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Variational Analysis For Some Frictional Contact Problems *

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abstract: We consider a class of evolutionary variational problems which describes the static frictional contact between a piezoelectric body and a conductive obstacle. The formulation is in a form of coupled system involving the displacement and electric potential fields. We provide the existence of unique weak solution of the problems. The proof is based on the evolutionary variational inequalities and Banach's fixed point theorem.

Key Words: Evolutionary variational inequality, Fixed point, Frictional contact, Piezoelectric material.

Contents

1. Introduction

In this paper we study a class of abstract evolutionary variational problem modelling the frictional contact of electro viscoelastic body with a conductive foundation. Frictional contact phenomena appear in everyday life and play a very important role in engineering structures and systems. The frictional contact conditions are various and may be complex, a considerable effort has been made in its modelling. An early study of a contact problem for elastic and viscoelastic materials within the framework of the variational inequalities was carried in reference works [4,11,13]. An extension to non convex energy functionals generated by non monotone laws was introduced in [12,15] and led to the so called hemivariational inequalities.

The piezoelectricity lie between the coupling of the mechanical and electrical material properties, it leads to the appearance of electric field when the mechanical

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stress is applied, and conversely in the presence of the electric potential the mechanical stress is generated. Among the materials that exhibit sufficient charge very few are those which are exploited in engineering controls equipement, it include quartz, Rochelle salt, lead titanate zirconate ceramics and so on.

A considerable interest was presented on the problems involving piezoelectric materials. The general model of electro elastic materials were studied in [10,21] and for the frictional contact of electro-elastic material, they were given in [1,9]. A few mathematical results arising in a study of a frictional contact of piezoelectric bodies, like in [8,14,17,18,22], under the assumption that the foundation is insulated, and in [2,7]. There is a need to expand the study the model for the process of frictional contact when the foundation is electrically conductive and the behavior of the piezoelectricity material are taken into account in the formulation of the electrical and mechanical boundary conditions. In the case of an eletro viscoelastic materiel the frictional contact of the body with a deformable and conductive foundation leads to a non smooth boundary conditions on the contact surface. The conductivity of the contact surface involves a coupling between the mechanical and the electrical unknowns so it leads to an ill posed mathematical problem (see [2,7]) . It is bypassed by the regularization of these electrical boundary conditions. It is proved in in the last references that there exists a unique weak solution for the frictional problem described by a normal compliance condition and version of Coulomb's law friction and a regularized electrical conditions.

This work is a continuation in this line of research and we study an abstract weak formulation of quasistatic frictional contact problem for an electro-viscoelastic material, in the framework of the MTCM, when the foundation is deformable and conductive and the friction is described by the normal compliance and versions of Coulomb's law. Our interest is to show that the abstract problem with regularised electric boundary conditions has a unique weak solution.

The paper is structured as follows. In section 2 we list some notations and assumptions on the problem data and state our main existence and uniqueness result. In section 3 we present the proof of the theorem. The arguments of the proof are based on the evolutionary variational inequalities and Banach's fixed point theorem. In section 4 we give an example of application of the abstract result.

The abstract weak formulation is given by the following problem $\mathcal{P}_V$.

Problem $\mathcal{P}_V$

Find a displacement field $u : [0,T] \to V$ and an electric potential $\varphi : [0,T] \to W$ such that

$$\begin{gathered}(A\dot{u}(t), v - \dot{u}(t))_V + (Bu(t), v - \dot{u}(t))_V + (E^*\varphi(t), v - \dot{u}(t))_V + \\ j(u(t), v) - j(u(t), \dot{u}(t)) \geq (f(t), v - \dot{u}(t))_V,\end{gathered} \tag{1.1}$$

for all $v \in V$ and $t \in [0,T]$,

$$(C\varphi(t), \zeta)_W - (Eu(t), \zeta)_W + (h(u(t), \varphi(t)), \zeta)_W = (q(t), \zeta)_W, \tag{1.2}$$

for all $\zeta \in W$ and $t \in [0,T]$, and

$$u(0) = u_0. \tag{1.3}$$

Here V and W are respectively spaces of admissible displacements and of electric potentiels, there are Hilbert spaces. The operators A, B, E, E^*, C and are respectively related to the electro viscoelastic constitutive law. The operators A, B and E are defined on V, the operators C and $E^*$are defined on W. The functionals J and h are respectively determined by the machanical and electrical boundary conditions on the contact surface. The data f is related to the traction forces and to the body forces. The data q is related to the densities of volume and surface free electric charges. The fonction u_0 is the initial data of the displacement field u. Here $[0,T]$ is the interval of the observation. The dot above u denotes the derivative of the displacement u with respect to the variable t.

