{ds_*} = -\lambda'$, we can write as follows

$$k_g = -\frac{\cosh\theta}{\lambda\lambda'} \tag{4.16}$$

$$k_n = -\frac{\sinh\theta}{\lambda\lambda'}. \tag{4.17}$$

By considering 4.16 and 4.17, we have

$$k_g^2 - k_n^2 = \frac{1}{(\lambda\lambda')^2}. \tag{4.18}$$

ii) From 4.12 and 4.13 and using the previous equations, we have

$$k_{g_*} = -\frac{1}{\lambda} = -\frac{1}{-s+c}; c \in \mathbb{R}_0. \tag{4.19}$$

By taking the derivative of the equation $\frac{k_n}{k_g} = \tanh\theta$ with respect to s_*, we get

$$\left(\frac{k_n}{k_g}\right)'\left(\frac{k_n}{k_g}\right) = \frac{d\theta}{ds}. \tag{4.20}$$

By subsitituting 4.20 in 4.15 and making necessary calculations, we obtain

$$k_{n_*} = \left(\left(\frac{k_n}{k_g}\right)'\left(\frac{k_n}{k_g}\right) - \lambda'\tau_g\right). \tag{4.21}$$

□

Theorem 4.5. *Let the pair $\{y(s),\ y_*(s_*)\}$ be an involute-evolute D-pair in the Minkowski 3-space E_1^3. Then the relation between the geodesic curvature k_g and the normal curvature k_n of $y(s)$ is given as follows:*

$$\left(\frac{k_n}{k_g}\right)'\left(\frac{k_n}{k_g}\right)k_g + \frac{k_n}{-s+c} - k_n' = 0$$

or

$$\left(\frac{k_n}{k_g}\right)'\left(\frac{k_n}{k_g}\right)k_n + \frac{k_n}{-s+c} - k_g' = 0.$$

Proof. By taking the derivative of the equation 4.5, we have

$$\begin{aligned} k_{g_*}\overrightarrow{g_*}+k_{n_*}\overrightarrow{n_*} &= (k_n^2-k_g^2)\lambda\frac{ds}{ds_*}\overrightarrow{T} \\ &+((\lambda\frac{d^2s}{ds_*^2}-(\frac{ds}{ds_*})^2)k_n+\lambda\frac{ds}{ds_*}k_n'+\lambda\frac{ds}{ds_*}k_g\tau_g)\overrightarrow{n} \\ &+(\lambda\frac{ds}{ds_*}(k_g'+k_n\tau_g)-k_n(\frac{ds}{ds_*})^2+k_n\lambda\frac{d^2s}{ds_*^2})\overrightarrow{g}. \end{aligned}$$

Furthermore, using equations $\frac{ds}{ds_*}=-\lambda'$, $\frac{d^2s}{ds_*^2}=-\lambda''$ and $\lambda'=-1$, $\lambda''=0$, we get

$$k_{g_*}\overrightarrow{g_*}+k_{n_*}\overrightarrow{n_*}=(k_n^2-k_g^2)(-\lambda\lambda')\overrightarrow{T}$$

$$+(-(\lambda')^2k_n+(-\lambda\lambda')k_n'-\lambda\lambda'k_g\tau_g)\overrightarrow{n}+(-\lambda\lambda'(k_g'+k_n\tau_g)-k_n(\lambda')^2)\overrightarrow{g}. \quad (4.22)$$

By the help of 4.7 and 4.22, we can write as follows,

$$(k_n^2-k_g^2)(-\lambda\lambda')=(-k_g\cosh\theta+k_n\sinh\theta)\left(-\lambda'\right) \quad (4.23)$$

$$(-(\lambda')^2k_n+(-\lambda\lambda')k_n'-\lambda\lambda'k_g\tau_g)=(\frac{d\theta}{ds}+\tau_g)\left(-\lambda'\right)\cosh\theta \quad (4.24)$$

$$(-\lambda\lambda'(k_g'+k_n\tau_g)-k_n(\lambda')^2)=(\frac{d\theta}{ds}+\tau_g)\left(-\lambda'\right)\sinh\theta. \quad (4.25)$$

By considering 4.23, 4.24 and 4.25, we have

$$(k_n^2-k_g^2)\lambda=(k_n\sinh\theta-k_g\cosh\theta) \quad (4.26)$$

$$(\lambda'k_n+\lambda k_n'+\lambda k_g\tau_g)=(\frac{d\theta}{ds}+\tau_g)\cosh\theta \quad (4.27)$$

$$(\lambda(k_g'+k_n\tau_g)+k_n\lambda')=(\frac{d\theta}{ds}+\tau_g)\sinh\theta. \quad (4.28)$$

By substituting 4.16 and 4.20 in 4.27, we can obtain

$$\left(\frac{k_n}{k_g}\right)'\left(\frac{k_n}{k_g}\right)k_g+\frac{k_n}{-s+c}-k_n'=0. \quad (4.29)$$

