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A Variation on Strongly Ideal Lacunary Ward Continuity

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ABSTRACT: The main purpose of this paper is to introduce the concept of strongly ideal lacunary quasi-Cauchyness of order (α, β) of sequences of real numbers. Strongly ideal lacunary ward continuity of order (α, β) is also investigated. Interesting results are obtained.

Key Words: Sequences, Series, Summability, Compactness, Continuity.

Contents

1. Introduction

A family I of subsets of $\mathbb{N}$, the set of positive integers, is said to be an *ideal* if I is additive *i.e.* $A,\ B \in I$ implies $A \cup B \in I$ and hereditary, *i.e.* $A \in I$, $B \subset A$ implies $B \in I$. An ideal I is called *non-trivial* if $I \neq 2^{\mathbb{N}}$, and an ideal I is said to be *admissible* if $I \supset \{\{n\} : n \in \mathbb{N}\}$. A non-empty family of sets $F \subseteq 2^X$ is said to be a *filter* of X if and only if (i) $\phi \notin F$, (ii) $A, B \in F$ implies $A \cap B \in F$ and (iii) $A \in F$, $A \subset B$ implies $B \in F$. The concept of ideal convergence (or I-convergence) of real sequences was introduced by Nuray and Ruckle in [31] who called it generalized statistical convergence as a generalization of statistical convergence which is a generalization of ordinary convergence ([25], [28], [34], [26], [4], [7], [8], [22], [37], [38], [6]), and also independently by Kostyrko, Šalát, and Wilczyński in [29]. Some further results connected with the notion of the I-convergence can be found in ([24], [30], [35], [32], [33], [40], [10], [11]). Throughout the paper, I will denote a non-trivial admissible ideal of $\mathbb{N}$. By a lacunary sequence we mean an increasing sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience. In the sequel, we will always assume that $\lim inf_r\ q_r > 1$. In [27], the notion of N_θ convergence was introduced, and studied by Freedman, Sember, and Raphael. A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if and only if it preserves Cauchy sequences. Using the idea of continuity of a real function in terms of sequences, many kinds of continuities were introduced and investigated,

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not all but some of them we recall in the following: slowly oscillating continuity ([12]), quasi-slowly oscillating continuity ([23]), Δ-quasi-slowly oscillating continuity ([13]), ward continuity ([14]), δ-ward continuity ([15]), δ^2-ward continuity ([2]), contra $\delta-\beta-$continuity ([1]), statistical ward continuity, ([8], [9], [7]), lacunary statistical ward continuity, ([43], [42], [39]), λ-statistically ward continuity ([16]), ideal ward continuity ([10]) and Abel continuity ([17]) which enabled some authors to obtain some characterizations of uniform continuity in terms of sequences in the sense that a function, on a special subset of $\mathbb{R}$, preserves certain types of sequences (see [3], [41], [18], [23]). The concept of lacunary I-convergence of sequences was introduced and investigated in [40]. A sequence of (x_k) of real numbers is said to be $N_\theta(I)$-convergent to a real number L (lacunary I-convergent in the statement of [40], or strongly ideal lacunary convergent to a real number L), if there is a real number L such that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \geq \varepsilon \right\} \in I$$

for each $\varepsilon > 0$. When a sequence (x_k) is $N_\theta(I)$-convergent to a real number L it is written $N_\theta(I) - \lim x_k = L$. Recently, the concepts of N_θ-ward compactness of a subset E of $\mathbb{R}$, and N_θ-ward continuity of a real function are introduced, and investigated in [19].

The purpose of this paper is to introduce and investigate $N_\alpha^\beta(\theta, I)$-ward continuity, and prove interesting theorems.

2. Main Results

In this section, by defining the notion of $N_\alpha^\beta(\theta, I)$-convergence we introduce and investigate $N_\alpha^\beta(\theta, I)$-sequential continuity, and $N_\alpha^\beta(\theta, I)$-ward continuity for $0 < \alpha \leq \beta \leq 1$.

Definition 2.1. *Let $0 < \alpha \leq \beta \leq 1$. A sequence of (x_k) of real numbers is said to be $N_\alpha^\beta(\theta, I)$-convergent to a real number L (strongly ideal lacunary convergent to a real number L of order (α, β)), if there is a real number L such that*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left(\sum_{k \in I_r} |x_k - L| \right)^\beta \geq \varepsilon \right\} \in I$$

for each $\varepsilon > 0$. When a sequence (x_k) is $N_\alpha^\beta(\theta, I)$-convergent to a real number L it is written $N_\alpha^\beta(\theta, I) - \lim x_k = L$.