2. Preliminaries and notations

We consider a body which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with a smooth boundary $\partial\Omega = \Gamma$ and a unit outward normal ν.We denote by Γ_C the contact boundary, and we use the usual notation u_ν for the normal component of vector u. We consider that spaces V and H are real Hilbert spaces satisfying $V \subset \mathcal{H} \subset V'$ and $W \subset H \subset W'$ with continuous and dense injections, where $H = L^2(\Omega)$ and $\mathcal{H} = \left(L^2(\Omega)\right)^d$. We define inner product on V and on W by

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}\,, \qquad (\varphi, \varsigma)_W = (\nabla\varphi, \nabla\varsigma)_H\,,$$

where ε is linear map defined from V to $\mathcal{H}$.We suppose that we may apply the Sobolev trace theorem, it means that there exists two constants c_0 and $\tilde{c}_0$, depending only on Ω, and parts of Γ, such that

$$|\zeta|_{L^2(\Gamma_C)} \leq c_0 \, |\zeta|_W\,, \quad \forall \zeta \in W, \tag{2.1}$$

$$|v|_{(L^2(\Gamma_C))^d} \leq \tilde{c}_0 \, |v|_V\,, \quad \forall v \in V. \tag{2.2}$$

Let the Hilbert spaces $L^p(0,T;H)$ and $W^{1,p}(0,T;V)$ $1 \leq p \leq +\infty$,

$$L^p(0,T;H) \;=\; \{\boldsymbol{u} \mid \boldsymbol{u} :]0,T[\to H\}\,, \tag{2.3}$$

$$W^{1,p}(0,T;V) \;=\; \left\{\boldsymbol{u} \in L^p(0,T;V), \;\; \dot{\boldsymbol{u}} = \frac{d\boldsymbol{u}(t)}{dt} \in L^p(0,T;V)\right\}. \tag{2.4}$$

The spaces $C(0,T;X)$ and $C^1(0,T;X)$ are respectively continuous and differentiable continuous functions from $[0,T]$ into X with a respective norms :

$$|f|_{C(0,T;X)} = \max_{t\in[0,T]} |f|_X \quad \text{and} \quad |f|_{C^1(0,T;X)} = \max_{t\in[0,T]} |f|_X + \max_{t\in[0,T]} \left|\dot{f}\right|_X.$$

The functional h given in the equation (1.2) is defined with the Riesz representation theorem by

$$(h(u,\varphi), \zeta)_W = \int_{\Gamma_C} \psi(u_\nu - g)\phi_L(\varphi - \varphi_0)\zeta \, da, \tag{2.5}$$

where φ_0 is the potential of the electric foundation. The function ϕ_L is given by

$$\phi_L(s) = \begin{cases} -L & \text{if } s < -L, \\ s & \text{if } -L \leq s \leq L, \\ L & \text{if } s > L, \end{cases} \tag{2.6}$$

here L is a large positive constant, it may be arbitrarily large, higher than any possible peak voltage in the system. The function ψ is given bellow.

We list the assumptions on the problem's data. We assume that A is nonlinear strongly monotone and Lipshitz continuous operator on V, and B is a nonlinear Lipshitz continuous operator on V such that

$$\left\{ \begin{array}{l} \text{(a) } A : V \to V. \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \qquad |Au_1 - Au_2|_V \leq L_A \, |u_1 - u_2|_V \,, \quad \forall u_1, u_2 \in V. \\ \text{(c) There exists } m_A > 0 \text{ such that} \\ \qquad (Au_1 - \ Au_2, u_1 - u_2)_V \geq m_A \, |u_1 - u_2|_V^2 \,, \quad u_1, u_2 \in V. \end{array} \right. \tag{2.7}$$

$$\left\{ \begin{array}{l} \text{(a) } B : V \to V. \\ \text{(b) There exists } L_B > 0 \text{ such that} \\ \qquad |Bu_1 - Bu_2|_V \leq L_B \, |u_1 - u_2|_V \qquad \forall \, u_1, u_2 \in V, \end{array} \right. \tag{2.8}$$

for more details see [20].
The linear operators E^* and E satisfy

$$\left\{ \begin{array}{l} \text{(a) } E^* : W \to V, \\ \text{(b) there exists } C_{E^*} > 0 \text{ such that} \\ \qquad |E^* v|_V \leq C_{E^*} \, |v|_W \,, \quad \forall \, u \in W, \\ \text{(c) } E : V \to W \\ \qquad |Eu|_W \leq C_E \, |u|_V \,, \quad \forall \, u \in V. \end{array} \right. \tag{2.9}$$

