## Recent Results for the Logarithmic Keller-Segel-Fisher/KPP System [*]

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ABSTRACT: We consider a Keller-Segel type chemotaxis model with logarithmic sensitivity and logistic growth. It is a $2 \times 2$ system describing the interaction of cells and a chemical signal. We study Cauchy problem with finite initial data, i.e., without the commonly used smallness assumption on initial perturbations around a constant ground state. We survey a sequence of recent results by the authors on the existence of global-in-time solution, long-time behavior, vanishing coefficient limit and optimal time decay rates of the solution.

Key Words: Chemotaxis, Logarithmic sensitivity, Logistic growth, Cauchy problem, Classical solution, Global well-posedness, Long-time behavior, Vanishing coefficient limit, Optimal time-decay.

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## 1. Introduction

We consider a Keller-Segel type chemotaxis model with logarithmic sensitivity and logistic growth:

$$\begin{cases} c_t = \varepsilon c_{xx} - \mu u c - \sigma c, \\ u_t + \chi[u(\ln c)_x]_x = D u_{xx} + au(1 - \frac{u}{K}). \end{cases} \tag{1.1}$$

Here the unknown functions $c = c(x,t)$ and $u = u(x,t)$ are functions of the space variable $x \in \mathbb{R}$ and time variable $t \in \mathbb{R}^+$, representing the concentration of a chemical signal and the density of a cellular population, respectively. The system parameters are interpreted as follows.

- $\varepsilon \geq 0$: diffusion coefficient of chemical signal;

- $\mu \neq 0$: coefficient of density-dependent production/degradation rate of chemical signal;

- $\sigma \geq 0$: natural degradation rate of chemical signal;

- $\chi \neq 0$: coefficient of chemotactic sensitivity;

- $D > 0$: diffusion coefficient of cellular population;

- $a > 0$: natural growth rate of cellular population;

- $K > 0$: typical carrying capacity of cellular population.

The system describes the dynamics when certain biological organism releases or consumes a chemical signal in the local environment while both entities are naturally diffusing and reacting. The system includes logarithmic chemotactic response of cells to the signal, and some or all of the following mechanisms: random walk/diffusion, consumption/deposition of the chemical by cells, natural degradation of the chemical, and the logistic growth of cells. In particular, the logarithmic sensitivity function in (1.1) accounts for the Fechner's law, which states that subjective sensation is proportional to the logarithm of the stimulus intensity. Such a sensitivity function was used in the original Keller-Segel model [1].

The model was first proposed without the logistic growth term by Othmer and Stevens to describe the movement of a chemotactic population that deposits a chemical signal to modify the local environment for succeeding passages [7]:

$$\begin{cases} c_t = \varepsilon c_{xx} - \mu u c - \sigma c, \\ u_t + \chi[u(\ln c)_x]_x = D u_{xx}. \end{cases} \tag{1.2}$$

That is, $\chi < 0$ and $\mu < 0$ were assumed in (1.2). Such a model has found applications in cancer research [3]. Biologically, the sign of $\chi$ dictates whether the chemotactic movement is attractive ($\chi > 0$) or repulsive ($\chi < 0$). Therefore, it makes perfect sense to have $\chi > 0$ and $\mu > 0$ in (1.2), as it describes that the

cells are attracted to and consume the chemical. Since there is no difference in the analysis of these two scenarios, we assume $\chi\mu > 0$ throughout this paper. Mathematically, the non-diffusive part of the transformed system to be discussed below is hyperbolic in biologically relevant regimes when $\chi\mu > 0$, while it may change type when $\chi\mu < 0$ [15].

For Othmer-Stevens' model (1.2), the logarithmic function can be removed by the inverse Hopf-Cole transformation [2]:

$$v = (\ln c)_x = \frac{c_x}{c}. \tag{1.3}$$

Under the new variables, (1.2) is converted into a system of hyperbolic-parabolic conservation laws:

$$\begin{cases} v_t + (\mu u - \varepsilon v^2)_x = \varepsilon v_{xx}, \\ u_t + \chi(uv)_x = D u_{xx}, \end{cases} \tag{1.4}$$

noting $\varepsilon \geq 0$ and $D > 0$.

