Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 38** 7 (2020): 85–97.
ISSN-00378712 in press
doi:10.5269/bspm.v38i7.46259

On (f, I) −Lacunary Statistical Convergence of Order α of Sequences of Sets

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abstract: In this paper we introduce the concepts of Wijsman (f, I) −lacunary statistical convergence of order α and Wijsman strongly (f, I) −lacunary statistical convergence of order α, and investigated between their relationship.

Key Words: I−convergence, Wijsman statistical convergence, Lacunary sequence.

Contents

1. Introduction

The concept of statistical convergence was introduced by Steinhaus [37] and Fast [17]. Schoenberg [32] established some basic properties of statistical convergence and studied the concept as a summability method. Later on it was further investigated from the sequence space point of view and linked with summability theory by Altınok et al. ([2], [3]), Bhardwaj and Dhawan [4], Caserta et al. [5], Çınar et al. [9], Connor [12], Çakallı et al. ([6], [7], [8]), Çolak et al. ([10], [11], [23]), Et et al. ([13], [14], [15], [16]), Fridy [19], Gadjiev and Orhan [21], Işık and Akbaş [22], Salat [29], Savaş and Et [31], Şengül [34], Taylan [38], and many others. Nuray and Rhoades [28] extended the notion to statistical convergence of sequences of sets and gave some basic theorems. Ulusu et al. ([39], [40]) defined Wijsman lacunary statistical convergence of sequence of sets, and considered its relation with Wijsman statistical convergence.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience. In recent years, lacunary sequences have been studied in ([6], [18], [20], [33], [35], [36], [39], [40], [41]).

The notion of a modulus was given by Nakano [26]. A modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

i) $f(x) = 0$ if and only if $x = 0$,

2010 *Mathematics Subject Classification:* 40A05, 40C05, 46A45.
Submitted January 16, 2019. Published March 27, 2019

ii) $f(x+y) \leq f(x) + f(y)$ for $x, y \geq 0$,
iii) f is increasing,
iv) f is continuous from the right at 0.

It follows that f must be continuous in everywhere on $[0, \infty)$. A modulus may be unbounded or bounded.

Aizpuru et al. [1] defined $f-$density of a subset $E \subset \mathbb{N}$ for any unbounded modulus f by

$$d^f(E) = \lim_{n\to\infty} \frac{f(|\{k \leq n : k \in E\}|)}{f(n)}, \text{if the limit exists}$$

and defined $f-$statistical convergence for any unbounded modulus f by

$$d^f(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}) = 0$$

and we write it as $S(f) - \lim x_k = \ell$ or $x_k \to \ell\,(S(f))$.

Let X be a non-empty set. Then a family of sets $I \subseteq 2^X$ (power sets of X) is said to be an *ideal* if I is additive *i.e.* $A, B \in I$ implies $A \cup B \in I$ and hereditary, *i.e.* $A \in I$, $B \subset A$ implies $B \in I$.

A non-empty family of sets $F \subseteq 2^X$ is said to be a *filter* of X if and only if (i) $\phi \notin F$, (ii) $A, B \in F$ implies $A \cap B \in F$ and (iii) $A \in F$, $A \subset B$ implies $B \in F$.

An ideal $I \subseteq 2^X$ is called *non-trivial* if $I \neq 2^X$.

A non-trivial ideal I is said to be *admissible* if $I \supset \{\{x\} : x \in X\}$.

If I is a non-trivial ideal in X $(X \neq \phi)$ then the family of sets $F(I) = \{M \subset X : (\exists A \in I)(M = X \setminus A)\}$ is a filter of X, called the *filter associated with I*.

Let (X, d) be a metric space. For any non-empty closed subset A_k of X, we say that the sequence $\{A_k\}$ is bounded if $\sup_k d(x, A_k) < \infty$ for each $x \in X$. In this case we write $\{A_k\} \in L_\infty$.

Throughout the paper I will stand for a non-trivial admissible ideal of $\mathbb{N}$.

The idea of $I-$convergence of real sequences was introduced by Kostyrko et al. [24] and also independently by Nuray and Ruckle [27] (who called it generalized statistical convergence) as a generalization of statistical convergence. Later on $I-$convergence was studied in ([7], [25], [30], [35], [40]).

