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On Binary Operation Graphs

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ABSTRACT: Labeled graphs are the the graphs that their vertices or edges or to both assigned labels of integers according to a certain conditions. Given a graph $G = (V, E)$, a vertex labeling is a function of V to a set of labels. A graph with such a function defined is called a vertex-labeled graph. The concept of labeled graph generally refers to a vertex-labeled graph with all labels distinct. In this work, new results of graph labeling " binary operation labeling " are determined. Binary operation labeling is a type of labeling graphs by vertices.

Key Words: Binary operation graphs, Maximal binary operation graphs.

Contents

1. Introduction

Let $G = (V, E)$ be a finite and simple undirected graph of order p and and size q. In any graph $deg(v)$ is, the number of edges that incident on v in G. Any notion or definition which is not found here could be found in [14].

A graph G is said to be labeled (numbered) if each vertex u in G is assigned to a non negative integer $f(u)$. Such a graph may equivalently be labeled by the consecutive integers $\{1, \cdots , p\}$. For many applications, the edges or vertices are given labels that are meaningful in the associated domain. For example, the edges may be assigned weights representing the "cost" of traversing between the incident vertices. The work here is focus on studing vertex labeling.
Rosa [18] in 1967 started graph labeling principles. Graham and Sloane [13] have also contributed into these principles. For a survey on graph labeling see Gallian [10]. See also, [6], [7], [11], [12], [15], [19], [20], [21] and [1].

Binary operation labeling was defined in [2] which is a vertices labeling scheme. Let $G = (V, E)$ be a (p, q)-graph and let $f : V(G) \to 1, 2, \cdots , p$ be a bijection function. Binary operation graphs are graphs admits a binary operation labeling. When a binary operation graph has a maximum number of edge then it is a maximal binary operation graph. In this paper, new results of "Binary operation graphs "are introduced.
Labeled graphs can be use as a modeling tool for numerous applications such as:

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circuit design, coding theory, x-ray crystallography, radar , communication network addressing, secret sharing schemes, and models for constraint programming over finite domains,for more details–see [3], [4], [5], [8], [9], [16], [17], [22] , [23]and [24].

2. Binary operation labeling

Definition 2.1. *, [2] Let $G = (V, E)$ be a graph of order p be a binary operation graph if there exists a bijection function $f : V(G) \to 1, 2, \ldots, p$, where the induced edge function $f^* : E(G) \to N$ is defined as*

$$f^*(vu) = \begin{Bmatrix} (f(v) + f(u))/2; \text{ if } f(v) \text{ and } f(u) \text{ are both odd or both are even (a)} \\ \\ (f(v)f(u))/2; \text{ if } f(v) \text{ is odd and } f(u) \text{ is even or vice versa (b)} \end{Bmatrix} \tag{2.1}$$

Remark 2.2. *[2] Let u be any vertex in a graph having a maximal binary operation labeling where the adjacent vertices to u are divided into two sets, the first contains all vertices which are adjacent to u expressed by 1(a) and denoted by S_u, and the second contains all vertices that are adjacent to u which expressed by 1(b) and denoted by M_u. That means*

$$S_u = \begin{Bmatrix} (f(u)/2) + i; \ i = 0, \ldots, \lceil p/2 \rceil - 1, i \neq f(u)/2; \text{ if } f(u) \text{ and } f(v) \text{ are even} \\ \\ (f(u) + (2i+1))/2; \ i = 1, \ldots, \lceil p/2 \rceil, i \neq \lfloor f(u)/2 \rfloor; \text{ if } f(u) \text{ and } f(v) \text{ are odd} \end{Bmatrix}$$

and

$$M_u = \begin{Bmatrix} f(u)/2 + if(u); \ i = 0, \ldots, \lceil p/2 \rceil - 1; \text{ if } f(v) \text{ is odd and f(u) is even} \\ \\ 2if(u); \ i = 1, \ldots, \lfloor p/2 \rfloor; \text{ if } f(v) \text{ is even and f(u) is odd} \end{Bmatrix}$$

3. Main results

Theorem 3.1. *In any binary operation graph $G, u \in G$ the maximum degree $\triangle(G) \leq p - t - 1$, where*

$$t = \begin{Bmatrix} \lfloor p/2f(u) \rfloor; \text{ if } f(u) \text{ is even (a)} \\ \lfloor (p + f(u))/2f(u) \rfloor - 1; \text{ if } f(u) \text{ is odd (b)} \end{Bmatrix} \tag{3.1}$$