Similarly, the inverse Hopf-Cole transformation (1.3) converts the Keller-Segel-Fisher/KPP model (1.1) into a system of hyperbolic-parabolic balance laws:

$$\begin{cases} v_t + (\mu u - \varepsilon v^2)_x = \varepsilon v_{xx}, \\ u_t + \chi(uv)_x = D u_{xx} + au(1 - \frac{u}{K}). \end{cases} \tag{1.5}$$

Equation (1.5) can be further simplified by using re-scaled and/or dimensionless variables:

$$\tilde{t} = \frac{\chi\mu K}{D}t, \qquad \tilde{x} = \frac{\sqrt{\chi\mu K}}{D}x, \qquad \tilde{v} = \text{sign}(\chi)\sqrt{\frac{\chi}{\mu K}}v, \qquad \tilde{u} = \frac{u}{K}. \tag{1.6}$$

After dropping the tilde accent, we arrive at

$$\begin{cases} v_t + (u - \varepsilon_2 v^2)_x = \varepsilon_1 v_{xx}, \\ u_t + (uv)_x = u_{xx} + ru(1 - u), \end{cases} \tag{1.7}$$

where the new parameters are

$$r = \frac{aD}{\chi\mu K} > 0, \qquad \varepsilon_1 = \frac{\varepsilon}{D} \geq 0, \qquad \varepsilon_2 = \frac{\varepsilon}{\chi}. \tag{1.8}$$

## 1.1. Cauchy Problem

Now we consider the Cauchy problem of (1.1):

$$\begin{cases} c_t = \varepsilon c_{xx} - \mu uc - \sigma c, \\ u_t + \chi[u(\ln c)_x]_x = D u_{xx} + au(1 - \frac{u}{K}), \quad x \in \mathbb{R}, \ t \in \mathbb{R}^+, \\ (c, u)(x, 0) = (c_0, u_0)(x), \quad x \in \mathbb{R}, \end{cases} \tag{1.9}$$

or equivalently, the Cauchy problem of (1.7) (after the change of variables and rescaling):

$$\begin{cases} v_t + (u - \varepsilon_2 v^2)_x = \varepsilon_1 v_{xx}, \\ u_t + (uv)_x = u_{xx} + ru(1 - u), \end{cases} \quad x \in \mathbb{R}, \ t \in \mathbb{R}^+, \tag{1.10}$$

$$(v, u)(x, 0) = (v_0, u_0)(x), \quad x \in \mathbb{R}.$$

The Cauchy datum $(v_0, u_0)$ is assumed to be a perturbation of a constant equilibrium state $(\bar{v}, \bar{u})$.

To be an equilibrium state we need $\bar{u} = 0$ or $\bar{u} = 1$. It is clear that the former is unstable. Therefore, we set $\bar{u} = 1$. To discuss $\bar{v}$ we recall the definition of $v$ in (1.3), which gives us

$$v_0 = \frac{c_0'}{c_0}, \qquad c_0(x) = c_0(0)e^{\int_0^x v_0(y)\,dy} \text{ with } c_0(0) > 0, \tag{1.11}$$

where for simplicity we have omitted the scaling constant $\text{sign}(\chi)\sqrt{\chi/\mu K}$ from (1.6). If $v_0 - \bar{v} \in L^1(\mathbb{R})$ while $\bar{v} \gtrless 0$, we have

$$\int_0^\infty v_0(y)\,dy = \pm\infty, \quad \int_{-\infty}^0 v_0(y)\,dy = \pm\infty.$$

Therefore, from (1.11) we have $c_0(x) \to \infty$ either as $x \to \infty$ or as $x \to -\infty$, depending on $\bar{v} > 0$ or $\bar{v} < 0$. For physically interesting problems we consider $\lim_{x\to\pm\infty} c_0(x) = c_\pm$ with $0 < c_\pm < \infty$. Therefore, we take $\bar{v} = 0$. In summary,

$$\lim_{x\to\pm\infty}(v_0, u_0) = (v_\pm, u_\pm) = (0, 1). \tag{1.12}$$

### 1.2. Small Data Solutions

Cauchy problem (1.10) fits in the general framework of hyperbolic-parabolic balance laws:

$$w_t + f(w)_x = [B(w)w_x]_x + q(w), \quad x \in \mathbb{R}, \ t \in \mathbb{R}^+, \tag{1.13}$$

$$w(x, 0) = w_0(x), \quad x \in \mathbb{R},$$

where $w, f, q \in \mathbb{R}^n$, and $B \in \mathbb{R}^{n \times n}$. The unknown function $w = w(x, t)$ represents density functions of physical quantities; $f$ is the flux function; $B$ is the viscosity matrix, representing viscosity, heat conduction, and species diffusion; and $q$ is for external force, relaxation, chemical reaction, etc. We assume that $f$, $B$ and $q$ are smooth functions of $w$. In applications, $B$ and $q'$ (the Jacobian matrix of $q$) are usually rank deficient. The system can be extended to multi space dimensions:

$$w_t + \sum_{j=1}^m f_j(w)_{x_j} = \sum_{j,k=1}^m \left[B_{jk}(w)w_{x_k}\right]_{x_j} + q(w), \qquad x \in \mathbb{R}^m, \ t \in \mathbb{R}^+, \tag{1.14}$$

$$w(x, 0) = w_0(x), \quad x \in \mathbb{R}^m$$

for $m \geq 1$, where $w, f_j, q \in \mathbb{R}^n$ and $B_{jk} \in \mathbb{R}^{n \times n}$.