2. Main Results

In this section, we give relations between the concepts of Wijsman $(f, I)-$lacunary statistical convergence of order α and Wijsman strongly $(f, I)-$lacunary statistical convergence of order α of sequences of sets.

Definition 2.1. *Let f be an unbounded modulus, (X, d) be a metric space, θ be a lacunary sequence, $\alpha \in (0, 1]$ and $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $(f, I)-$lacunary statistically convergent to A of order α (or $S_\theta^\alpha(f, I_w)-$convergent to A) if for each $\varepsilon > 0$, $\delta > 0$ and $x \in X$,*

$$\left\{r \in \mathbb{N} : \frac{1}{f(h_r)^\alpha} f(|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \geq \delta\right\}$$

belongs to I. In this case, we write $A_k \longrightarrow A\left(S_\theta^\alpha(f, I_w)\right)$. For $\theta = (2^r)$, we shall write $S^\alpha(f, I_w)$ instead of $S_\theta^\alpha(f, I_w)$ and in the special cases $\alpha = 1$ and $\theta = (2^r)$ we shall write $S(f, I_w)$ instead of $S_\theta^\alpha(f, I_w)$.

As an example, consider the following sequence:

$$A_k = \begin{cases} \{2x\}, & k_{r-1} < k < k_{r-1} + \sqrt{h_r} \\ \{0\}, & \text{otherwise} \end{cases}.$$

Let $(\mathbb{R}, d)$ be a metric space such that for $x, y \in X$, $d(x, y) = |x - y|$, $A = \{1\}, x > 1$, $f(x) = x$ and $\alpha = 1$. Since

$$\frac{1}{f(h_r)^\alpha} f\left(|\{k \in I_r : |d(x, A_k) - d(x, \{1\})| \geq \varepsilon\}|\right) \geq \delta$$

belongs to I, the sequences $\{A_k\}$ is Wijsman (f, I) −lacunary statistically convergent to $\{1\}$ of order α.

Definition 2.2. *Let f be an unbounded modulus, (X, d) be a metric space, θ be a lacunary sequence, $\alpha \in (0, 1]$ and $I \subseteq 2^{\mathbb{N}}$ be an admissible ideal of subsets of $\mathbb{N}$. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is said to be Wijsman strongly (f, I) −lacunary statistically convergent to A of order α (or $N_\theta^\alpha[f, I_w]$ −convergent to A) if for each $\varepsilon > 0$ and $x \in X$,*

$$\left\{r \in \mathbb{N} : \frac{1}{f(h_r)^\alpha} \sum_{k \in I_r} f\left(|d(x, A_k) - d(x, A)|\right) \geq \varepsilon\right\}$$

belongs to I. In this case, we write $A_k \longrightarrow A\left(N_\theta^\alpha[f, I_w]\right)$. For $\theta = (2^r)$, we shall write $N^\alpha[f, I_w]$ instead of $N_\theta^\alpha[f, I_w]$ and in the special cases $\alpha = 1$ and $\theta = (2^r)$ we shall write $N[f, I_w]$ instead of $N_\theta^\alpha[f, I_w]$.

As an example, consider the following sequence:

$$A_k = \begin{cases} \left\{\frac{3xk}{4}\right\}, & k_{r-1} < k < k_{r-1} + \sqrt{h_r} \\ \{0\}, & \text{otherwise} \end{cases}.$$

Let $(\mathbb{R}, d)$ be a metric space such that for $x, y \in X$, $d(x, y) = |x - y|$, $A = \{1\}, x > 1$, $f(x) = x$ and $\alpha = 1$. Since

$$\frac{1}{f(h_r)^\alpha} \sum_{k \in I_r} f\left(|d(x, A_k) - d(x, \{1\})|\right) \geq \varepsilon,$$

the sequences $\{A_k\}$ is Wijsman strongly (f, I) −lacunary statistically convergent to $\{1\}$ of order α.