Proof. Let u_j be any vertex in G such that $f(u_j) = j; j = 1, 2, \ldots, n$. To calculate the maximum degree of this vertex there are two cases as follows.
(i) If $f(u_j)$ is even, then all vertices of even labels which are adjacent to vertex u_j are expressed by Eq.(1(a)). So all edge labels are increasing, then there are no repeated edge labels. Also all vertices of odd labels adjacent to vertex u_j are expressed by Eq.(1(b)), so labels of these edges are also increasing; again there are no repeated edge labels. The repeated edge labels can occur when two adjacent vertices to vertex u_j one is even labeled, and the other is odd labeled.
Now, if vertex u_j joins with vertex u_1 then we get $f^*(u_j v_1) = j/2$ and $f^*(u_j u_h) \neq$

$j/2 \; \forall \; h$, where h is even. So, in this case there are no repeated edge labels. Next join occurs when we join vertex u_j with vertex u_3 then we get $f^*(u_ju_3) = 3j/2 = f^*(u_ju_{2j})$, therefore, we get the first repeated edge label. By the same procedure we can get the second repeated edge label$f^*(u_jv_5) = 5j/2 = f^*(u_jv_{4j})$. Thus, in general

$f^*(u_ju_{2t+1}) = (2t+1)j/2 = f^*(u_ju_{2kj})$, and these edge labels exist if $p \geq 2tj$ that means $t \leq p/2j$, so the maximum value of t that satisfies this equation is $t = \lfloor p/2j \rfloor$.

(ii) If j is odd, then all vertices of odd labels join with vertex u_j by Eq.(1(a)). So all edge labels are increasing, then there are no repeated edge labels. Also all vertices of even labels can join with vertex u_j according to Eq.(1(b)), so labels of these edges are also increasing, again there are no repeated edge labels. The repeated edges can occur when we join vertex u_j with two vertices, one of them of is odd labeled and the other is even labeled. Now, if vertex u_j joins with vertex v_2 then we get $f^*(u_ju_2) = j$ which can be got only if $f^*(u_ju_j) = j$. So, in this case there are no repeated edge labels, since G is a simple graph. The next join occurs when vertex u_j joins with vertex u_4 then we get $f^*(u_jv_4) = 2j = f^*(u_ju_3j)$, therefore, we get the first repeated edge label. By the same procedure we can get the second repeated edge label $f^*(u_ju_6) = 3j = f^*(u_ju_{5j})$. Thus, in general $f^*(u_ju_{2t}) = 2j = f^*(u_ju_{(2t-1)j})$, and these edge labels exist if $n \geq (2t-1)j$ that means $(2k-1) \leq p/j$, so the maximum t which satisfies this equation is $t = \lfloor (p+j)/2j \rfloor$. Thus, the number of repeated edges is $\lfloor (p+f(u_j))/(2f(u_j)) \rfloor - 1$, since $t \geq 2$. □

Theorem 3.2. *Let G be a graph having maximal binary operation labeling then G has at most one vertex of degree $p-1$.*

Proof. If u is a vertex of even label such that $2f(u) \leq p$, then $deg(u) < p-1$, according to Theorem 3.1. If v is a vertex of odd label such that $3f(v) \leq p$, then $deg(v) < p-1$, according to Theorem 3.1.

Now, consider $D_e = \{u; 2f(u) > p$ and $D_o = v; 3f(v) > p\}$

Case 1: If p is even, then there are two cases as follows.

(i) If $p/2$ is even , then

Claim 1.

1. $(p/2)+1 \in S_{u_r}] \cap S_{u_s} if \; u_r, u_s \in D_e$.
2. $\begin{Bmatrix} p/2; \text{ if } f^*(v_rv_s) \neq p/2 \\ (p/2)-2; \text{ if } f^*(v_rv_s) = p/2 \end{Bmatrix} \in S_{v_r} \cap S_{v_s}$ if $v_r, v_s \in D_o$
3. $p/2 \in S_{v_r} \cap S_{u_s}$, if $v_r \in D_o, u_p \neq u_s \in D_e$.
4. $(p/2)+2 \in S_{v_r} \cap S_{u_p}$ if $v_r \in D_o$ *and* $u_p \in D_e$.

Proof of Claim 1.

1. $|D_e| = p/4$, since the elements in $D_e = \{u_{n-2k}; k = 0, \ldots, (p/4) - 1\}$, it is clear that $u_{2k+2} \in G$ exists for all k and $f^*(u_{p-2k}u_{2k+2}) = (p/2) + 1$ and $\forall u_r, u_s \in D_e$, $f^*(u_r u_s) \neq (p/2) + 1$, since $u_{2k+2} \notin D_e\ \forall k$.