The operator C is linear such that

$$\left\{ \begin{array}{l} \text{(a) } C : W \to W, \\ \text{(b) there exists } M_C > 0 \text{ such that} \\ \qquad |C\tau|_W \leq M_C \, |\tau|_W \,, \quad \forall \, \tau \in W, \\ \text{(c) there exists } L_C > 0 \text{ such that} \\ \qquad (C\tau, C\tau)_W \geq m_C \, |\tau|_W^2 \,, \quad \forall \, \tau \in W. \end{array} \right. \tag{2.10}$$

The function ψ satisfies

$$\left\{ \begin{array}{l} \text{(a) } \psi : \Gamma_C \times \mathbb{R} \to \mathbb{R}_+. \\ \text{(b) } \exists \, L_\psi > 0 \text{ such that } |\psi(x, u_1) - \psi(x, u_2)| \leq L_\psi |u_1 - u_2| \\ \qquad \forall u_1, u_2 \, \in \mathbb{R}, \ \text{a.e. } x \in \Gamma_C. \\ \text{(c) } \exists \, M_\psi > 0 \text{ such that } |\psi(x, u)| \leq M_\psi \ \forall \, u \in \mathbb{R}, \ \text{a.e. } x \in \Gamma_C. \\ \text{(e) } x \mapsto \psi(x, u) \text{ is measurable on } \Gamma_C, \text{ for all } u \in \mathbb{R}. \\ \text{(e) } x \mapsto \psi(x, u) = 0, \text{ for all } u \leq 0. \end{array} \right. \tag{2.11}$$

The functional $j : X \times X \to \mathbb{R}$ satisfies:

$$\begin{cases} \text{(a) } j(u,\cdot) \text{ is convex and l.s.c. on } V \text{ for all } u \in V, \\ \text{(b) There exists } m > 0 \text{ such that} \\ \qquad j(u_1,v_2) - j(u_1,v_1) + j(u_2,v_1) - j(u_2,v_2) \\ \qquad\qquad \leq m \, |u_1 - u_2|_V \, |v_1 - v_2|_V \quad \forall\, u_1,u_2,v_1,v_2 \in V. \end{cases} \tag{2.12}$$

Where

$$f \in W^{1,p}(0,T;V), \tag{2.13}$$

$$q \in W^{1,p}(0,T;H), \tag{2.14}$$

$$u_0 \in V. \tag{2.15}$$

Theorem 2.1. *Assume that (2.7)–(2.15) hold. Then there exists a unique solution of the problem $\mathcal{P}_V$. Moreover, the solution satisfies*

$$u \in W^{2,p}(0,T;V), \quad \varphi \in W^{1,p}(0,T;W). \tag{2.16}$$

3. Proof of the main result

The proof of Theorem 2.1 is carried out in several steps and is based on the following abstract result for evolutionary variational inequalities.

Let X be a real Hilbert space with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\cdot\|_X$, and consider the following problem, find a function $x : [0,T] \to X$ such that

$$\begin{aligned} &(A\dot{x}(t), y - \dot{x}(t))_X + (Bx(t), y - \dot{x}(t))_X + j(x(t), y) - j(x(t), \dot{x}(t)) \\ &\geq (f, y - \dot{x}(t))_X \quad \forall\, y \in X,\ t \in [0,T], \end{aligned} \tag{3.1}$$

$$x(0) = x_0. \tag{3.2}$$

To solve problem (3.1) and (3.2) we need the following assumptions.
The operator $A : X \to X$ is strongly monotone and Lipschitz continuous, i.e.,

$$\begin{cases} \text{(a) there exists } m_A > 0 \text{ such that} \\ \qquad (Ax_1 - Ax_2, x_1 - x_2)_X \geq m_A \|x_1 - x_2\|_X^2, \ \forall\, x_1,x_2 \in X, \\ \text{(b) there exists } L_A > 0 \text{ such that} \\ \qquad \|Ax_1 - Ax_2\|_X \leq L_A \|x_1 - x_2\|_X, \ \forall\, x_1,x_2 \in X. \end{cases} \tag{3.3}$$

The nonlinear operator $B : X \to X$ is Lipschitz continuous, i.e., there exists $L_B > 0$ such that

$$\|Bx_1 - Bx_2\|_X \leq L_B \|x_1 - x_2\|_X \quad \forall\, x_1,x_2 \in X. \tag{3.4}$$

The functional $j : X \times X \to \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) } j(x,\cdot) \text{ is convex and l.s.c. on } X \text{ for all } x \in X, \\ \text{(b) There exists } m > 0 \text{ such that} \\ \qquad j(x_1,y_2) - j(x_1,y_1) + j(x_2,y_1) - j(x_2,y_2) \\ \qquad\quad \leq m \|x_1 - x_2\|_X \|y_1 - y_2\|_X, \ \forall\, x_1,x_2,y_1,y_2 \in X. \end{cases} \tag{3.5}$$