Theorem 2.1. *Let f be an unbounded modulus, (X, d) be a metric space, $\theta = (k_r)$ be a lacunary sequence, $\alpha \in (0, 1]$ and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X, then $N_\theta^\alpha[f, I_w]$ is a proper subset of $S_\theta^\alpha(f, I_w)$.*

Proof. The inclusion part of proof is easy. In order to show that the inclusion $N_\theta^\alpha[f, I_w] \subseteq S_\theta^\alpha(f, I_w)$ is proper, let θ be given and we define a sequence $\{A_k\}$ as follows

$$A_k = \begin{cases} \{x^2\}, & k = 1, 2, 3, ..., [\sqrt{h_r}] \\ \{0\}, & \text{otherwise} \end{cases} .$$

Let $(\mathbb{R}, d)$ be a metric space such that for $x, y \in X$, $d(x, y) = |x - y|$, $f(x) = \sqrt{x}$. We have for every $\varepsilon > 0, x > 0$ and $\frac{1}{2} < \alpha \leq 1$,

$$\frac{1}{f(h_r)^\alpha} f(|\{k \in I_r : |d(x, A_k) - d(x, \{0\})| \geq \varepsilon\}|) \leq \frac{f([\sqrt{h_r}])}{f(h_r)^\alpha} = \frac{[\sqrt{h_r}]^{\frac{1}{2}}}{\sqrt{h_r}^\alpha} = \frac{[\sqrt{h_r}]^{\frac{1}{2}}}{h_r^{\frac{\alpha}{2}}},$$

and for any $\delta > 0$ we get

$$\left\{r \in \mathbb{N} : \frac{1}{f(h_r)^\alpha} f(|\{k \in I_r : |d(x, A_k) - d(x, \{0\})| \geq \varepsilon\}|) \geq \delta\right\} \subseteq \left\{r \in \mathbb{N} : \frac{[\sqrt{h_r}]^{\frac{1}{2}}}{h_r^{\frac{\alpha}{2}}} \geq \delta\right\}.$$

Since the set on the right-hand side is a finite set and so belongs to I, it follows that for $\frac{1}{2} < \alpha \leq 1$, $A_k \to \{0\}(S_\theta^\alpha(f, I_w))$.

On the other hand, for $\alpha = 1$ and $x > 0$,

$$\begin{aligned} \frac{1}{f(h_r)^\alpha} \sum_{k \in I_r} f(|d(x, A_k) - d(x, \{0\})|) &= \frac{f(x^2 - 2x)[\sqrt{h_r}]}{f(h_r)^\alpha} \\ &= \frac{\sqrt{(x^2 - 2x)}[\sqrt{h_r}]}{\sqrt{h_r}^\alpha} \to 1 \end{aligned}$$

and for $0 < \alpha < 1$

$$\frac{\sqrt{(x^2 - 2x)}[\sqrt{h_r}]}{\sqrt{h_r}^\alpha} \to \infty.$$

Hence we have

$$\begin{aligned} &\left\{r \in \mathbb{N} : \frac{1}{f(h_r)^\alpha} \sum_{k \in I_r} f(|d(x, A_k) - d(x, \{0\})|) \geq 0\right\} \\ &= \left\{r \in \mathbb{N} : \frac{\sqrt{(x^2 - 2x)}[\sqrt{h_r}]}{\sqrt{h_r}^\alpha} \geq 0\right\} \\ &= \{a, a+1, a+2, ...\} \end{aligned}$$

for some $a \in \mathbb{N}$ which belongs to $F(I)$, since I is admissible. So,

$$A_k \nrightarrow \{0\}(N_\theta^\alpha[f, I_w]).$$

□

From Theorem 2.1 we have the following results.

Corollary 2.2. *i) Let α and β be two fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then $N^{\alpha}[f, I_w] \subseteq S^{\beta}(f, I_w)$,*

ii) Let α be a fixed real numbers such that $0 < \alpha \leq 1$, then $N^{\alpha}[f, I_w] \subseteq S(f, I_w)$,

iii) Let θ be a lacunary sequence, then $N_{\theta}[f, I_w] \subseteq S_{\theta}(f, I_w)$.

Theorem 2.3. *Let $\theta = (k_r)$ be a lacunary sequence and α be a fixed real number such that $0 < \alpha \leq 1$. If $\lim_{r\to\infty} \dfrac{f(h_r)^{\alpha}}{f(k_r)^{\alpha}} > 0$, then $S^{\alpha}(f, I_w) \subset S_{\theta}^{\alpha}(f, I_w)$.*

Proof. If $A_k \to A(S^{\alpha}(f, I_w))$, then for every $\varepsilon > 0$, for each $x \in X$, and for sufficiently large r, we have

$$\frac{1}{f(k_r)^{\alpha}} f(|\{k \leq k_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \geq$$
$$\frac{f(h_r)^{\alpha}}{f(k_r)^{\alpha}} \frac{1}{f(h_r)^{\alpha}} f(|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|).$$