2. $|D_o| = \{\lceil n/3 \rceil\}$, since the element in $D_o = v_{p-1-2k}; k = 0, \ldots, \lceil n/3 \rceil - 1$, it is clear that v_{2k+1} exists for all k and $f^*(v_{p-1-2k}v_{2k+1}) = p/2$. If $v_{2k+1} \in D_o$ for some k then we take $f^*(v_{p-1-2k}v_{2k-3}) = (p/2) - 2$.

3. A vertex u_{2k} joins with all vertices u_{p-2k}, $k = 0, \ldots, (n/4) - 1$ in D_e and $f^*(v_{p-2k}u_{2k}) = p/2$. A vertex u_{2k+1} joins with all vertices $v_{p-1-2k}, k = 0, \ldots, \lceil p/3 \rceil - 1 in D_o$ and $f^*(v_{p-1-2k}v_{2k+1}) = p/2$. A vertex u_4 joins with vertex u_p in D_e and$f^*(u_p u_4) = (p/2) + 2$. A vertex v_{2k+5} joins with all vertices v_{p-1-2k} in D_o and $f^*(v_{p-1-2k}v_{2k+5}) = (p/2) + 2$.

(ii) If $p/2$ is odd , then Claim 2.

1. $\left\{\begin{matrix} (p/2)+2;\ \text{if } f^*(u_r u_s) \neq (p/2)+2 \\ p/2; if f^*(u_r u_s) = (p/2)+2 \end{matrix}\right\} \in S_{u_r} \cap S_{u_s}$ if $u_r, u_s \in D_e$

2. $\left\{\begin{matrix} (p/2)+1;\ \text{if } f^*(v_r v_s) \neq p/2+1 \\ (p/2)-1;\ \text{if } f^*(v_r v_s) = (p/2)+1 \end{matrix}\right\} \in S_{v_r} \cap S_{v_s}$ if $v_r, v_s \in D_o$

3. $(p/2) + 1 \in S_{(v_r)} \cap S_{(u_s)}$, if $v_r \in D_o, u_{(p/2)+1} \neq u_s \in D_e$

4. $(p/2) + 2 \in S_{v_r} \cap S_{u_{(p/2)+1}}$, if $v_r \in D_o, u_{(p/2)+1} \in D_e$

Proof of Claim 2.

1. $|D_e| = \lceil p/4 \rceil$, since $D_e = \{u(p-2k); k = 0, \ldots, \lceil p/4 \rceil - 1\}$, it is clear that u_{2k+4} exists for all k and $f^*(u_{p-2k}u_{2k+4}) = (p/2) + 2$. If $u_{2k+4} \in D_e$ for some k then we take u_{2k} where $f^*(u_{p-2k}u_{2k}) = p/2$.

2. $|D_o| = \lceil p/4 \rceil$, since $D_o = \{v_{p-1-2k}; k = 0, \ldots, \lceil p/4 \rceil - 1\}$, it is clear that v_{2k+3} exists for all k and $f^*(v_{p-1-2k}v_{2k+3}) = (p/2) + 1$. If $v_{2k+3} \in D_o$ for some k then we take v_{2k-1}, where $f^*(v_{p-1-2k}v_{2k-1}) = (p/2) - 1$

3. Vertex v_{2k+2} joins with all vertices v_{p-2k} in D_e and $f^*(v_{p-2k}v_{2k+2}) = (p/2)+1$. A vertex v_{2k+3} joins with all vertices u_{p-1-2k} *in* $D_o, k = 0, \ldots, \lceil p/4 \rceil - 1$, and $f^*(v_{p-1-2k}v_{2k+3}) = (p/2) + 1$

4. Vertex $u_{(p/2)+3}$ joins with vertex $u_{(p/2)+1}$ in D_e and $f^*(u_{p/2+3}u_{p/2+1}) = (p/2) + 2$. A vertex v_{2k+5} joins with all vertices $v_{p-1-2k}, k = 0, \ldots, \lceil p/4 \rceil - 1$, *in* D_o and $f^*(v_{p-1-2k}v_{2k+5}) = (p/2) + 2$.

Case 2: If p is odd, then
Claim 3.

