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An Absolute Matrix Summability of Infinite Series and Fourier Series

Şebnem Yildiz

ABSTRACT: The aim of this paper is to generalize a main theorem concerning weighted mean summability to absolute matrix summability which plays a vital role in summability theory by using quasi-f-power sequences.

Key Words: Summability factors, Absolute matrix summability, Fourier series, Infinite series, Hölder inequality, Minkowski inequality.

Contents

1 Introduction 49

2 The Known Results 50

3 The Main Results 52

4 Proof of Theorem 3.1 53

5 An application of absolute matrix summability to Fourier series 56

6 Applications 57

1. Introduction

Definition 1.1. *[1] A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$.*

Definition 1.2. *[17] A positive sequence $X = (X_n)$ is said to be quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = \{f_n(\sigma, \beta)\} = \left\{n^\sigma (logn)^\beta, \beta \geq 0, 0 < \sigma < 1\right\}$.*

If we take $\beta = 0$, then we have a quasi-σ-power increasing sequence. Every almost increasing sequence is a quasi-σ-power increasing sequence for any non-negative σ, but the converse is not true for $\sigma > 0$ (see [13]).

Definition 1.3. *For any sequence (λ_n) we write that $\Delta^2\lambda_n = \Delta\lambda_n - \Delta\lambda_{n+1}$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. The sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty$.*

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Let $\sum a_n$ be a given infinite series with the partial sums (s_n). By u_n^α and t_n^α we denote the nth Cesàro means of order α, with $\alpha > -1$, of the sequence (s_n) and (na_n), respectively, that is (see [8])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v \quad \text{and} \quad t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{1.1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)...(\alpha+n)}{n!} = O(n^\alpha), \qquad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \tag{1.2}$$

Definition 1.4. *[10], [12] The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if*

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\alpha - u_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty. \tag{1.3}$$

If we take $\alpha = 1$, then $|C, \alpha|_k$ summability reduces to $|C, 1|_k$ summability. Let (p_n) be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1). \tag{1.4}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.5}$$

defines the sequence (t_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) (see [11]).

Definition 1.5. *[2] The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if*

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid t_n - t_{n-1} \mid^k < \infty. \tag{1.6}$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $\left|\bar{N}, p_n\right|_k$ summability is the same as $|C, 1|_k$ (resp. $\mid \bar{N}, p_n \mid$) summability.

2. The Known Results

The following theorems are known dealing with the $\left|\bar{N}, p_n\right|_k$ summability factors of infinite series.

Theorem 2.1. *[14] Let (X_n) be a almost increasing sequence. If the sequences (X_n), (λ_n), and (p_n) satisfy the conditions*

$$\lambda_m X_m = O(1) \quad as \quad m \to \infty, \tag{2.1}$$

$$\sum_{n=1}^{m} nX_n|\Delta^2\lambda_n| = O(1) \quad as \quad m \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{m} \frac{P_n}{n} = O(P_m) \tag{2.3}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n}|t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{2.4}$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{2.5}$$

then the series $\sum a_n\lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

Theorem 2.2. *[6] Let (X_n) be a quasi-σ-power increasing sequence. If the sequences (X_n), (λ_n) and (p_n) satisfy the conditions (2.1)-(2.3), and*

$$\sum_{n=1}^{m} \frac{p_n}{P_n}\frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{2.6}$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{2.7}$$

then the series $\sum a_n\lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.

Later on, Bor has proved the following theorem by taking quasi-f-power increasing sequence instead of a quasi-σ-power increasing sequence.

Theorem 2.3. [7] *Let (X_n) be a quasi-f-power increasing sequence. If the sequences (X_n), (λ_n) and (p_n) satisfy all the conditions of Theorem 2.2, then the series $\sum a_n\lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$.*

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv}s_v, \quad n = 0, 1, ... \tag{2.8}$$

Definition 2.4. *[16] The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if*

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty, \tag{2.9}$$

If we take $p_n = 1$ for all values of n, then we have $|A|_k$ summability (see [18]). And also if we take $a_{nv} = \frac{p_v}{P_n}$, then we have $\left|\bar{N}, p_n\right|_k$ summability. Furthermore, if we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n, then $|A, p_n|_k$ summability reduces to $|C, 1|_k$ summability (see [10]).