When writing (1.10) in the form of (1.13), we have $n = 2$,

$$w = \begin{pmatrix} v \\ u \end{pmatrix}, \quad f(w) = \begin{pmatrix} u - \varepsilon_2 v^2 \\ uv \end{pmatrix}, \quad B = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q(w) = \begin{pmatrix} 0 \\ ru(1-u) \end{pmatrix}.$$

It is clear that $B$ is rank one if $\varepsilon_1 = 0$, and $q'$ is rank one.

A set of structural conditions has been proposed by the first author to (1.14), which includes (1.13), see [10]. This set of conditions is able to capture common properties of several important systems, such as Navier-Stokes equations for compressible flows, Euler equations with damping, equations for gas flows in translational and vibrational non-equilibrium, and in particular, the logarithmic Keller-Segel-Fisher/KPP model (1.10). The set contains an extended concept of entropy condition, which implies symmetrization of (1.14). (Therefore, (1.10) is symmetrizable.) The rest of the set consists of block structures of the viscosity terms and the lower order term, respectively, and the Kawashima-Shizuta condition on strong coupling [9]. Under this set of conditions we are able to answer basic questions in PDE theory in the case of small data solutions, as to be detailed below.

Suppose that the aforementioned structural conditions are satisfied. We consider the Cauchy datum $w_0$ as a perturbation of a constant equilibrium state $\bar{w}$: $q(\bar{w}) = 0$. Existence of (unique) solution to (1.14) global-in-time is established in [10] if $w_0 - \bar{w}$ is small in $H^s(\mathbb{R}^m)$, $s > m/2 + 1$. For multi space dimensions $m \geq 2$, optimal $L^2$ time decay rates are obtained in [11]: If $w_0 - \bar{w}$ is small in $H^s(\mathbb{R}^m) \cap L^1(\mathbb{R}^m)$, $s > m/2 + 1$, then $\|D_x^l(w - \bar{w})\|_{L^2}(t)$ has decay rate $(t+1)^{-m/4-l/2}$ for $0 \leq l \leq s-2$. Also for multi space dimensions $m \geq 2$ and under the same assumptions, we can show that $w$ is time-asymptotically approximated in $L^2$ by $w^*$, the solution of the corresponding linear system with the same initial datum [12]. Optimal $L^2$ decay rates can be obtained for one space dimension $m = 1$ as well [13]: If $w_0 - \bar{w}$ is small in $H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, $s \geq 4$, then $\|D_x^l(w - \bar{w})\|_{L^2}(t)$ has decay rate $(t + 1)^{-1/4-l/2}$ for $0 \leq l \leq s - 2$. We note that all $L^2$ results can be extended to $L^p$ results, $2 \leq p \leq \infty$, in a straightforward manner by applying Gagliardo-Nirenberg inequality [6].

The direct application of results under the general setting (1.13) or (1.14) to Keller-Segel-Fisher/KPP model (1.10) gives us the following: If $(v_0, u_0 - 1)$ is small in $H^s(\mathbb{R})$, $s \geq 2$, then (1.10) has a unique solution $(v, u)(x, t)$ global-in-time. If $(v_0, u_0 - 1)$ is small in $H^s(\mathbb{R}) \cap L^1(\mathbb{R})$, $s \geq 4$, then $\|D_x^l v\|_{L^2}(t)$ has decay rate $(t + 1)^{-1/4-l/2}$ for $0 \leq l \leq s - 2$. The variable $u$ actually has better rates towards $\bar{u} = 1$: $\|D_x^l(u-1)\|_{L^2}(t)$ decays at the rate $(t + 1)^{-3/4-l/2}$ for $0 \leq l \leq s - 4$. This is because $u$ is the faster decaying part in the solution resulted from the logistic growth term. In the general setting (1.13) or (1.14), there is also a faster decaying part in $w$ towards $\bar{w}$ due to the lower order term $q(w)$. For a discussion and the formulation of this part readers are referred to [10,11,13]. The construction of asymptotic solutions in one space dimension is intrinsically different from multi space dimensions, and is necessary to contain nonlinear Burgers waves if the system is nonlinear in nature. While we do not have a general theory for (1.13), we are able to obtain an asymptotic solution for (1.10) by ad hoc consideration [14], again,

under smallness assumption on the initial perturbation. The asymptotic solution for $v$ is a heat kernel or a Burgers wave (depending on $\varepsilon = 0$ or $\varepsilon > 0$), carrying the same mass as $v_0$, while for $u - 1$ it is a constant multiple of the first spatial derivative of that wave. Recently, the study has been extended to the pointwise sense, both in space and in time, for the case of non-diffusive chemical, $\varepsilon = 0$, [8].