1. $\left\{\begin{matrix}\lceil p/2\rceil+1; \text{ if } f^*(u_ru_s)\neq\lceil p/2\rceil+1\\ \lceil p/2\rceil-1; \text{ if } f^*(u_ru_s)=\lceil p/2\rceil+1\end{matrix}\right\}\in S_{u_r}\cap S_{u_s}$ if $u_r,u_s\in D_e$

2. $\left\{\begin{matrix}\lceil p/2\rceil; \text{ if } f^*(v_rv_s)\neq\lceil p/2\rceil\\ \lceil p/2\rceil-2; \text{ if } f^*(v_rv_s)=\lceil p/2\rceil\end{matrix}\right\}\in S_{v_r}\cap S_{v_s}$ if $v_r,v_s\in D_o$

3. $\lceil p/2\rceil\in S_{v_r}\cap S_{u_s}, if\ v_r\in D_o, u_{\lceil p/2\rceil}\neq u_s\in D_e$

4. $\lceil p/2\rceil+2\in S_{v_r}\cap S_{u_s},$ if $v_r\in D_o, u_{\lceil p/2\rceil}\in D_e$

Proof of Claim 3.

1. $|D_e|=\lceil p/4\rceil$, since $D_e=\{u_{p-1-2k}; k=0,\ldots,\lceil p/4\rceil-1\}$, it is clear that u_{2k+4} exists for all k and $f^*(u_{p-1-2k}u_{2k+4})=(n+1)/2+1=\lceil p/2\rceil+1$. If $u_{2k+4}\in D_e$ for some k then we take $f^*(u_{p-1-2k}u_{2k})=\lfloor p/2\rfloor-1$.

2. $|D_o|=\lceil n/4\rceil$, since $D_o=\{v_(p-2k); k=0,\ldots,\lceil p/4\rceil\}$, it is clear that v_{2k+1} exists for all k and $f^*(v_{p-2k}v_{2k+1})=(p+1)/2=\lceil p/2\rceil$ If $v_{2k+1}\in D_o$ for some k then we take $f^*(v_{2k-3}v_{2k-3})=\lceil p/2\rceil-2$.

3. A vertex u_{2k+2} joins with all vertices u_{p-1-2k} in D_e and $f^*(u_{p-1-2k}u_{2k+2})=\lceil p/2\rceil$. A vertex v_{2k+1} joins with all vertices v_{p-2k} in D_o and $f^*(v_{p-2k}v_{2k+1})=\lceil p/2\rceil$.

4. A vertex $u_{\lceil p/2\rceil+2}$ joins with vertex $u_{\lceil p/2\rceil}$ in D_e and $f^*(u_{\lceil p/2\rceil}u_{\lceil p/2\rceil+4})=\lceil p/2\rceil+2$. A vertex v_{2k+5} joins with all vertices v_{p-2k} in D_o and $f^*(v_{p-2k}v_{2k+5})=\lceil p/2\rceil+2$. Then if we take any vertex from either D_e or D_o of degree $(p-1)$, we cannot find another vertex with degree equal to $(p-1)$.

□

Lemma 3.3. *Let G the maximal binary operation graph , if $deg(u)=1$, $u\in G$ then the vertex labeling $f(u)$ satisfies the following conditions:*

(i) $f(u)<2p/(p-2)$; $f(u)<p-2$, if p is odd and $f(u)$ is even or vice versa
(ii) $f(u)<2p/(p-3)$; $f(u)<p-3$, if p and $f(u)$ are both odd or both are even.

Proof. Let $f(u_j)=j; j=1,2,\ldots,p$. In the previous theorem we proved that the repeated edges labels belong to $M_{u_j}\cap S_{u_j}$, so if any value in M_{u_j} is greater than the maximum value in S_{u_j}, then this value does not belong to the intersection sets. Let $u_j\in V$, and there are two vertices in G such that the values of edges that join these vertices with the vertex u_j are greater than the maximum value of S_{u_j}. So these values are not repeated, thus $deg(u_j)\geq 2$. Now we have two cases.
Case 1: If j is even then there are two cases as follows.
(i) If p is odd, then if $f^*(u_ju_{p-2)})$ does not belong to intersection sets, then we get

two values which do not belong to intersection sets, that means $deg(u_j) \geq 2$, and $f^*(u_j u_{p-2}) \geq p$, since $p-1$ is the maximum value in S_{v_j}, so $(j(p-2))/2 \geq p$ which implies that $j \geq 2p/(n-2)$.
Therefore, when $j < 2p/(p-2)$, the vertex u_j can join with at most one vertex.
(ii) If p is even then by same the manner in (i) if $f^*(u_j u_{p-3}) \geq n$ then $deg(u_j) \geq 2$ and $(j(p-3))/2 \geq p$, which implies that $j \geq 2n/(n-3)$. Thus, when $j < 2p/(p-3)$, the vertex u_j can join with at most one vertex.
Case 2: If j is odd then there are two cases as follows.
(i) If p is even, similar to case 1 (i).
(ii) If p is odd, similar to case 1 (ii). □