Lemma 2.2. *Any $N_\alpha^\beta(\theta, I)$-convergent sequence (x_k) has an ordinary convergent subsequence (x_{k_n}) with $N_\alpha^\beta(\theta, I) - \lim x_k = \lim x_{k_n}$.*

Proof. The proof follows from the details of Theorem 3.1 of [40], so is omitted. □

A sequential method is called subsequential if a sequence (x_k) is summable by the sequential method to L, then there exists a convergent subsequence (x_{k_n}) with the limit L (see [5], [20], and [21]). Now we are going to prove the following theorem which will be used throughout the paper.

Theorem 2.3. *The method $N_\alpha^\beta(\theta, I)$ is regular and subsequential.*

Proof. To prove that the sequential method $N_\alpha^\beta(\theta, I)$ is regular, take any convergent sequence (x_k) of points in X with the ordinary limit L, i.e. $\lim_{k \longrightarrow \infty} x_k = L$. Let $\varepsilon > 0$. Then there exists a positive integer $n_0 \in \mathbb{N}$ such that $|x_k - L| < \varepsilon$ for $k \geq n_0$. Thus it follows from the admissibility of I that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left(\sum_{k \in I_r} |x_k - L| \right)^\beta \geq \varepsilon \right\} \in I.$$

Thus $(x_k) \in N_\alpha^\beta(\theta, I)$, so the sequential method $N_\alpha^\beta(\theta, I)$ is regular. The proof of the subsequentiality of $N_\alpha^\beta(\theta, I)$ follows from Lemma 2.2, so the proof of the theorem is completed. □

Definition 2.4. *A subset A of $\mathbb{R}$ is called $N_\alpha^\beta(\theta, I)$-sequentially compact if any sequence of points in A has an $N_\alpha^\beta(\theta, I)$-convergent subsequence with $N_\alpha^\beta(\theta, I)$-limit in A.*

Theorem 2.5. *A subset of $\mathbb{R}$ is $N_\alpha^\beta(\theta, I)$-sequentially compact if and only if it is sequentially compact in the ordinary sense.*

Proof. Although the proof follows Corollary 6 of [20], we give a direct proof for completeness. Let A be a subset of $\mathbb{R}$. Since any convergent sequence is $N_\alpha^\beta(\theta, I)$-convergent, it is easily seen that sequential compactness implies $N_\alpha^\beta(\theta, I)$-sequential compactness. To prove the converse suppose that A is $N_\alpha^\beta(\theta, I)$-sequentially compact. If $x = (x_n)$ is a sequence of points in A, then it has an $N_\alpha^\beta(\theta, I)$-convergent subsequence (x_{n_k}) of the sequence x with $N_\alpha^\beta(\theta, I) - lim x = L \in A$. It follows from Theorem 2.3 that the sequence (x_n) has a convergent subsequence of (x_{n_k}) with a limit in A. This completes the proof of the theorem. □

Definition 2.6. *A function f defined on a subset A of $\mathbb{R}$ is $N_\alpha^\beta(\theta, I)$-sequentially continuous at a point $x_0 \in A$ if, given a sequence $x = (x_n)$ of points in A, $N_\alpha^\beta(\theta, I)-$ $lim x = x_0$ implies that $N_\alpha^\beta(\theta, I) - lim f(x) = f(x_0)$. If f is $N_\alpha^\beta(\theta, I)$-sequentially continuous at every point of A, then f is called to be $N_\alpha^\beta(\theta, I)$-sequentially continuous on A.*

As a matter of fact, we see in the following theorem that the set of all $N_\alpha^\beta(\theta, I)$ sequentially continuous functions is equal to the set of continuous functions.

Theorem 2.7. *A function is $N_\alpha^\beta(\theta, I)$-sequentially continuous if and only if it is continuous in the ordinary sense.*

Proof. The proof follows from Lemma 1 and Corollary 9 in [21], so is omitted. □

Definition 2.8. *A sequence (x_k) of points in $\mathbb{R}$ is called $N_\alpha^\beta(\theta, I)$-quasi-Cauchy (or strongly ideal lacunary quasi-Cauchy) if*

$$N_\alpha^\beta(\theta, I) - \lim_{k\to\infty} \Delta x_k = 0,$$

i.e.