Finally, we assume that

$$f \in C(0,T,X), \quad x_0 \in X. \tag{3.6}$$

Theorem 3.1. *Let (3.3)–(3.6) hold. Then,*

(1) there exists a unique solution $x \in C^1([0,T], X)$ of problem (3.1) and (3.2),

(2) if x_1 and x_2 are two solutions of (3.1) and (3.2) corresponding to the data $f_1, f_2 \in C([0,T]; X)$, then there exists $c > 0$ such that

$$\|\dot{x}_1(t) - \dot{x}_2(t)\|_X \leq c\left(\|f_1(t) - f_2(t)\|_X + \|x_1(t) - x_2(t)\|_X\right) \quad \forall t \in [0,T], \tag{3.7}$$

(3) if, moreover, $f \in W^{1,p}(0,T;X)$, for some $p \in [1,\infty]$, then the solution satisfies $x \in W^{2,p}(0,T;X)$.

The proof of the existence and uniqueness of the solution is a standard result of evolutionary variational inequalities it can be found for example in [5,16].

We turn now to the proof of Theorem 2.1. To this end we assume that (2.7)–(2.9) and (2.12)-(2.15) hold, and with the same techniques of the fixed point theorem developped in [7] we prove the existence and uniqueness of the solution u of inequality (1.1).which satisfies the three results of the Theorem 3.1.

Let $\eta \in C([0,T], V)$ be given, and consider the following problem .

Problem $\mathcal{P}^1_\eta$.

Find a displacement field $u_\eta : [0,T] \to V$ such that

$$\begin{gathered}(A(\dot{u}_\eta(t)), v - \dot{u}_\eta(t))_V + (B(u_\eta(t)), v - \dot{u}_\eta(t))_V + (\eta(t), v - \dot{u}_\eta(t))_V +, \\ j(u_\eta(t), v) - j(u_\eta(t), \dot{u}_\eta(t)) \geq (f(t), v - \dot{u}_\eta(t))_V, \quad \forall v \in V,\ t \in [0,T], \end{gathered} \tag{3.8}$$

$$u_\eta(0) = u_0. \tag{3.9}$$

We have the following result for $\mathcal{P}^1_\eta$.

Lemma 3.2. *(1) There exists a unique solution $u_\eta \in C^1([0,T]; V)$ to the problem (3.8) and (3.9).*

(2) If u_1 and u_2 are two solutions of (3.8) and (3.9) corresponding to the data η_1, $\eta_2 \in C([0,T]; V)$, then there exists $c > 0$ such that

$$|\dot{u}_1(t) - \dot{u}_2(t)|_V \leq c\left(|\eta_1(t) - \eta_2(t)|_V + |u_1(t) - u_2(t)|_V\right) \quad \forall t \in [0,T]. \tag{3.10}$$

(3) If, moreover, $\eta \in W^{1,p}(0,T;V)$ for some $p \in [1,\infty]$, then the solution satisfies $u_\eta \in W^{2,p}(0,T;V)$.

Proof: We apply Theorem 3.1, we take $X = V$, with the inner product $(\cdot,\cdot)_V$ and the associated norm $|\cdot|_V$. The Riesz representation theorem allows us to define $f_\eta : [0,T] \to V$ by

$$(f_\eta(t), v)_V = (f(t) - \eta(t), v)_V, \tag{3.11}$$

for all $u, v \in V$ and $t \in [0,T]$. Assumptions (2.7), (2.8) and (2.12) imply that the operators A and B and j satisfy respectively conditions (3.3), (3.4) and (3.5).

Moreover, since the function $f \in W^{1,p}(0,T;V)$ and keeping in mind that $\eta \in C([0,T];V)$, we deduce from the expression (3.11) that $f_\eta \in C([0,T];V)$. We note that (2.15) shows that condition (3.6) is satisfied. Hence the Lemma 3.2 is a concequence of Theorem 3.1 . □

In the next step we use the solution $u_\eta \in C^1([0,T],V)$ of the problem $\mathcal{P}^1_\eta$, obtained in Lemma 3.2, to construct the following variational problem for the electrical potential.

Problem $\mathcal{P}^2_\eta$.