3. The Main Results

Recently some papers have been done concerning absolute matrix summability of infinite series and Fourier series (see [3]-[5], [15], [19]-[27]). The aim of this paper is to generalize Theorem 2.3 for $|A, p_n|_k$ summability method for these series by taking quasi-f-power increasing sequence instead of a quasi-σ-power increasing sequence.
Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{3.1}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{3.2}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{3.3}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{3.4}$$

Using this notation we have the following theorem.

Theorem 3.1. *Let (X_n) be a quasi-f-power increasing sequence. Let $k \geq 1$ and $A = (a_{nv})$ be a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \ n = 0, 1, ..., \tag{3.5}$$

$$a_{n-1,v} \geq a_{nv}, \ \text{for } n \geq v + 1, \tag{3.6}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right) \tag{3.7}$$

$$\sum_{v=1}^{n-1} \frac{1}{v} \hat{a}_{n,v+1} = O(a_{nn}). \tag{3.8}$$

If the sequences (X_n), (λ_n) and (p_n) satisfy all the conditions of Theorem 2.3, then the series $\sum a_n \lambda_n$ is summable $|A, p_n|_k$, $k \geq 1$.

It may be remarked that if we take $A = (\bar{N}, p_n)$, the conditions (3.5)-(3.7) are satisfied automatically and the condition (3.8) is satisfied by the condition (2.3). We need the following lemmas for the proof of our theorem.

Lemma 3.2. [3] *Under the conditions of Theorem 2.1 we have that*

$$nX_n|\Delta\lambda_n| = O(1) \quad as \quad n \to \infty, \tag{3.9}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty. \tag{3.10}$$

4. Proof of Theorem 3.1

Let X_n be a be a quasi-f-power increasing sequence and (I_n) denotes the A-transform of the series $\sum_{n=1}^{\infty} a_n\lambda_n$. Then, we have

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v.$$

Applying Abel's transformation to this sum, we have that

$$\begin{aligned}
\bar{\Delta} I_n =& \sum_{v=1}^{n} \hat{a}_{nv} a_v \lambda_v \frac{v}{v} = \sum_{v=1}^{n-1} \Delta(\frac{\hat{a}_{nv}\lambda_v}{v}) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn}\lambda_n}{n} \sum_{r=1}^{n} r a_r \\
=& \sum_{v=1}^{n-1} \Delta(\frac{\hat{a}_{nv}\lambda_v}{v})(v+1)t_v + \hat{a}_{nn}\lambda_n \frac{n+1}{n} t_n \\
=& \sum_{v=1}^{n-1} \bar{\Delta} a_{nv} \lambda_v t_v \frac{v+1}{v} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \Delta\lambda_v t_v \frac{v+1}{v} \\
&+ \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \frac{t_v}{v} + a_{nn}\lambda_n t_n \frac{n+1}{n} \\
=& I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid I_{n,r} \mid^k < \infty, \quad \text{for} \quad r = 1, 2, 3, 4. \tag{4.1}$$

First, by applying Hölder's inequality with indices k and k', where $k > 1$ and $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \mid I_{n,1} \mid^k \leq& \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left\{ \sum_{v=1}^{n-1} |\frac{v+1}{v}| \left|\bar{\Delta} a_{nv}\right| |\lambda_v||t_v| \right\}^k \\
= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} & \sum_{v=1}^{n-1} \left|\bar{\Delta} a_{nv}\right| |\lambda_v|^k |t_v|^k \times \left\{ \sum_{v=1}^{n-1} \left|\bar{\Delta} a_{nv}\right| \right\}^{k-1},
\end{aligned}$$

using

$$\Delta\hat{a}_{nv} = \hat{a}_{nv} - \hat{a}_{n,v+1} = \bar{a}_{nv} - \bar{a}_{n-1,v} - \bar{a}_{n,v+1} + \bar{a}_{n-1,v+1} = a_{nv} - a_{n-1,v},$$

and from (3.5) and (3.6) we have

$$\begin{aligned}\sum_{v=1}^{n-1} |\bar{\Delta} a_{nv}| &= \sum_{v=1}^{n-1} |a_{nv} - a_{n-1,v}| = \sum_{v=1}^{n-1} (a_{n-1,v} - a_{nv}) \\ &= \sum_{v=0}^{n-1} a_{n-1,v} - a_{n-1,0} - \sum_{v=0}^{n} a_{nv} + a_{n0} + a_{nn} \\ &= 1 - a_{n-1,0} - 1 + a_{n0} + a_{nn} \leq a_{nn},\end{aligned}$$