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left(\sum_{k\in I_r} |\Delta x_k| \right)^\beta \geq \varepsilon \right\} \in I$$

for every $\varepsilon > 0$.

We note that any quasi-Cauchy sequence is $N_\alpha^\beta(\theta, I)-$quasi-Cauchy, so any convergent sequence is $N_\alpha^\beta(\theta, I)-$quasi-Cauchy. Any Cauchy sequence is $N_\alpha^\beta(\theta, I)-$quasi-Cauchy, but the converse is not always true. Sum of two $N_\alpha^\beta(\theta, I)-$quasi-Cauchy sequence is $N_\alpha^\beta(\theta, I)-$quasi-Cauchy. Subsequence of an $N_\alpha^\beta(\theta, I)-$quasi-Cauchy sequence need not be $N_\alpha^\beta(\theta, I)-$quasi-Cauchy as well. Now we introduce the definition of $N_\alpha^\beta(\theta, I)-$ward compactness of a subset of X.

Definition 2.9. *A subset A of $\mathbb{R}$ is called $N_\alpha^\beta(\theta, I)-$ward compact (or strongly ideal lacunary ward compact of order (α, β)) if any sequence of points in A has an $N_\alpha^\beta(\theta, I)-$quasi-Cauchy subsequence.*

The union of two $N_\alpha^\beta(\theta, I)-$ward compact subset of $\mathbb{R}$ is $N_\alpha^\beta(\theta, I)-$ward compact, the intersection of any number of $N_\alpha^\beta(\theta, I)-$ward compact subsets is $N_\alpha^\beta(\theta, I)-$ward compact, sum of two $N_\alpha^\beta(\theta, I)-$ward compact subset of $\mathbb{R}$ is $N_\alpha^\beta(\theta, I)-$ward compact, and any finite subset of $\mathbb{R}$ is $N_\alpha^\beta(\theta, I)-$ward compact. These observations lead us to the following:

Theorem 2.10. *A subset A of $\mathbb{R}$ is bounded if and only if it is $N_\alpha^\beta(\theta, I)-$ward compact.*

Proof. Let A be any bounded subset of $\mathbb{R}$ and (x_n) be any sequence of points in A. (x_n) is also a sequence of points in $\overline{A}$ where $\overline{A}$ denotes the closure of A. As $\overline{A}$ is sequentially compact there is a convergent subsequence (x_{n_k}) of (x_n) (no matter the limit is in A or not). This subsequence is $N_\alpha^\beta(\theta, I)-$convergent since $N_\theta-$method is regular. Hence (x_{n_k}) is $N_\alpha^\beta(\theta, I)-$quasi-Cauchy. To prove that $N_\alpha^\beta(\theta, I)-$ward compactness implies boundedness, suppose that A is unbounded. If it is unbounded above, then one can construct a sequence (x_n) of numbers in A such that $x_{n+1} > k_n + x_n$ for each positive integer n. Then the sequence (x_n) does not have any $N_\alpha^\beta(\theta, I)-$quasi-Cauchy subsequence, so A is not $N_\alpha^\beta(\theta, I)-$ward compact. If A is bounded above and unbounded below, then similarly we obtain that A is not $N_\alpha^\beta(\theta, I)-$ward compact. This completes the proof of the theorem.

It easily follows from the preceding theorem that a closed subset of $\mathbb{R}$ is $N_\alpha^\beta(\theta, I)-$ward compact if and only if it is $N_\alpha^\beta(\theta, I)-$sequentially compact; and a closed subset of $\mathbb{R}$ is $N_\alpha^\beta(\theta, I)-$ward compact if and only if it is statistically ward compact. □

Definition 2.11. *A function defined on a subset A of R is called $N_\alpha^\beta(\theta, I)-$ward continuous (or strongly ideal lacunary ward continuous) if it preserves $N_\alpha^\beta(\theta, I)-$quasi-Cauchy sequences, i.e. $(f(x_n))$ is an $N_\alpha^\beta(\theta, I)-$quasi-Cauchy sequence whenever (x_n) is.*

We note that a composite of two $N_\alpha^\beta(\theta, I)-$ward continuous functions is $N_\alpha^\beta(\theta, I)-$ward continuous, and we prove in the following that sum of two $N_\alpha^\beta(\theta, I)-$ward continuous functions is $N_\alpha^\beta(\theta, I)-$ward continuous.