Find an electrical potential $\varphi_\eta : [0,T] \to W$ such that

$$(C\left(\varphi_\eta(t)\right), \zeta)_W - (E(u_\eta(t)), \zeta)_W + (h(u_\eta(t), \varphi_\eta(t)), \zeta)_W = (q(t), \zeta)_W, \quad \forall \zeta \in W,\ t \in [0,T]. \tag{3.12}$$

For the well-posedness of problem $\mathcal{P}^2_\eta$ we have

Lemma 3.3. *There exists a unique solution $\varphi_\eta \in W^{1,p}(0,T;W)$ which satisfies* (3.12).
Moreover, if φ_{η_1} and φ_{η_2} are the solutions of (3.12) *corresponding to η_1, $\eta_2 \in C([0,T];V)$ then, there exists $c > 0$, such that*

$$|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)|_W \le c\, |u_{\eta_1}(t) - u_{\eta_2}(t)|_V \quad \forall t \in [0,T]. \tag{3.13}$$

Before giving the proof of the theorem, note that the regularity of the solution of the electrical problem was given in [2,7] under a small condition ($M_\psi < \frac{m_C}{c_0^2}$), we will see through our proof that this small condition is omitted

Proof:

Let $t \in [0,T]$, the problem $\mathcal{P}^2_\eta$ can be written in the following operaror form. Find $\varphi_\eta : [0,T] \to W$ such that for $\forall\, t \in [0,T]$

$$\mathcal{C}_\eta(t)\varphi_\eta(t) = q_\eta(t),$$

where $\mathcal{C}_\eta(t) : W \to W$ is a nonlinear operator given by the Riesz representation theorem

$$(\mathcal{C}_\eta(t)\varphi, \zeta)_W = (C(\varphi), \zeta)_W + (h(u_\eta(t), \varphi), \zeta)_W, \tag{3.14}$$
$$(q_\eta(t), \zeta)_W = (E(u_\eta(t)), \zeta)_W + (q(t), \zeta)_W, \tag{3.15}$$

for all $\varphi, \zeta \in W$. Let $\varphi_1, \varphi_2 \in W$, then the assumptions on the coercivity of the operator C ((2.10)(c)), the positivity of the fonction ψ (see (2.11)) and on the monotonicity of the function ϕ_L defined by (2.6) imply that

$$(\mathcal{C}_\eta(t)\varphi_1 - \mathcal{C}_\eta(t)\varphi_2, \varphi_1 - \varphi_2)_W \tag{3.16}$$
$$\ge m_C\, |\varphi_1 - \varphi_2|^2_W + \int_{\Gamma_C} \psi(u_{\eta\nu}(t) - g)\big(\phi_L(\varphi_1 - \varphi_0) - \phi_L(\varphi_2 - \varphi_0)\big)(\varphi_1 - \varphi_2)\, da$$
$$\ge m_C\, |\varphi_1 - \varphi_2|^2_W\,. \tag{3.17}$$

Using (2.10)(b), the bound $|\psi(u_i - g)| \le M_\psi$, the Lipshitz continuity of ϕ_L given by (2.6) and the Sobolev trace theorem (2.1) we have

$$\begin{aligned}
&(\mathcal{C}_\eta(t)\varphi_1 - \mathcal{C}_\eta(t)\varphi_2, \zeta)_W \\
&\le M_C\, |\varphi_1 - \varphi_2|_W\, |\zeta|_W + M_\psi \int_{\Gamma_C} |\varphi_1 - \varphi_2|\, |\zeta|\, da \quad \forall \zeta \in W, \\
&\le (M_C + M_\psi c_0^2)\, |\varphi_1 - \varphi_2|_W\, |\zeta|_W\,,
\end{aligned}$$

this implies that,

$$|\mathcal{C}_\eta(t)\varphi_1 - \mathcal{C}_\eta(t)|_W \le (M_C + M_\psi c_0^2)\, |\varphi_1 - \varphi_2|_W\,. \tag{3.18}$$

This shows that the operator $\mathcal{C}_\eta(t)$ is a strongly monotone Lipschitz continuous operator on W and therefore, there exists a unique element $\varphi_\eta(t) \in W$ solution of $\mathcal{C}_\eta(t)\varphi_\eta(t) = q(t)$.

The definition of the operator (3.14) means that $\varphi_\eta(t) \in W$ is the unique solution of the nonlinear variational equation (3.12).

We show next that $\varphi_\eta \in W^{1,p}(0,T;W)$. Let $t_1, t_2 \in [0,T]$ and, for the sake of simplicity, we write $\varphi_\eta(t_i) = \varphi_i$, $u_{\eta\nu}(t_i) = u_i$, $q_\eta(t_i) = q_i$, for $i = 1,2$. Using (2.10)(a) and (2.9), we find

$$\begin{aligned}
&m_C\, |\varphi_1 - \varphi_2|_W^2 + \int_{\Gamma_C} \psi(u_1 - g)\phi_L(\varphi_1 - \varphi_0) - \psi(u_2 - g)\phi_L(\varphi_2 - \varphi_0)\, (\varphi_1 - \varphi_2)\, da \\
&\le c_E\, |u_1 - u_2|_V\, |\varphi_1 - \varphi_2|_W + |q_1 - q_2|_W\, |\varphi_1 - \varphi_2|_W\,.
\end{aligned} \tag{3.19}$$