and using $\sum_{n=v+1}^{m+1} |\bar{\Delta} a_{nv}| \leq a_{vv}$ we have,

$$\begin{aligned}\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} | I_{n,1} |^k &= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1} \left\{\sum_{v=1}^{n-1} |\bar{\Delta} a_{nv}||\lambda_v|^k |t_v|^k\right\} \\ &= O(1) \sum_{v=1}^{m} |\lambda_v|^{k-1} |\lambda_v||t_v|^k \sum_{n=v+1}^{m+1} |\bar{\Delta} a_{nv}| \\ &= O(1) \sum_{v=1}^{m} \frac{1}{X_v^{k-1}} |\lambda_v||t_v|^k a_{vv} \\ &= O(1) \sum_{v=1}^{m-1} \Delta|\lambda_v| \sum_{r=1}^{v} a_{rr} \frac{|t_r|^k}{X_r^{k-1}} + O(1)|\lambda_m| \sum_{v=1}^{m} a_{vv} \frac{|t_v|^k}{X_v^{k-1}} \\ &= O(1) \sum_{v=1}^{m-1} |\Delta\lambda_v| X_v + O(1)|\lambda_m| X_m \\ &= O(1) \quad \text{as} \quad m \to \infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.1. Also, we have that

$$\begin{aligned}&\sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} | I_{n,2} |^k \leq \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left\{\sum_{v=1}^{n-1} |\frac{v+1}{v}||\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right\}^k \\ &= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left\{\sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\Delta\lambda_v||t_v| \frac{X_v}{X_v}\right\}^k \\ &= O(1) \sum_{n=2}^{m+1} \left(\frac{P_n}{p_n}\right)^{k-1} \left\{\sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\Delta\lambda_v| X_v \frac{1}{X_v^k} |t_v|^k\right\} \times \left\{\sum_{v=1}^{n-1} \hat{a}_{n,v+1} |\Delta\lambda_v| X_v\right\}^{k-1}\end{aligned}$$

$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\left\{\sum_{v=1}^{n-1}\hat{a}_{n,v+1}|\Delta\lambda_v|X_v\frac{1}{X_v^k}|t_v|^k\right\}\times\left\{\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v\right\}^{k-1}$$
$$= O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{1}{X_v^{k-1}}\frac{1}{v}|t_v|^k\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1} = O(1)\sum_{v=1}^{m}v|\Delta\lambda_v|\frac{1}{vX_v^{k-1}}|t_v|^k$$
$$= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\frac{|t_r|^k}{rX_r^{k-1}} + O(1)m|\Delta\lambda_m|\sum_{r=1}^{m}\frac{|t_r|^k}{rX_r^{k-1}}$$
$$= O(1)\sum_{v=1}^{m-1}|\Delta(v|\Delta\lambda_v|)|X_v + O(1)m|\Delta\lambda_m|X_m$$
$$= O(1)\sum_{v=1}^{m-1}vX_v|\Delta^2\lambda_v| + O(1)\sum_{v=1}^{m-1}X_v|\Delta\lambda_v| + O(1)m|\Delta\lambda_m|X_m$$
$$= O(1) \quad \text{as} \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.1. Furthermore, as in $I_{n,1}$, we have

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} | I_{n,3} |^k \leq \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|}{v}\right\}^k$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left\{\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|^k\frac{|t_v|^k}{v}\right\}\times\left\{\sum_{v=1}^{n-1}\frac{1}{v}\hat{a}_{n,v+1}\right\}^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}\sum_{v=1}^{n-1}|\lambda_{v+1}||\lambda_{v+1}|^{k-1}\frac{|t_v|^k}{v}\hat{a}_{n,v+1}$$
$$= O(1)\sum_{v=1}^{m}\frac{|t_v|^k}{v}\frac{1}{X_v^{k-1}}|\lambda_{v+1}|\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1}$$
$$= O(1)\sum_{v=1}^{m}\frac{|t_v|^k}{v}\frac{1}{X_v^{k-1}}|\lambda_{v+1}|\sum_{n=v+1}^{m+1}\hat{a}_{n,v+1}$$
$$= O(1)\sum_{v=1}^{m}\frac{1}{X_v^{k-1}}|\lambda_{v+1}|\frac{|t_v|^k}{v}$$
$$= O(1) \quad \text{as} \quad m\to\infty,$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.1. Again, as in $I_{n,1}$, we have that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,4}|^k = O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^k|\lambda_n|^k|t_n|^k$$

$$= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1} a_{nn}^{k-1}a_{nn}|\lambda_n|^{k-1}|\lambda_n||t_n|^k$$

$$= O(1)\sum_{n=1}^{m} a_{nn}\frac{1}{X_n^{k-1}}|\lambda_n||t_n|^k = O(1) \quad \text{as} \quad m\to\infty,$$

by virtue of hypotheses of the Theorem 3.1 and Lemma 3.1. This completes the proof of Theorem 3.1.