### 1.3. Large Data Solutions

It is challenging to remove the smallness assumption on the initial perturbation. For instance, there is no general global existence theory for (1.14) or (1.13) if $w_0 - \bar{w}$ is finite in $H^s(\mathbb{R}^m)$ for an appropriate $s$. In fact, for many important physical systems in the form of (1.14) or (1.13), such as Navier-Stokes equations for compressible flows, the global existence of solutions with finite initial perturbations is a long standing obstacle. As an effort in this direction, in recent years there are a few studies on this issue for specific, simpler systems, such as the Keller-Segel model (1.4) with $\varepsilon \geq 0$. In particular, the results reported in [4,5] showed that for any given constant state $(0, \bar{u})$ with $\bar{u} > 0$, and any initial datum $(v_0, u_0)$ such that $u_0 \geq 0$ and $(v_0, u_0 - \bar{u}) \in H^2(\mathbb{R})$ (no smallness assumption on the perturbation), 1) there exists a unique global-in-time classical solution to the Cauchy problem of (1.4) with either $\varepsilon = 0$ or $\varepsilon > 0$; 2) the solution converges to the constant state asymptotically as time goes to infinity; and 3) the chemically diffusive solution ($\varepsilon > 0$) converges to the non-diffusive solution ($\varepsilon = 0$) as $\varepsilon \to 0$ with certain convergence rate. Moreover, under smallness assumptions on the low frequency part of the initial perturbations, the explicit (algebraic) decay rates of the solutions to (1.4) toward the constant state are identified by the method of weighted energy estimate.

Our contribution in this regard is some recent studies on the Keller-Segel-Fisher/KPP model (1.10) (hence (1.9)) concerning the existence of global-in-time solution, long-time behavior, vanishing coefficient limit and optimal time decay rates of the solution, without or relaxing the smallness assumption on the initial perturbation [15,16]. The results are to be detailed in next section. We want to point out that the solution picture in this scenario is completely different from the one for Keller-Segel model (1.4). The proofs of our results are carefully crafted by exploring the fine structure of the logistic growth term. We would also like to remark that we are able to obtain optimal time decay rates for large data solutions of (1.10) when the chemical is non-diffusive ($\varepsilon = 0$). As a comparison, all of the existing results on (non-optimal) decay rates for (1.4) require some smallness assumption on the initial perturbation, no matter $\varepsilon > 0$ or $\varepsilon = 0$, [4,5].

## 2. Main Results and Discussion for the Transformed System

**Notation 2.1.** *Throughout this paper, $\| \cdot \|$, $\| \cdot \|_\infty$ and $\| \cdot \|_{H^s}$ denote the norms of the usual Lebesgue measurable function spaces $L^2$, $L^\infty$ and Hilbert space $H^s$, respectively, on $\mathbb{R}$. We use $\|(f_1, f_2, ..., f_N)\|_X^2$ to denote the sum $\sum_{i=1}^N \|f_i\|_X^2$. Unless otherwise specified, $c_i$, $i \in \mathbb{Z}^+$, denotes a generic positive constant, whose value may vary line by line according to the context.*

Recall that we are considering (1.10), the Cauchy problem of the transformed and simplified logarithmic Keller-Segel-Fisher/KPP model, which we copy below for convenience:

$$\begin{cases} v_t + (u - \frac{\varepsilon}{\chi}v^2)_x = \frac{\varepsilon}{D}v_{xx}, \\ u_t + (uv)_x = u_{xx} + ru(1-u), \end{cases} \quad x \in \mathbb{R}, \ t \in \mathbb{R}^+, \tag{2.1}$$
$$(v,u)(x,0) = (v_0, u_0)(x), \quad x \in \mathbb{R}.$$

Here the parameter $r > 0$ is defined in (1.8), and the Cauchy datum $(v_0, u_0)$ satisfies (1.12).