Theorem 2.12. *The sum of two $N_\alpha^\beta(\theta, I)-$ward continuous functions is $N_\alpha^\beta(\theta, I)-$ward continuous.*

Proof. Let f and g be $N_\alpha^\beta(\theta, I)-$ward continuous functions on a subset A of $\mathbb{R}$, and (x_n) be an $N_\alpha^\beta(\theta, I)-$quasi-Cauchy sequence of points in A. Take any $\varepsilon > 0$. Since f and g are $N_\alpha^\beta(\theta, I)-$ward continuous,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left(\sum_{k\in I_r} |\Delta f(x_k)|\right)^\beta \geq \frac{\varepsilon}{2}\right\} \in I$$

and

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left(\sum_{k\in I_r} |\Delta g(x_k)|\right)^\beta \geq \frac{\varepsilon}{2}\right\} \in I.$$

Therefore it follows from the non-triviality and admissibility of I that

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left(\sum_{k\in I_r} |\Delta(f(x_k) + g(x_k))|\right)^\beta \geq \varepsilon\right\} \in I$$

since

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left(\sum_{k\in I_r} |\Delta(f(x_k) + g(x_k))|\right)^\beta \geq \varepsilon\right\}$$
$$\subset \left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left(\sum_{k\in I_r} |\Delta f(x_k)|\right)^\beta \geq \frac{\varepsilon}{2}\right\} \bigcup \left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left(\sum_{k\in I_r} |\Delta g(x_k)|\right)^\beta \geq \frac{\varepsilon}{2}\right\}.$$

This completes the proof of the theorem. □

It is easy to see that if f is $N_\alpha^\beta(\theta, I)-$ward continuous, then $-f$ is also $N_\alpha^\beta(\theta, I)-$ward continuous. Furthermore if f is $N_\alpha^\beta(\theta, I)-$ward continuous and λ is a constant real number, then λf is also $N_\alpha^\beta(\theta, I)-$ward continuous, and f is $N_\alpha^\beta(\theta, I)-$ward continuous , then $|f|$ is also $N_\alpha^\beta(\theta, I)-$ward continuous. On the other hand, we prove in the following that the maximum of two $N_\alpha^\beta(\theta, I)-$ward continuous functions is $N_\alpha^\beta(\theta, I)-$ward continuous.

Theorem 2.13. *Maximum of two $N_\alpha^\beta(\theta, I)-$ward continuous functions is $N_\alpha^\beta(\theta, I)-$ward continuous.*

Proof. Let f and g be $N_\alpha^\beta(\theta, I)-$ward continuous functions, then it follows from Theorem 2.12 that $f+g$ and $f-g$ are $N_\alpha^\beta(\theta, I)-$ward continuous. Hence it follows from the equality

$$max\{f, g\} = \frac{1}{2}(f + g + |f - g|)$$

that $max\{f, g\}$ is $N_\alpha^\beta(\theta, I)-$ward continuous. This completes the proof of the theorem. □

We note that sum and maximum of finite number of $N_\alpha^\beta(\theta, I)-$ward continuous functions are $N_\alpha^\beta(\theta, I)-$ward continuous.

Theorem 2.14. *$N_\alpha^\beta(\theta, I)-$ward continuous image of any $N_\alpha^\beta(\theta, I)-$ward compact subset of the domain is $N_\alpha^\beta(\theta, I)-$ward compact.*

Proof. Assume that f is an $N_\alpha^\beta(\theta, I)-$ward continuous function on a subset A of $\mathbb{R}$, and B is an $N_\alpha^\beta(\theta, I)-$ward compact subset of A. Let $(f(x_n))$ be any sequence of points in $f(B)$ where $x_n \in B$ for each positive integer n. $N_\alpha^\beta(\theta, I)-$ward compactness of B implies that there is an $N_\alpha^\beta(\theta, I)-$quasi Cauchy subsequence $(\gamma_k) = (x_{n_k})$ of (x_n). Write $(t_k) = (f(\gamma_k))$. As f is $N_\alpha^\beta(\theta, I)-$ward continuous, $(f(\gamma_k))$ is $N_\alpha^\beta(\theta, I)-$quasi-Cauchy which is a subsequence of the sequence $(f(x_n))$. This completes the proof of the theorem. □

Theorem 2.15. *If f is $N_\alpha^\beta(\theta, I)-$ward continuous on a subset A of $\mathbb{R}$, then it is $N_\alpha^\beta(\theta, I)-$sequentially continuous on A.*