5. An application of absolute matrix summability to Fourier series

Let f be a periodic function with period 2π and integrable (L) over $(-\pi,\pi)$. Without any loss of generality the constant term in the Fourier series of f can be taken to be zero, so that

$$f(t) \sim \sum_{n=1}^{\infty}(a_n cosnt + b_n sinnt) = \sum_{n=1}^{\infty} C_n(t). \tag{5.1}$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)cos(nt)dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)sin(nt)dt.$$

We write

$$\phi(t) = \frac{1}{2}\left\{f(x+t)+f(x-t)\right\}, \tag{5.2}$$

$$\phi_\alpha(t) = \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1}\phi(u)\,du, \quad (\alpha>0). \tag{5.3}$$

It is well known that if $\phi(t)\in\mathcal{BV}(0,\pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the $(C,1)$ mean of the sequence $(nC_n(x))$ (see [9]).
The Fourier series play an important role in many areas of applied mathematics and mechanics. Using these series, Bor has obtained the following main result.

Theorem 5.1. *[6] Let (X_n) be a quasi-σ-power increasing sequence. If $\phi_1(t)\in\mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 2.2, then the series $\sum C_n(x)\lambda_n$ is summable $|\bar{N},p_n|_k$, $k\geq 1$.*

Theorem 5.2. *[7] Let (X_n) be a quasi-f-power increasing sequence. If $\phi_1(t)\in\mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 2.3, then the series $\sum C_n(x)\lambda_n$ is summable $|\bar{N},p_n|_k$, $k\geq 1$.*

We now apply the above theorems to the weighted mean in which $A=(a_{nv})$ is defined as $a_{nv} = \frac{p_v}{P_n}$ when $0\leq v\leq n$, where $P_n = p_0+p_1+...+p_n$. Therefore, it is well known that

$$\bar{a}_{nv} = \frac{P_n - P_{v-1}}{P_n} \quad \text{and} \quad \hat{a}_{n,v+1} = \frac{p_n P_v}{P_n P_{n-1}}.$$

We can obtain new results dealing with absolute matrix summability of Fourier series in the following manner.

Theorem 5.3. *Let A be a positive normal matrix satisfying the conditions of Theorem 3.1. Let (X_n) be a quasi-σ-power increasing. If $\phi_1(t) \in \mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 3.1, then the series $\sum C_n(x)\lambda_n$ is summable $|A, p_n|_k$, $k \geq 1$.*

Theorem 5.4. *Let A be a positive normal matrix satisfying the conditions of Theorem 3.1. Let (X_n) be a quasi-f-power increasing sequence. If $\phi_1(t) \in \mathcal{BV}(0,\pi)$, and the sequences (p_n), (λ_n), and (X_n) satisfy the conditions of Theorem 3.1, then the series $\sum C_n(x)\lambda_n$ is summable $|A, p_n|_k$, $k \geq 1$.*

6. Applications

We may now ask whether there are some examples other than weighted mean methods of matrices A that satisfy the hypotheses of Theorem 3.1. and by applying Theorem 3.1, Theorem 5.3 and Theorem 5.4 to weighted mean so, the following results can be easily verified.
1. If we take $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, Theorem 5.3 and Theorem 5.4, then we have Theorem 2.3, Theorem 5.1 and Theorem 5.2.
2. If we take $p_n = 1$ for all values of n in Theorem 3.1, Theorem 5.3 and Theorem 5.4, then we have a new result dealing with $|A|_k$ summability.
3. If we take $a_{nv} = \frac{p_v}{P_n}$ and $p_n = 1$ for all values of n in Theorem 3.1, Theorem 5.3 and Theorem 5.4, then we have a new result concerning $|C,1|_k$ summability.

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Şebnem Yildiz,
Department of Mathematics,
Ahi Evran University,
Kırşehir, Turkey.
E-mail address: `sebnemyildiz@ahievran.edu.tr and sebnem.yildiz82@gmail.com`