By using 4.17 and 4.20 in 4.28, we can obtain

$$\left(\frac{k_n}{k_g}\right)'\left(\frac{k_n}{k_g}\right)k_n+\frac{k_n}{-s+c}-k_g'=0. \quad (4.30)$$

□

Corollary 4.6. *Let M and M_* be oriented surfaces in Minkowski 3-space E_1^3 and the arc length spacelike curves $y(s)$ and $y_*(s_*)$ lying fully on M and M_*, respectively. Denote the Darboux frames of $y(s)$ and $y_*(s_*)$ by $\{T, g, n\}$ and $\{T_*, g_*, n_*\}$, respectively. If the pair $\{y, y_*\}$ is an involute-evolute D-pair. Then the following relations hold:*

1) If y is an asymptotic line, then $k_g = \theta = cons\tan t, k_{n_*} = \tau_g, k_{g_*} = \frac{-1}{-s+c}$.

2) If y is a principal line, then $k_{n_*} = \left(\frac{k_n}{k_g}\right)' \left(\frac{k_n}{k_g}\right)$ and $k_{g_*} = \frac{-1}{-s+c}$.

3) If y is a geodesic curve, then there is not $\{y(s), y_*(s_*)\}$ involute-evolute D-pair in the Minkowski 3-space E_1^3.

Where θ is the angle between the vectors T_* and g at the corresponding points of $y(s)$ and $y_*(s_*)$ and k_g, k_n, τ_g are called the geodesic curvature, the normal curvature and geodesic torsion, respectively.

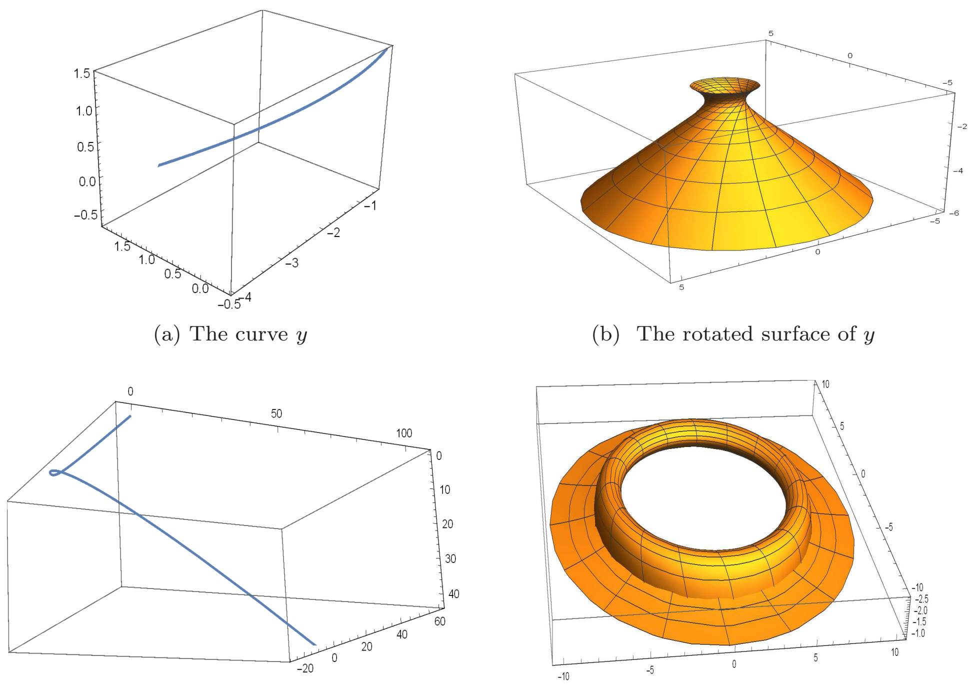
Example 4.7. *The curve*

$$y(s) = \left((e^s+1)\sinh s + \frac{e^{-s}}{2}, \frac{3}{2} - e^s, \frac{e^{-s}}{2} - (e^s+1)\cosh s\right)$$

is spacelike in the Minkowski 3*-space* E_1^3 *with arc length parameter* s *and this curve is an evolute of* $y_*(s)$*, for* $a = \frac{1}{2}, k = 1$*, the curve* $y_*(s)$ *is given as following*

$$y_*(s) = \begin{pmatrix} (1+(d-s)e^s)\sinh s + \frac{e^{-s}}{2} + (-s+c)e^s(\cosh s + \frac{1}{2} + e^{-2s}), \\ (-s+m)e^s - \frac{1}{2}, \\ \frac{e^{-s}}{2} - (1+(d-s)e^s)\cosh s - (-s+c)e^s(\sinh s + \frac{1}{2}) \end{pmatrix};$$

with $d, m, c \in \mathbb{R}_0$.



(a) The curve y (b) The rotated surface of y

(c) The curve y_* (d) The rotated surface of y_*

Figure 1: Graphics of curves y and y_*

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