For $\delta > 0$, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{f(h_r)^{\alpha}} f(|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \geq \delta \right\}$$
$$\subseteq \left\{ r \in \mathbb{N} : \frac{1}{f(k_r)^{\alpha}} f(|\{k \leq k_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \geq \frac{\delta f(h_r)^{\alpha}}{f(k_r)^{\alpha}} \right\} \in I.$$

This completes the proof. □

Theorem 2.4. *Let $\theta = (k_r)$ be a lacunary sequence and the parameters α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then the inclusion $N_{\theta}^{\alpha}[f, I_w] \subseteq N_{\theta}^{\beta}[f, I_w]$ is strict.*

Proof. The inclusion part of proof is easy. To show that the inclusion is strict define $\{A_k\}$ such that for $(\mathbb{R}, d)$, $x > 1$, $f(x) = x$ and $A = \{0\}$,

$$A_k = \begin{cases} \{5x + 2\}, & k_{r-1} < k < k_{r-1} + \sqrt{h_r} \\ \{0\}, & \text{otherwise} \end{cases}.$$

Then $\{A_k\} \in N_{\theta}^{\beta}[f, I_w]$ for $\frac{1}{2} < \beta \leq 1$ but $\{A_k\} \notin N_{\theta}^{\alpha}[f, I_w]$ for $0 < \alpha \leq \frac{1}{2}$. □

Theorem 2.5. *Let $\theta = (k_r)$ be a lacunary sequence and the parameters α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$, then the inclusion $S_{\theta}^{\alpha}(f, I_w) \subseteq S_{\theta}^{\beta}(f, I_w)$ is strict.*

Proof. The inclusion part of proof is easy. To show that the inclusion is strict define $\{A_k\}$ such that for $X = \mathbb{R}^2$ and $f(x) = x$

$$A_k = \begin{cases} (x, y) \in \mathbb{R}^2, x^2 + (y-1)^2 = k^2, & \text{if } k \text{ is square} \\ \{(0,0)\}, & \text{otherwise} \end{cases}.$$

Then $\{A_k\} \in S_\theta^\beta(f, I_w)$ for $\frac{1}{2} < \beta \leq 1$ but $\{A_k\} \notin S_\theta^\alpha(f, I_w)$ for $0 < \alpha \leq \frac{1}{2}$. □

Theorem 2.6. *Let $\theta = (k_r)$ be a lacunary sequence and α be a fixed real number such that $0 < \alpha \leq 1$. If $\lim_{r\to\infty} \inf \dfrac{f(h_r)^\alpha}{f(k_r)} > 0$, then $S(f, I_w) \subseteq S_\theta^\alpha(f, I_w)$.*

Proof. The proof is similar to that of Theorem 2.3. □

Theorem 2.7. *Let (X, d) be a metric space and A, A_k (for all $k \in \mathbb{N}$) be non-empty closed subsets of X and α be a fixed real number such that $0 < \alpha \leq 1$. If $\theta = (k_r)$ is a lacunary sequence with $\limsup \frac{f(k_j - k_{j-1})^\alpha}{f(k_{r-1})^\alpha} < \infty$ $(j = 1, 2, ..., r)$, then $A_k \to A(S_\theta^\alpha(f, I_w))$ implies $A_k \to A(S^\alpha(f, I_w))$.*

Proof. If $\limsup \frac{f(k_j - k_{j-1})^\alpha}{f(k_{r-1})^\alpha} < \infty$, then without any loss of generality, we can assume that there exists a $0 < B_j < \infty$ such that $\frac{f(k_j - k_{j-1})^\alpha}{f(k_{r-1})^\alpha} < B_j$, $(j = 1, 2, ..., r)$ for all $r \geq 1$. Suppose that $A_k \to A(S_\theta^\alpha(f, I_w))$ and for $\varepsilon, \delta, \delta_1 > 0$ define the sets

$$C = \left\{ r \in \mathbb{N} : \frac{1}{f(h_r)^\alpha} f(|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) < \delta \right\}$$

and

$$T = \left\{ r \in \mathbb{N} : \frac{1}{f(n)^\alpha} f(|\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) < \delta_1 \right\}.$$