Adding and subtracting the same quantity $\int_{\Gamma_C} \psi(u_1 - g)\phi_L(\varphi_2 - \varphi_0)\, (\varphi_1 - \varphi_2)$ in the left side of the inequality (3.19) we get

$$\begin{aligned}
&m_C\, |\varphi_1 - \varphi_2|_W^2 + \int_{\Gamma_C} \psi(u_1 - g)\big(\phi_L(\varphi_1 - \varphi_0) - \phi_L(\varphi_2 - \varphi_0)\big)\, (\varphi_1 - \varphi_2)\, da \\
&+ \int_{\Gamma_C} \big(\psi(u_1 - g) - \psi(u_2 - g)\big)\phi_L(\varphi_2 - \varphi_0)\, (\varphi_1 - \varphi_2)\, da \\
&\le c_E\, |u_1 - u_2|_V\, |\varphi_1 - \varphi_2|_W + |q_1 - q_2|_W\, |\varphi_1 - \varphi_2|_W\,;
\end{aligned}$$

this yeilds to

$$\begin{aligned}
&m_C\, |\varphi_1 - \varphi_2|_W^2 + \int_{\Gamma_C} \psi(u_1 - g)\big(\phi_L(\varphi_1 - \varphi_0) - \phi_L(\varphi_2 - \varphi_0)\big)\, (\varphi_1 - \varphi_2)\, da \\
&\le c_E\, |u_1 - u_2|_V\, |\varphi_1 - \varphi_2|_W + |q_1 - q_2|_W\, |\varphi_1 - \varphi_2|_W \\
&\quad + \int_{\Gamma_C} |\psi(u_1 - g) - \psi(u_2 - g)\phi_L(\varphi_2 - \varphi_0)|\, |\varphi_1 - \varphi_2|\, da,
\end{aligned}$$

the positivity of ψ and the monotonicity ϕ_L leads to

$$\psi(u_1 - g)\big(\phi_L(\varphi_1 - \varphi_0) - \phi_L(\varphi_2 - \varphi_0)\big)\, (\varphi_1 - \varphi_2) \ge 0, \tag{3.20}$$

We use now the Lipshitz continuity of ψ, the bound $|\phi_L(\varphi_1 - \varphi_0)| \leq L$, and inequalities (2.2) and (2.1) to obtain

$$\begin{aligned} & m_C \, |\varphi_1 - \varphi_2|_W^2 \\ \leq \quad & \left[(c_E + L_\psi L c_0 \widetilde{c}_0) \, |u_1 - u_2|_V + |q_1 - q_2|_W \right] |\varphi_1 - \varphi_2|_W \, . \end{aligned} \tag{3.21}$$

It follows from inequality (3.21) that

$$|\varphi_1 - \varphi_2|_W \leq \frac{(c_E + L_\psi L c_0 \widetilde{c}_0)}{m_C} \, |u_1 - u_2|_V + \frac{1}{m_C} \, |q_1 - q_2|_W \, , \tag{3.22}$$

since $u_\eta \in C^1([0,T];\mathcal{H})$, and $q \in W^{1,p}(0,T;W)$ (2.14), inequality (3.22) implies that $\varphi_\eta \in W^{1,p}(0,T;W)$.

For all $t \in [0,T]$, let $\eta_1, \eta_2 \in C([0,T];V)$ and let $\varphi_{\eta_i} = \varphi_i$, $u_{\eta_i} = u_i$, $i = 1, 2$. We use (3.19) and the same arguments used in the proof of (3.21) we obtain

$$|\varphi_1(t) - \varphi_2(t)|_W \leq \frac{1}{m_\beta} (c_E + L_\psi L c_0 \widetilde{c}_0) \, |u_1 - u_2|_V \, .$$

This leads to (3.13), and this achieved the proof. □

We now consider the operator $\Lambda : C([0,T];V) \to C([0,T];V)$ defined by

$$\Lambda\eta(t) = E^* \varphi_\eta(t) \qquad \forall \, \eta \in C([0,T];V), \; t \in [0,T]. \tag{3.23}$$

For $\eta \in C([0,T];V)$, φ_η is the solution of the problem $\mathcal{P}^2_\eta$, by definition of E^* and the regularity of $\varphi_\eta(\ \varphi_\eta \in W^{1,2}(0,T,W))$ we have that $E^*\varphi_\eta \in C([0,T];V)$. The operator Λ is then well defined. We show now that Λ has a unique fixed point.