## 2.1. Global Well-Posedness

The first two theorems are concerned with the existence of global-in-time solution and its long time behavior when the initial datum has potentially large energy.

**Theorem 2.2** (Global Existence [15]). *Consider the Cauchy problem* (2.1), *where* $r, D > 0$, $\chi \neq 0$ *and* $\varepsilon \geq 0$ *are fixed constants. Suppose that the initial data satisfy* $u_0 > 0$ *and* $(v_0, u_0 - 1) \in H^2(\mathbb{R}) \times H^2(\mathbb{R})$. *Then there exists a unique solution to* (2.1) *for all* $t > 0$, *such that* $u(x,t) > 0$ *for* $x \in \mathbb{R}, t > 0$, *and*

$$\|(v, u - 1)\|_{H^2}^2(t) + \int_0^t \left( \varepsilon \|v_x\|_{H^2}^2 + \|u_x\|_{H^2}^2 \right)(\tau)\, d\tau \leq c_1,$$

$$\text{and} \quad \int_0^t \|v_x\|_{H^1}^2(\tau)\, d\tau \leq c_2(1 + \varepsilon),$$

*where the constants* $c_1$ *and* $c_2$ *are independent of* $t$ *and* $\varepsilon$, *and depend on* $r, D, \chi$ *and the initial data.*

**Theorem 2.3** (Long Time Behavior [15]). *Let the conditions of Theorem 2.2 hold. Then the unique global-in-time solution to* (2.1) *enjoys the following long time behavior:*

$$\lim_{t \to \infty} \left( \|v_x\|_{H^1} + \|u - 1\|_{H^2} + \|v\|_{C^1(\mathbb{R})} + \|u - 1\|_{C^1(\mathbb{R})} \right)(t) = 0,$$

*for any* $\varepsilon \geq 0$.

We see that there is no more smallness assumption on the initial perturbation in these theorems. For the restriction $u_0 > 0$ we recall the physical interpretation of $u$, which is the density function of a cellular population. Theorem 2.2 states that if $u$ is positive at the initial time, then it stays positive afterward.

## 2.2. Zero Diffusion Limit

In many applications the chemical is much less diffusive comparing to the cellular population, $\varepsilon \ll D$, [1]. Therefore, it is interesting to consider the solution limit as $\varepsilon \to 0$ and investigate whether the chemically non-diffusive model ($\varepsilon = 0$) is a good approximation of the diffusive one ($\varepsilon > 0$) in the process of vanishing

diffusion coefficient. For this we have our third theorem, which addresses the relationship between the two solutions, and characterizes the difference between the two in terms of $\varepsilon$.

**Theorem 2.4** (Zero Diffusion Limit and Convergence Rate [15]). *Let the conditions of Theorem 2.2 hold. Let $(v^\varepsilon, u^\varepsilon)$ and $(v^0, u^0)$ be the solutions to (2.1) with $\varepsilon > 0$ and $\varepsilon = 0$, respectively, and with the same initial data. Then for any fixed $t > 0$ we have*

$$\|(v^\varepsilon - v^0, u^\varepsilon - u^0)\|^2(t) \leq e^{c_3 t} c_4 \, \varepsilon^2 (\varepsilon + 1),$$
$$\|(v_x^\varepsilon - v_x^0, u_x^\varepsilon - u_x^0)\|^2(t) \leq e^{c_5 t} c_6 \, \varepsilon (\varepsilon^2 + \varepsilon + 1),$$

*where the constants $c_3, ..., c_6$ are independent of $t$ and $\varepsilon$.*

Theorem 2.4 shows that for fixed $t > 0$, the zeroth and first frequencies of the difference between the diffusive and non-diffusive solutions decay to zero, as $\varepsilon \to 0$, at different rates in terms of $\varepsilon$.

### 2.3. Time Decay Rates When $\mathbf{c_+ = c_-}$

If in addition $v_0 \in L^1(\mathbb{R})$, (1.11) implies

$$\lim_{x \to \pm\infty} c_0(x) = c_0(0) e^{\int_0^{\pm\infty} v_0(y)\, dy} \equiv c_\pm, \quad c_0(0) > 0, \tag{2.2}$$

where $0 < c_\pm < \infty$. Theorems 2.2-2.4 are true whether $c_+ \neq c_-$ or $c_+ = c_-$. To discuss time decay rates of the solution to the constant equilibrium state $(0, 1)$ we consider the case $c_+ = c_-$. The consideration of the other case involves an even more complicated approach, and is left to a future work.