It is obvious from our assumption that $C \in F(I)$, the filter associated with the ideal I. Further observe that

$$A_i = \frac{1}{f(h_i)^\alpha} f(|\{k \in I_i : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) < \delta$$

for all $i \in C$. Let $n \in \mathbb{N}$ be such that $k_{r-1} < n < k_r$ for some $r \in C$. Now we can write

$$
\begin{aligned}
& \frac{1}{f(n)^{\alpha}} f\left(\left|\left\{k \leq n : \left|d\left(x, A_k\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right|\right) \\
\leq\; & \frac{1}{f(k_{r-1})^{\alpha}} f\left(\left|\left\{k \leq k_r : \left|d\left(x, A_k\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right|\right) \\
\leq\; & \frac{1}{f(k_{r-1})^{\alpha}} f\left(\left|\left\{k \in I_1 : \left|d\left(x, A_k\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right|\right) + ... \\
& + \frac{1}{f(k_{r-1})^{\alpha}} f\left(\left|\left\{k \in I_r : \left|d\left(x, A_k\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right|\right) \\
=\; & \frac{f(k_1)^{\alpha}}{f(k_{r-1})^{\alpha}} \frac{1}{f(h_1)^{\alpha}} f\left(\left|\left\{k \in I_1 : \left|d\left(x, A_k\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right|\right) \\
& + \frac{f(k_2 - k_1)^{\alpha}}{f(k_{r-1})^{\alpha}} \frac{1}{f(h_2)^{\alpha}} f\left(\left|\left\{k \in I_2 : \left|d\left(x, A_k\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right|\right) \\
& + ... + \frac{f(k_r - k_{r-1})^{\alpha}}{f(k_{r-1})^{\alpha}} \frac{1}{f(h_r)^{\alpha}} f\left(\left|\left\{k \in I_r : \left|d\left(x, A_k\right) - d\left(x, A\right)\right| \geq \varepsilon\right\}\right|\right) \\
\leq\; & \sup_{i \in C} A_i . \frac{f(k_1)^{\alpha} + f(k_2 - k_1)^{\alpha} + ... + f(k_r - k_{r-1})^{\alpha}}{f(k_{r-1})^{\alpha}} \\
\leq\; & \sup_{i \in C} A_i \left(B_1 + B_2 + ... + B_r\right) < \delta \sum_{j=1}^{r} B_j .
\end{aligned}
$$

Choosing $\delta_1 = \frac{\delta}{\sum_{j=1}^{r} B_j}$ and in view of the fact that $\cup \{n : k_{r-1} < n < k_{r,}\ r \in C\} \subset T$ where $C \in F(I)$. This completes the proof of the theorem. □

Theorem 2.8. *Suppose $\theta' = (s_r)$ is a lacunary refinement of the lacunary sequence $\theta = (k_r)$ and $\alpha, \beta \in (0, 1]$ be fixed real numbers such that $\alpha \leq \beta$. Let $I_r = (k_{r-1}, k_r]$ and $J_r = (s_{r-1}, s_r], (r = 1, 2, 3, ...)$. If there exists $\epsilon > 0$ such that*

$$
\frac{f(|J_j|)^{\beta}}{f(|I_i|)^{\alpha}} \geq \epsilon \text{ for every } J_j \subseteq I_i \ ,
$$

then $A_k \to A\left(S_{\theta}^{\alpha}(f, I_w)\right)$ implies $A_k \to A\left(S_{\theta'}^{\beta}(f, I_w)\right)$.

Proof. For any $\varepsilon > 0$ and every J_j, we can find I_i such that $J_j \subseteq I_i$; then we have

$$\begin{aligned} & \frac{1}{f(|J_j|)^{\beta}} f(|\{k \in J_j : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \\ = & \left(\frac{f(|I_i|)^{\alpha}}{f(|J_j|)^{\beta}}\right)\left(\frac{1}{f(|I_i|)^{\alpha}}\right) f(|\{k \in J_j : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \\ \leq & \left(\frac{f(|I_i|)^{\alpha}}{f(|J_j|)^{\beta}}\right)\left(\frac{1}{f(|I_i|)^{\alpha}}\right) f(|\{k \in I_i : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \\ \leq & \left(\frac{1}{\epsilon}\right)\left(\frac{1}{f(|I_i|)^{\alpha}}\right) f(|\{k \in I_i : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|), \end{aligned}$$

and so

$$\begin{aligned} & \left\{r \in \mathbb{N} : \frac{1}{f(|J_j|)^{\beta}} f(|\{k \in J_j : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \geq \delta\right\} \\ \subseteq & \left\{r \in \mathbb{N} : \left(\frac{1}{f(|I_i|)^{\alpha}}\right) f(|\{k \in I_i : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \geq \delta\epsilon\right\} \in I. \end{aligned}$$