Proof. Let f be an $N_\alpha^\beta(\theta, I)-$ward continuous function on a subset A of $\mathbb{R}$, and (x_k) be an $N_\alpha^\beta(\theta, I)-$convergent sequence of points in A. Write $N_\alpha^\beta(\theta, I)-\lim_{k\to\infty} x_k = x_0$. Then the sequence

$$(x_1, x_0, x_2, x_0, ..., x_{n-1}, x_0, x_n, x_0, ...)$$

is $N_\alpha^\beta(\theta, I)-$convergent to x_0. Hence it is $N_\alpha^\beta(\theta, I)-$quasi-Cauchy.

As f is $N_\alpha^\beta(\theta, I)-$ward continuous, the sequence

$$(f(x_1), f(x_0), f(x_2), f(x_0), ..., f(x_{n-1}), f(x_0), f(x_n), f(x_0), ...)$$

is $N_\alpha^\beta(\theta, I)-$quasi-Cauchy. It follows from this that the sequence $(f(x_n))$ is $N_\alpha^\beta(\theta, I)-$convergent to $f(x_0)$. This completes the proof of the theorem. □

Corollary 2.16. *If f is $N_\alpha^\beta(\theta, I)-$ward continuous on a subset A of $\mathbb{R}$, then it is continuous on A.*

Proof. The proof easily follows from Theorem 2.15 and Theorem 2.7, so is omitted.

We see in the following theorem that the set of $N_\alpha^\beta(\theta, I)-$ward continuous functions is equal to the set of uniformly continuous functions on a bounded subset of $\mathbb{R}$. □

Theorem 2.17. *An $N_\alpha^\beta(\theta, I)-$ward continuous function on an $N_\alpha^\beta(\theta, I)-$ward compact subset A of $\mathbb{R}$ is uniformly continuous.*

Proof. Suppose that f is not uniformly continuous on A so that there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$ there are $x, y \in E$ with $|x-y| < \delta$ but $|f(x)-f(y)| \geq \varepsilon_0$. For each positive integer n, there exist x_n and y_n such that $|x_n - y_n| < \frac{1}{n}$, and $|f(x_n) - f(y_n)| \geq \varepsilon_0$. Since A is $N_\alpha^\beta(\theta, I)-$ward compact, there exists an $N_\alpha^\beta(\theta, I)-$quasi-Cauchy subsequence (x_{n_k}) of the sequence (x_n). It is clear that the corresponding subsequence (y_{n_k}) of the sequence (y_n) is also $N_\alpha^\beta(\theta, I)-$quasi-Cauchy, since $(y_{n_{k+1}} - y_{n_k})$ is a sum of three $N_\alpha^\beta(\theta, I)-$null sequences, i.e.

$$y_{n_{k+1}} - y_{n_k} = (y_{n_{k+1}} - x_{n_{k+1}}) + (x_{n_{k+1}} - x_{n_k}) + (x_{n_k} - y_{n_k}).$$

On the other hand, it follows from the equality $x_{n_{k+1}} - y_{n_k} = x_{n_{k+1}} - x_{n_k} + x_{n_k} - y_{n_k}$ that the sequence $(x_{n_{k+1}} - y_{n_k})$ is $N_\alpha^\beta(\theta, I)-$convergent to 0. Hence the sequence

$$(x_{n_1}, y_{n_1}, x_{n_2}, y_{n_2}, x_{n_3}, y_{n_3}, ..., x_{n_k}, y_{n_k}, ...)$$

is $N_\alpha^\beta(\theta, I)-$quasi-Cauchy. But the transformed sequence

$$(f(x_{n_1}), f(y_{n_1}), f(x_{n_2}), f(y_{n_2}), f(x_{n_3}), f(y_{n_3}), ..., f(x_{n_k}), f(y_{n_k}), ...)$$

is not $N_\alpha^\beta(\theta, I)-$quasi-Cauchy. Thus f does not preserve $N_\alpha^\beta(\theta, I)-$quasi-Cauchy sequences. This contradiction completes the proof of the theorem. □

Theorem 2.18. *Uniform limit of $N_\alpha^\beta(\theta, I)-$ward continuous functions is $N_\alpha^\beta(\theta, I)-$ward continuous, i.e., if (f_n) is a sequence of $N_\alpha^\beta(\theta, I)-$ward continuous functions on a subset A of $\mathbb{R}$ and (f_n) is uniformly convergent to a function f, then f is $N_\alpha^\beta(\theta, I)-$ward continuous on A.*