Assuming $c_+ = c_-$, (1.11) implies

$$\int_{-\infty}^\infty v_0(x)\, dx = \int_{-\infty}^\infty \frac{c_0'(x)}{c_0(x)}\, dx = 0. \tag{2.3}$$

Noting from the first equation of (2.1), $v$ is a conserved quantity. That is,

$$\int_{-\infty}^\infty v(x, t)\, dx = \int_{-\infty}^\infty v(x, 0)\, dx = 0. \tag{2.4}$$

This allows one to define the anti-derivative:

$$\psi(x, t) = \int_{-\infty}^x v(y, t)\, dy, \quad t \geq 0. \tag{2.5}$$

Our next theorem is on the time decay rates in this setting:

**Theorem 2.5** (Decay Rates [15]). *Let the conditions of Theorem 2.2 hold, and assume that $\psi_0 \equiv \psi(\cdot, 0) \in L^2(\mathbb{R})$.*

- When $\varepsilon = 0$, the unique global-in-time solution to (2.1) satisfies

$$(1+t)\|v\|^2(t) + (1+t)^2\|u-1\|^2(t) + \int_0^t (1+\tau)\|(v_x, u-1)\|^2(\tau)\, d\tau \le c_7,$$

$$(1+t)^2\|(v_x, u_x)\|^2(t) + \int_0^t (1+\tau)^2\|(v_{xx}, u_x)\|^2(\tau)\, d\tau \le c_8,$$

$$(1+t)^3\|(v_{xx}, u_{xx})\|^2(t) + \int_0^t (1+\tau)^3 \left(\varepsilon\|v_{xxx}\|^2 + \|u_{xx}\|_{H^1}^2\right)(\tau)\, d\tau \le c_9,$$
(2.6)

where the constants $c_7, c_8, c_9$ are independent of $t$.

- When $\varepsilon > 0$, let $N > 0$ be an arbitrarily fixed constant. Then there exists a constant $\delta > 0$, such that if $\|v_0'\|^2 + \|u_0'\|^2 \le N$ and

$$\|\psi_0\|^2 + \|v_0\|^2 + \|u_0 - 1\|^2 \le \delta,$$
(2.7)

then the global-in-time solution to (2.1) enjoys the same decay rates as in (2.6).

The statement of Theorem 2.5 is slightly different from the one in [15] due to a brief, additional iteration, see [16] for a justification. From the theorem we see that explicit time decay rates can be obtained for chemically non-diffusive model ($\varepsilon = 0$) with large data. For chemically diffusive model ($\varepsilon > 0$) we do need the smallness assumption (2.7) on $\psi_0$ and $(v_0, u_0 - 1)$. Comparing with what has been discussed in Section 1.2, however, we do not need the smallness on derivatives of $(v_0, u_0)$.

From (2.6) we see that the $L^2$ decay rate for $v$ is $(t+1)^{-1/2}$ while the one for $u - 1$ is $(t+1)^{-1}$. These are better than $(t+1)^{-1/4}$ and $(t+1)^{-3/4}$ given in Section 1.2. This is possible because Theorem 2.5 is for the case $c_+ = c_-$. In fact, under such a scenario the optimal rates are $(t+1)^{-3/4}$ and $(t+1)^{-5/4}$, respectively, as to be discussed next. Although the rates obtained in Theorem 2.5 via energy and weighted energy methods are not optimal, they are the starting point of an iteration scheme that finally leads to the optimal ones. That is, (2.6) is crucial in obtaining optimal time decay rates.

### 2.4. Optimal Time Decay Rates When $c_+ = c_-$

We now announce a new result on the optimal rates, which is resulted from an iteration via spectral analysis and Duhamel's principle, and based on Theorem 2.5:

**Theorem 2.6** (Optimal Decay Rates [16]). *Let the conditions of Theorem 2.2 hold, and assume that $\psi_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, $u_0 - 1 \in L^1(\mathbb{R})$.*

- When $\varepsilon = 0$, the unique global-in-time solution to (2.1) satisfies

$$(1+t)^{\frac{3}{4}}\|v\|(t) + (1+t)^{\frac{5}{4}}(\|v_x\| + \|u-1\|)(t) + (1+t)^{\frac{7}{4}}\|u_x\|(t) \le c_{10}, \quad (2.8)$$

where the constant $c_{10} > 0$ depends only on the system parameters and initial data.