The proof completes immediately. □

Theorem 2.9. *Suppose $\theta = (k_r)$ and $\theta' = (s_r)$ are two lacunary sequences and $\alpha, \beta \in (0, 1]$ be fixed real numbers such that $\alpha \leq \beta$. Let $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $(r = 1, 2, 3, ...)$ and $I_{ij} = I_i \cap J_j$, $i, j = 1, 2, 3, ...$ If there exists $\epsilon > 0$*

$$\frac{f(|I_{ij}|)^{\beta}}{f(|I_i|)^{\alpha}} \geq \epsilon \text{ for every } i, j = 1, 2, 3, ..., \text{provided } I_{ij} \neq \varnothing,$$

then $A_k \to A(S_{\theta}^{\alpha}(f, I_w))$ implies $A_k \to A\left(S_{\theta'}^{\beta}(f, I_w)\right)$.

Proof. Let $\theta'' = \theta' \cup \theta$. Then θ'' is a lacunary refinement of the lacunary sequence θ', also θ. Then interval sequence of θ'' is $\{I_{ij} = I_i \cap J_j : I_{ij} \neq \varnothing\}$. From Theorem 2.8, if $A_k \to A(S_{\theta}^{\alpha}(f, I_w))$, then $A_k \to A\left(S_{\theta''}^{\beta}(f, I_w)\right)$. Since θ'' is also a lacunary refinement of the lacunary sequence θ', we have that $A_k \to A\left(S_{\theta''}^{\alpha}(f, I_w)\right)$ implies $A_k \to A\left(S_{\theta'}^{\beta}(f, I_w)\right)$. □

Theorem 2.10. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let α and β be such that $0 < \alpha \leq \beta \leq 1$,*

(i) If

$$\lim_{r \to \infty} \inf \frac{f(h_r)^{\alpha}}{f(\ell_r)^{\beta}} > 0 \tag{1}$$

then $S_{\theta'}^{\beta}(f, I_w) \subseteq S_{\theta}^{\alpha}(f, I_w)$, *(ii) If*

$$\lim_{r\to\infty} \frac{f(\ell_r)}{f(h_r)^{\alpha}} = 0 \tag{2}$$

then $S_{\theta}^{\alpha}(f, I_w) \subseteq S_{\theta'}^{\beta}(f, I_w)$.

Proof. $i)$ Omitted.

(ii) Let (2) be satisfied. Since $I_r \subset J_r$, for $\varepsilon > 0$ we may write

$$\begin{aligned}
& \frac{1}{f(\ell_r)^{\beta}} f(|\{k \in J_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \\
= \; & \frac{1}{f(\ell_r)^{\beta}} f(|\{s_{r-1} < k \leq k_{r-1} : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\
& + |\{k_r < k \leq s_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| \\
& + |\{k_{r-1} < k \leq k_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \\
\leq \; & \frac{1}{f(\ell_r)^{\beta}} f(k_{r-1} - s_{r-1} + s_r - k_r) \\
& + \frac{1}{f(\ell_r)^{\beta}} f(|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \\
\leq \; & \frac{f(\ell_r - h_r)}{f(\ell_r)^{\beta}} + \frac{1}{f(\ell_r)^{\beta}} f(|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \\
\leq \; & \frac{f(\ell_r + h_r)}{f(\ell_r)^{\beta}} + \frac{1}{f(\ell_r)^{\beta}} f(|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \\
\leq \; & \frac{f(\ell_r) + f(h_r)}{f(h_r)^{\beta}} + \frac{1}{f(h_r)^{\alpha}} f(|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \\
\leq \; & \frac{f(\ell_r) + f(\ell_r)}{f(h_r)^{\alpha}} + \frac{1}{f(h_r)^{\alpha}} f(|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \\
\leq \; & \left(\frac{2f(\ell_r)}{f(h_r)^{\alpha}}\right) + \frac{1}{f(h_r)^{\alpha}} f(|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|)
\end{aligned}$$

for all $r \in \mathbb{N}$ and so

$$\begin{aligned}
& \left\{r \in \mathbb{N} : \frac{1}{f(\ell_r)^{\beta}} f(|\{k \in J_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \geq \delta\right\} \\
\subseteq \; & \left\{r \in \mathbb{N} : \frac{1}{f(h_r)^{\alpha}} f(|\{k \in I_r : |d(x, A_k) - d(x, A)| \geq \varepsilon\}|) \geq \delta - \frac{2f(\ell_r)}{f(h_r)^{\alpha}}\right\} \in I.
\end{aligned}$$