Proof. Let ε be a positive real number and (x_k) be any $N_\alpha^\beta(\theta, I)-$quasi-Cauchy sequence of points in A. By the uniform convergence of (f_n), there exists a positive integer N such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $x \in A$ whenever $n \geq N$. As f_N is $N_\alpha^\beta(\theta, I)-$ward continuous on A, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left(\sum_{k \in I_r} |f_N(x_{k+1}) - f_N(x_k)| \right)^\beta < \frac{\varepsilon}{3} \right\} \in F(I).$$

On the other hand, we have

$$\begin{aligned} & \frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}|f(x_{k+1})-f(x_k)|\right)^\beta \\ \leq & \frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}|f(x_{k+1})-f_N(x_{k+1})|\right)^\beta \\ & +\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}|f_N(x_{k+1})-f_N(x_k)|\right)^\beta+\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}|f_N(x_k)-f(x_k)|\right)^\beta \\ < & \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon. \end{aligned}$$

Therefore

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}|f(x_{k+1})-f(x_k)|\right)^\beta<\varepsilon\right\}\in F(I)$$

i.e.

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\left(\sum_{k\in I_r}|f(x_{k+1})-f(x_k)|\right)^\beta\geq\varepsilon\right\}\in I.$$

This completes the proof of the theorem. □

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References

1. M.A. Al Shumrani, S. Jafari, C. Özel and N. Rajesh, *A new type of contra-continuity via δ-β-open sets*. C. R. Acad. Bulgare Sci. **71**(7) (2018) 867-874.
2. N.L. Braha and H. Çakallı, *A new type continuity for real functions*. J. Math. Anal. **7**(6) (2016) 54-62.
3. D. Burton and J. Coleman, *Quasi-Cauchy Sequences*. Amer. Math. Monthly **117**(4) (2010) 328-333.
4. A. Caserta, G. Di Maio and L.D.R. Kočinac, *Statistical convergence in function spaces*. Abstr. Appl. Anal. **2011** (2011) Art. ID 420419, 11 pp. DOI: 10.1155/2011/420419

5. J. Connor and K.-G.Grosse-Erdmann, *Sequential definitions of continuity for real functions*. Rocky Mountain J. Math. **33**(1) (2003) 93-121.

6. J.S. Connor, *The Statistical and strong $p-$Cesàro convergence of sequences*. Analysis **8**(1-2) (1988) 47-63.

7. H. Çakallı, *Statistical quasi-Cauchy sequences*. Math. Comput. Modelling **54**(5-6) (2011) 1620-1624.

8. H. Çakallı, *Statistical ward continuity*. Appl. Math. Lett. **24**(10) (2011) 1724-1728.

9. H. Çakallı, *A new approach to statistically quasi Cauchy sequences*. Maltepe Journal of Mathematics **1**(1) (2019) 1-8.

10. H. Çakallı and B. Hazarika, *Ideal quasi-Cauchy sequences*. J. Inequal. Appl. **2012**(234) (2012) 11pp. doi:10.1186/1029-242X-2012-234.

11. H. Çakallı, *A variation on ward continuity*. Filomat **27**(8) (2013) 1545-1549.

12. H. Çakallı, *Slowly oscillating continuity*. Abstr. Appl. Anal., Hindawi Publ. Corp., New York, **2008** Article ID 485706, (2008) doi:10.1155/2008/485706.

13. H. Çakallı, *On Δ-quasi-slowly oscillating sequences*. Comput. Math. Appl. **62**(9) (2011) 3567-3574.

14. H. Çakallı, *Forward continuity*. J. Comput. Anal. Appl. **13**(2) (2011) 225-230.

15. H. Çakallı, *δ-quasi-Cauchy sequences*. Math. Comput. Modelling **53**(1-2) (2011) 397-401.

16. H. Çakallı, A. Sönmez and Ç.G. Aras, *λ-statistically ward continuity*. An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N.S.) **63**(2) (2017) 313-321.

17. H. Çakallı and M. Albayrak, *New type continuities via Abel convergence*. The Scientific World Journal **2014** (2014) Article ID 398379, 6 pages. http://dx.doi.org/10.1155/2014/398379.