Note 3.1. *$deg(u_1) \geq 2; \forall p > 2$, since $f^*(u_2 u_1) = 1$ and the edge with label 2 is gotten by either $f^*(u_1 u_3)$ or $f^*(u_1 u_4)$.*

Theorem 3.4. *There is at most one end vertex in any maximal binary operation graph.*

Proof. There are two cases that depend on p as follows.
Case 1: If p is even, then there are two cases as follows.
(i) If j is odd, then (a) If $p = 4$, then $deg(u_3) = 1$ as shown in Figure 1.

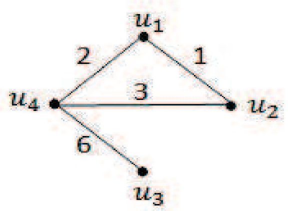

Figure 1

(b) $\forall p \geq 6$, according to Lemma 3.1 $j < 2n/(p-2) \leq 3$, thus there is no vertex of odd label that can join with only one vertex according to Note 3.3.
(ii) If j is even, then:
(a) If $p = 4$, then $deg(u_2) = 2$, since $f^*(u_2 v_1) = 1$ and we get label 3 by joining either u_2 with u_4 or u_2 with u_3. Hence, $deg(u_4) = 1$ as shown in Figure 2.

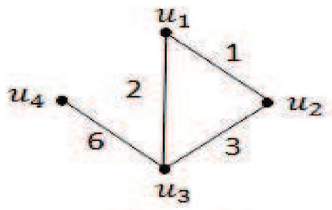

Figure 2

(b) $\forall n \geq 6$, then by Lemma 3.1 $j < 2n/(p-2) \leq 4$, so the only vertex that can join with one vertex is u_2. And it is clear that in case i(a) and case ii(a), u_3 and u_4 cannot get degree one on the same maximal binary operation graph.
Case 2: If p is odd, then there are two cases as follows:
(i) If j is odd, then (a) If $p = 3, 5$, then there is no vertex of degree one, since in $p = 3$, $f^*(u_3u_1) = 2$ and $f^*(u_3u_2) = 3$ are not repeated, so $deg(u_3) = 2$ and in $p = 5$, $f^*(u_3u_4) = 6$ and $f^*(u_3u_5) = 4$ are not repeated, so $deg(u_3) \geq 2$ and $f^*(u_5u_3) = 4$ and $f^*(u_5u_4) = 10$ are not repeated so $deg(u_5) \geq 2$.
(b) If $p = 7$, then by Lemma 3.1 the only vertex that can join with only one vertex is u_3, since $j < 2p/(p-3) < 4$, and in this case $deg(u_3) = 1$ as shown in Figure 3.

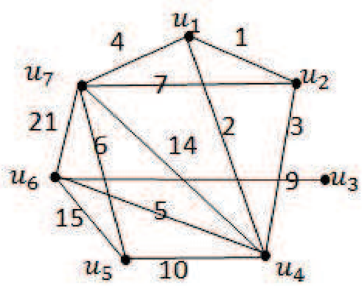

Figure 3

(c) $\forall p \geq 9$, by Lemma 3.1 there is no vertex of degree one, since $j < 2p/(n-3) \leq 3$.
(ii) If j is even, then
(a) If $p = 3, 5$, If $p = 3$ then there is no vertex of degree one, since $f^*(u_2u_1) = 1$ and $f^*(u_2u_3) = 3$ are not repeated, so $deg(u_2) = 2$. If $p = 5$, then the vertex u_2 joins with at least two vertices, since $f^*(u_2u_1) = 1$ and $f^*(u_2u_5) = 5$ are not repeated, so $deg(u_2) \geq 2$. The vertex u_4 joins with at least two vertices, since $f^*(u_4u_5) = 10$ and $f^*(u_4u_3) = 6$ are not repeated, so $deg(u_4) \geq 2$.

(b) $\forall p \geq 7$, by Lemma 3.1 the only vertex that can join with only one vertex is u_2 , since $j < 2p/(p-3) < 4$. The vertex u_2 joins with at least two vertices, since $f^*(u_2u_1) = 1$ and $f^*(u_2u_p) = p$ (by proof of Lemma 3.1) are not repeated, so $deg(u_2) \geq 2$.

□

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