- *When $\varepsilon > 0$, let $N > 0$ be an arbitrarily fixed constant. Then there exists a constant $\delta > 0$, such that if $\|v_0'\|^2 + \|u_0'\|^2 \leq N$ and*

$$\|\psi_0\|^2 + \|v_0\|^2 + \|u_0 - 1\|^2 \leq \delta,$$

  *then the global-in-time solution to* (2.1) *enjoys*

$$(1+t)^{\frac{3}{4}}\|v\|(t) + \sum_{k=0}^{1}(1+t)^{\frac{5}{4}+\frac{k}{2}}(\|D_x^{k+1}v\| + \|D_x^k(u-1)\|)(t) + (1+t)^{\frac{9}{4}}\|u_{xx}\|(t) \leq c_{11},$$
  
$$(2.9)$$

  *where the constant $c_{11} > 0$ depends only on the system parameters and initial data.*

## 3. Main Result and Discussion for the Original System

Recall the original system of Keller-Segel-Fisher/KPP model (1.9):

$$\begin{cases} c_t = \varepsilon c_{xx} - \mu u c - \sigma c, \\ u_t + \chi[u(\ln c)_x]_x = Du_{xx} + au(1 - \frac{u}{K}), \end{cases} \quad x \in \mathbb{R}, \; t \in \mathbb{R}^+,$$
$$(c, u)(x, 0) = (c_0, u_0)(x), \quad x \in \mathbb{R},$$
$$(3.1)$$

where $\chi, \mu \neq 0$, $D, a, K > 0$ and $\varepsilon, \sigma \geq 0$ are constants, with $\chi\mu > 0$. The variable $u$ here is the same as the one in (2.1) except for the rescaling defined in (1.6). The variable $c$, on the other hand, is related to $v$ in (2.1) by the transform (1.3):

$$v = (\ln c)_x = \frac{c_x}{c}, \tag{3.2}$$

and the rescaling (1.6). Therefore, as long as $v_0 = c_0'/c_0$ and $u_0$ satisfy the assumptions of Theorems 2.2-2.6, corresponding conclusions from Section 2 apply. In particular, properties of the cellular population density function $u$ in Section 2 directly apply to $u$ in (3.1). To obtain properties of the chemical concentration function $c$, however, additional iteration based on the first equation of (3.1) and the results in Section 2 is needed. Here we announce a new result in this regard:

**Theorem 3.1** (Optimal Decay Rates in Original Variables [16]). *Suppose that the initial data satisfy $c_0(x) > 0$, $c_0 - \bar{c} \in H^3(\mathbb{R}) \cap L^1(\mathbb{R})$, with a constant $\bar{c} > 0$, $u_0 > 0$, and $u_0 - K \in H^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Then there exists a unique solution to* (3.1) *for all $t \geq 0$. The solution satisfies $c(x, t) > 0$ and $u(x, t) > 0$ for all $x \in \mathbb{R}$ and $t \geq 0$. Writing*

$$c(x, t) = e^{-(\mu K + \sigma)t}\tilde{c}(x, t), \tag{3.3}$$

*the solution has the following decay property:*

- *When $\varepsilon = 0$, for $t \geq 0$ we have*

$$\sum_{k=0}^{2}(1+t)^{\frac{1}{4}+\frac{k}{2}}\|D_x^k(\tilde{c} - \bar{c})\|(t) + \sum_{k=0}^{1}(1+t)^{\frac{5}{4}+\frac{k}{2}}\|D_x^k(u - K)\|(t) \leq c_{12}, \quad (3.4)$$

  *where the constant $c_{12} > 0$ depends only on the system parameters and initial data.*

- *When $\varepsilon > 0$, let $N > 0$ be an arbitrarily fixed constant. Then there exists a constant $\delta > 0$, such that if $\|c_0''\|^2 + \|u_0'\|^2 \leq N$ and*

$$\|c_0 - \bar{c}\|_{H^1}^2 + \|u_0 - K\|^2 \leq \delta,$$

*then the global-in-time solution to (3.1) enjoys*

$$\sum_{k=0}^{3}(1+t)^{\frac{1}{4}+\frac{k}{2}}\|D_x^k(\tilde{c}-\bar{c})\|(t) + \sum_{k=0}^{2}(1+t)^{\frac{5}{4}+\frac{k}{2}}\|D_x^k(u-K)\|(t) \leq c_{13}, \quad (3.5)$$

*where the constant $c_{13} > 0$ depends only on the system parameters and initial data.*

The zero mass restriction (2.3) on $v_0$ becomes $c_+ = c_- = \bar{c}$ in the original variable. That is, the initial perturbation $(c_0 - \bar{c}, u_0 - K)$ in Theorem 3.1 is truly generic in the sense that its mass can be anything finite, not necessary zero. From (3.3), when $\mu < 0$ and $\sigma > 0$, $c$ exponentially grows in time if $-\mu K > \sigma$, and exponentially decays if $-\mu K < \sigma$. It is interesting to observe that in the critical case of $-\mu K = \sigma$, $c - \bar{c}$ decays in $L^2$ algebraically, with the rate $(t+1)^{-1/4}$. This is consistent to that of a heat kernel, the solution to the diffusion equation with an initial point-mass. Algebraic rates like this have not been obtained before for the Keller-Segel model (1.2).