18. H. Çakallı and A. Sönmez, *Slowly oscillating continuity in abstract metric spaces*. Filomat **27**(5) (2013) 925-930.

19. H. Çakallı, *N_θ-ward continuity*. Abstr. Appl. Anal., Hindawi Publ. Corp., New York, **2012** Article ID 680456, (2012) 8 pp. doi:10.1155/2012/680456.

20. H. Çakallı, *Sequential definitions of compactness*. Appl. Math. Lett. **21**(6) (2008) 594-598.

21. H. Çakallı, *On G-continuity*. Comput. Math. Appl. **61**(2) (2011) 313-318.

22. H. Çakallı, M. Et and H. Şengül, *A variation on N_θ ward continuity*. Georgian Mathematical Journal doi: https://doi.org/10.1515/gmj-2018-0037.

23. I. Çanak and M. Dik, *New Types of Continuities*. Abstr. Appl. Anal., Hindawi Publ. Corp., New York, **2010** Article ID 258980, (2010) doi:10.1155/2010/258980.

24. P. Das, E. Savaş and S.Kr. Ghosal, *On generalizations of certain summability methods using ideals*. Appl. Math. Lett. **24**(9) (2011) 1509-1514.

25. H. Fast, *Sur la convergence statistique*. Colloq. Math. **2** (1952) 241-244.

26. J. Fridy, *On statistical convergence*. Analysis **5** (1985) 301-313.

27. A.R. Freedman, J.J. Sember and M.Raphael, *Some Cesaro-type summability spaces*. Proc. London Math. Soc. **37**(3) (1978) 508-520.

28. A.D. Gadjiev and C. Orhan, *Some approximation theorems via statistical convergence*. Rocky Mountain J. Math. **32**(1) (2002) 129-138.

29. P. Kostyrko, T. Šalát and W. Wilczyński, *$I-$convergence*. Real Anal. Exchange **26**(2) (2000/2001) 669-686.

30. P. Kostyrko, M. Mačaj, M. Sleziak and T. Šalát, *I-convergence and extremal I-limit points*. Math. Slovaca **55**(4) (2005) 443-464.

31. F. Nuray and W.H. Ruckle, *Generalized statistical convergence and convergence free spaces*. J. Math. Anal. Appl. **245**(2) (2000) 513-527.

32. T. Salát, B.C. Tripathy and M. Ziman, *On some properties of I-convergence*. Tatra Mt. Math. Publ. **28**(2) (2004) 279-286.

33. T. Šalát, B.C. Tripathy and M. Ziman, *On I-convergence field*. Ital. J. Pure Appl. Math. **17** (2005) 45-54.

34. T. Šalát, *On statistically convergent sequences of real numbers*. Math. Slovaca **30**(2) (1980) 139-150.

35. E. Savaş and P. Das, *A generalized statistical convergence via ideals*. Appl. Math. Lett. **24**(6) (2011) 826-830.

36. H. Şengül, H. Çakallı and M. Et, $N_{\alpha}^{\beta}(\theta, I)-$ *ward continuity*. AIP Conference Proceedings 2086, 030038 (2019); doi: https://doi.org/10.1063/1.5095123.

37. H. Şengül and M. Et, *On (λ, I)-statistical convergence of order α of sequences of function*. Proc. Nat. Acad. Sci. India Sect. A **88**(2) (2018) 181-186.

38. H. Şengül and M. Et, *On I-lacunary statistical convergence of order α of sequences of sets*. Filomat **31**(8) (2017) 2403-2412.

39. I. Taylan, *Abel statistical delta quasi Cauchy sequences of real numbers*. Maltepe Journal of Mathematics, **1**(1) (2019) 18-23.

40. B.C. Tripathy, B. Hazarika and B. Choudhary, *Lacunary I-Convergent Sequences* Kyungpook Math. J. **52**(4) (2012) 473-482.

41. R.W. Vallin, *Creating slowly oscillating sequences and slowly oscillating continuous functions*. With an appendix by Vallin and H. Cakalli, Acta Math. Univ. Comenianae **80**(1) (2011) 71-78.

42. Ş. Yıldız, *Lacunary statistical p-quasi Cauchy sequences*. Maltepe Journal of Mathematics **1**(1) (2019) 9-17.

43. S. Yildiz, *Variations on lacunary statistical quasi Cauchy sequences*. AIP Conference Proceedings 2086, 030045 (2019); doi: https://doi.org/10.1063/1.5095130.

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