Lemma 3.4. *There exists a unique* $\bar{\eta} \in C([0,T];V)$ *such that* $\Lambda\bar{\eta} = \bar{\eta}$.

Proof: Let η_1, $\eta_2 \in C([0,T];V)$ and denote by u_i and φ_i the functions u_{η_i} and φ_{η_i} obtained in Lemmas 3.2 and 3.3, for $i = 1, 2$. Let $t \in [0,T]$, using (3.23) and (2.9) we obtain

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_V \leq c \, |\varphi_1(t) - \varphi_2(t)|_W \, ,$$

keeping in mind (3.13), we find

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_V \leq c \, |u_1(t) - u_2(t)|_V \, . \tag{3.24}$$

On the other hand (3.10) yields

$$|\dot{u}_1(t) - \dot{u}_2(t)|_V \leq c \Big(\, |\eta_1(t) - \eta_2(t)|_V + |u_1(s) - u_2(s)|_V \, \Big).$$

Since $u_i \in C^1(0,T,V)$, thus

$$u_i(t) = u_0 + \int_0^t \dot{u}_i(s) \, ds,$$

then

$$|u_1(t) - u_2(t)|_V \le \int_0^t |\dot u_1(s) - \dot u_2(s)|_V \, ds, \tag{3.25}$$

and

$$|\dot u_1(t) - \dot u_2(t)|_V \le c\Big(|\eta_1(t) - \eta_2(t)|_V + \int_0^t |\dot u_1(s) - \dot u_2(s)|_V \, ds\Big).$$

It follows now from a Gronwall's type lemma [3], that

$$\int_0^t |\dot u_1(s) - \dot u_2(s)|_V \, ds \le c \int_0^t |\eta_1(s) - \eta_2(s)|_V \, ds. \tag{3.26}$$

Combining (3.24)–(3.26), we deduce

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_V \le c \int_0^t |\eta_1(s) - \eta_2(s)|_V \, ds.$$

Reiterating this inequality n times implies that the operator $\Lambda^n = \underbrace{\Lambda \circ \Lambda \circ \cdots \circ \Lambda}_{n \text{ times}}$ satisfies

$$|\Lambda^n\eta_1(t) - \Lambda^n\eta_2(t)|_{C([0,T];V)} \le \frac{c^n}{n!} |\eta_1(t) - \eta_2(t)|_{C([0,T];V)} \,.$$

This inequality shows that for a sufficiently large n the operator Λ^n is a contraction on the Banach space $C([0,T];V)$ and therefore, there exists a unique element $\bar\eta \in C([0,T];V)$ such that $\Lambda\bar\eta = \bar\eta$.

When we replace η by $\bar\eta$ the unique solution of the equation $\Lambda\eta = \eta$ in problems $\mathcal{P}^1_\eta, \mathcal{P}^2_\eta$ and taking into account that $\Lambda\bar\eta = E^*\varphi_{\bar\eta}$, then the couple $(u_{\bar\eta}, \varphi_{\bar\eta})$ becomes the unique solution of the respective problems $\mathcal{P}^1_{\bar\eta}$ and $\mathcal{P}^2_{\bar\eta}$, therefore $(u_{\bar\eta}, \varphi_{\bar\eta})$ is the unique solution of the problem $\mathcal{P}_V$. The regularity (2.16) follows from Lemmas 3.2 and 3.3. The proof of the Theorem 2.1 is now complete. □

4. Application

We choose a model of the frictional contact of an electro-viscoelastic material with deformable and conductive foundation which descibed by the problem P :

Problem P Find a displacement field $\boldsymbol{u} : \Omega \times [0,T] \to \mathbb{R}^d$, an electric potential $\varphi : \Omega \times [0,T] \to \mathbb{R}$, an electric displacement field, such that

$$\boldsymbol{\sigma} \; = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{G}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \mathcal{E}^*\nabla\varphi \text{ in } \Omega \times (0,T)\,, \tag{4.1}$$

$$\boldsymbol{D} = \mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}) - \gamma\nabla\varphi \qquad \text{in } \Omega \times (0,T)\,, \tag{4.2}$$

$$Div\ \boldsymbol{\sigma} + \boldsymbol{f}_0 \; = 0 \quad \text{ in } \Omega \times (0,T)\,, \tag{4.3}$$

$$div\ \boldsymbol{D} - \boldsymbol{q}_0 = 0 \qquad \text{in } \Omega \times \times (0,T)\,, \tag{4.4}$$

$$\boldsymbol{u} \; = 0, \qquad \text{on } \Gamma_1 \times (0,T)\,, \tag{4.5}$$

$$\boldsymbol{\sigma}\nu = \boldsymbol{f}_2 \quad \text{on } \Gamma_2 \times (0,T)\,, \tag{4.6}$$