## 4. Conclusion

We have surveyed some recent results and announced new ones on the qualitative behavior of large data classical solutions to a Keller-Segel type chemotaxis model (1.1), which has logarithmic sensitivity and logistic growth. The model can be transformed into a system of hyperbolic-parabolic balance laws (1.7) by the inverse Hopf-Cole transformation. We considered the Cauchy problem of (1.7) on $\mathbb{R}$ with the initial data being perturbed around the constant equilibrium state $(v_\pm, u_\pm) = (0, 1)$. We have results on existence of global-in-time solution, time asymptotic stability of the constant equilibrium state, and vanishing diffusion limit of the diffusive solution as $\varepsilon \to 0$. We also have optimal time decay rates of the solution to the constant equilibrium state.

Based on the results of the Cauchy problem of (1.7) we are able to go back to the original physical variables, and consider the Cauchy problem of (1.1). In particular, we have optimal time decay rates of the chemical concentration function and the cell density function towards their constant background values.

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## References

1. Keller, E.F. and Segel, L.A., *Traveling bands of chemotactic bacteria: a theoretical analysis*, J. Theor. Biol. 30, 235-248, (1971).

2. Levine, H.A. and Sleeman, B.D., *A system of reaction diffusion equations arising in the theory of reinforced random walks*, SIAM J. Appl. Math. 57, 683-730, (1997).

3. Levine, H.A., Sleeman, B.D. and Nilsen-Hamilton, M., *A mathematical model for the roles of pericytes and macrophages in the initiation of angiogenesis. I. The role of protease inhibitors in preventing angiogenesis*, Math. Biosci. 168, 71-115, (2000).

4. Li, D., Pan, R. and Zhao, K. *Quantitative decay of a one-dimensional hybrid chemotaxis model with large data*, Nonlinearity 28, 2181-2210, (2015).

5. Martinez, V., Wang, Z. and Zhao, K., *Asymptotic and viscous stability of large-amplitude solutions of a hyperbolic system arising from biology*, Indiana Univ. Math. J. 67, 1383-1424, (2018).

6. Nirenberg, L., *On elliptic partial differential equations*, Ann. Scuola Norm. Sup. Pisa 13, 115-162 (1959).

7. Othmer, H. and Stevens, A., *Aggregation, blowup and collapse: The ABC's of taxis in reinforced random walks*, SIAM J. Appl. Math. 57, 1044-1081, (1997).

8. Rugamba, J. and Zeng, Y., *Pointwise Asymptotic Behavior of a Chemotaxis Model*, Math. Comput. Appl., 23 Paper No. 56, 12 pp. (2018).

9. Shizuta, Y. and Kawashima, S., *Systems of equations of hyperbolic-parabolic type with applications to the discrete Boltzmann equation*, Hokkaido Math. J. 14, 249-275 (1985).

10. Zeng, Y., *Global existence theory for general hyperbolic-parabolic balance laws with application*, J. Hyperbolic Differ. Equ. 14, 359-391, (2017).

11. Zeng, Y., *$L^p$ decay for general hyperbolic-parabolic systems of balance laws*, Discrete Contin. Dyn. Syst. Ser. A 38, 363-396, (2018).

12. Zeng, Y., *Asymptotic behavior of solutions to general hyperbolic-parabolic systems of balance laws in multi space dimensions*, Pure Appl. Math. Q., special issue in honor of Prof. Chi-Wang Shu's 60th birthday, 14, 161–192 (2018).

13. Zeng, Y., *$L^p$ time asymptotic decay for general hyperbolic-parabolic balance laws with applications*, Submitted.

14. Zeng, Y., *Hyperbolic-parabolic balance laws: asymptotic behavior and a chemotaxis model*, Comm. Appl. Anal., (2017 SEARCDE Proceedings), to appear.

15. Zeng, Y. and Zhao, K., *On the logarithmic Keller-Segel-Fisher/KPP system*, Discrete Contin. Dyn. Syst. Ser. A, 39, 5365–5402 (2019).

16. Zeng, Y. and Zhao, K., Optimal decay rates for a chemotaxis model with logistic growth, logarithmic sensitivity and density-dependent production/consumption rate. J. Differential Equations, to appear.

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