This gives that $S_{\theta}^{\alpha}(f, I_w) \subseteq S_{\theta'}^{\beta}(f, I_w)$. □

Theorem 2.11. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, α and β be fixed real numbers such that $0 < \alpha \leq \beta \leq 1$. Then we have*

(i) If (1) *holds then* $N_{\theta'}^{\beta}[f, I_w] \subset N_{\theta}^{\alpha}[f, I_w]$,
(ii) If $\lim_{r\to\infty} \frac{\ell_r}{f(h_r)^{\alpha}} = 0$ *holds and* $\{A_k\} \in L_\infty$ *then* $N_{\theta}^{\alpha}[f, I_w] \subset N_{\theta'}^{\beta}[f, I_w]$.

Proof. (i) Omitted.

(ii) Let suppose that (2) holds. Since $\{A_k\} \in L_\infty$ then there exists some $M > 0$ such that $|d(x, A_k) - d(x, A)| \leq M$ for all k. Now, since $I_r \subseteq J_r$ and $h_r \leq \ell_r$ for all $r \in \mathbb{N}$, we may write

$$\begin{aligned}
& \frac{1}{f(\ell_r)^{\beta}} \sum_{k\in J_r} f(|d(x, A_k) - d(x, A)|) \\
= & \frac{1}{f(\ell_r)^{\beta}} \sum_{k\in J_r - I_r} f(|d(x, A_k) - d(x, A)|) \\
& + \frac{1}{f(\ell_r)^{\beta}} \sum_{k\in I_r} f(|d(x, A_k) - d(x, A)|) \\
\leq & \frac{(\ell_r - h_r) f(M)}{f(\ell_r)^{\beta}} + \frac{1}{f(\ell_r)^{\beta}} \sum_{k\in I_r} f(|d(x, A_k) - d(x, A)|) \\
\leq & \frac{\ell_r f(M)}{f(h_r)^{\alpha}} + \frac{1}{f(h_r)^{\alpha}} \sum_{k\in I_r} f(|d(x, A_k) - d(x, A)|)
\end{aligned}$$

for every $r \in \mathbb{N}$ and so

$$\begin{aligned}
& \left\{ r \in \mathbb{N} : \frac{1}{f(\ell_r)^{\beta}} \sum_{k\in J_r} f(|d(x, A_k) - d(x, A)|) \geq \varepsilon \right\} \\
\subseteq & \left\{ r \in \mathbb{N} : \frac{1}{f(h_r)^{\alpha}} \sum_{k\in I_r} f(|d(x, A_k) - d(x, A)|) \geq \varepsilon - \frac{\ell_r f(M)}{f(h_r)^{\alpha}} \right\} \in I.
\end{aligned}$$

□

Theorem 2.12. *Let* $\theta = (k_r)$ *and* $\theta' = (s_r)$ *be two lacunary sequences such that* $I_r \subseteq J_r$ *for all* $r \in \mathbb{N}$, α *and* β *be fixed real numbers such that* $0 < \alpha \leq \beta \leq 1$. *Let* (1) *holds, if a sequence is strongly* $N_{\theta'}^{\beta}[f, I_w]$*−summable to* A, *then it is* $S_{\theta}^{\alpha}(f, I_w)$*−statistically convergent to* A.

Proof. Omitted. □

3. Acknowledgments

The authors acknowledge that some of the results were presented at the 2nd International Conference of Mathematical Sciences, 31 July 2018-6 August 2018, (ICMS 2018) Maltepe University, Istanbul, Turkey, and the statements of some results in this paper will be appeared in AIP Conference Proceeding of 2nd International Conference of Mathematical Sciences, (ICMS 2018) Maltepe University, Istanbul, Turkey ([35]).

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