$$\boldsymbol{\sigma}_\nu = -\,p\,(u_\nu - g)\ , \text{on } \Gamma_3 \times (0,T)\,, \tag{4.7}$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau\| \le \mu p_\tau(u_\nu - g), \text{ on } \Gamma_3 \times (0,T)\,, \\ \|\boldsymbol{\sigma}_\tau\| < \mu p_\tau(u_\nu - g) \Longrightarrow \dot{\mathbf{u}}_\tau = \mathbf{0,} \\ \|\boldsymbol{\sigma}_\tau\| = \mu p_\tau(u_\nu - g) \Longrightarrow \exists\lambda > 0 \text{ such that } \boldsymbol{\sigma}_\tau = -\lambda\dot{\mathbf{u}}_\tau, \end{cases} \tag{4.8}$$

$$\varphi = 0 \quad \text{on } \Gamma_a \times (0,T)\,, \tag{4.9}$$

$$\boldsymbol{D}\cdot\nu = q_b \quad \text{on}\,\Gamma_b \times (0,T)\,, \tag{4.10}$$

$$\boldsymbol{D}\cdot\nu = k\psi\,(u_\nu - g)\,\phi_L\,(\varphi - \varphi_0) \ \text{ on } \Gamma_3 \times (0,T)\,, \tag{4.11}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \qquad \text{in } \Omega. \tag{4.12}$$

First, Equations. (4.1) and (4.2) represent the nonlinear electro-viscoelastic constitutive law in which $\boldsymbol{\sigma} = (\sigma_{ij})$ is the stress tensor, $\mathcal{A}$ and $\mathcal{G}$ are the viscosity and elasticity operators, whereas $\varepsilon(\mathbf{u}) = (e_{ij}(\mathbf{u}))$ denotes the linearized strain tensor, respectively, $\mathcal{E} = (e_{ijk})$ is the third-order piezoelectric tensor, $\mathcal{E}^*$ is its transpose, $\gamma = (\gamma_{ij})$ denotes the electric permittivity tensor and $\mathbf{D}$ is the electric displacement vector.

The equations (4.3) and (4.4) are the equilibrium equations, in equation (4.3) we suppose the process is quasistatic. Here the conditions (4.5) and (4.6) are the displacement and traction boundary conditions, respectively conditions (4.7) and (4.8), represent frictionl contact condition on Γ_3 described by the normal compliance function p; such that $p\,(r) = 0$ when $r \le 0$, g is the initial gap and the condition, $u_\nu - g \ge 0$ represents the penetration of body in the foundation.The friction bound $\mu p_\tau(u_\nu - g)$ is the maximum value of the modulus of the tangential tensor. Conditions (4.7) and (4.8) were used in several studies as in [6,19]. The expressions (4.9) and (3.8) are boundary conditions on electric potential φ and displacement field D on Γ_a and Γ_b. On part of the boundary Γ_3, and during the process of contact the normal of electric displacement field is assumed to be proportional to the difference between the potential of foundation φ_0 and the body's surface potential, given by condition (4.11). Finally, (4.12) is the initial condition on displacement.

Next denote by $\mathbb{S}^d$ the space of second order of symmetric tensors on $\mathbb{R}^d$ $(d = 1,2,3)$ and by $(\cdot)$ and $|\cdot|$ respectively the scalar product and the Euclidean norm in $\mathbb{S}^d$ (resp in $\mathbb{R}^d$).

$$\begin{aligned} &\boldsymbol{u}.\boldsymbol{v} = u_i v_i \qquad |u| = (u.u)^{\frac{1}{2}} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d, i = 1, \cdots d. \\ &\boldsymbol{\sigma}.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \qquad |\tau| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d, i = 1, \cdots d, j = 1, \cdots d. \end{aligned}$$

Here and bellow the indices i, j run between 1 and d and the summation convention over repeated indices is adopted. Let $\Omega \subset \mathbb{R}^d$, we shall use the notation

$$\begin{array}{l} H = \left\{ \boldsymbol{u} = (u_i) \mid u_i \in L^2(\Omega) \right\} = \left(L^2(\Omega) \right)^d, \\ \mathcal{W} = \left\{ \boldsymbol{D} \in H \mid div\boldsymbol{D} \in L^2(\Omega) \right\}, \\ \mathcal{H} = \left\{ \boldsymbol{\sigma} = (\sigma)_{ij} \mid \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \right\}, \\ H_1 = \left\{ \boldsymbol{u} = (u_i) \mid \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H} \right\}, \\ \mathcal{H}_1 = \left\{ \boldsymbol{\sigma} \in \mathcal{H} \mid Div\boldsymbol{\sigma} \in H \right\}, \end{array}$$

with $\boldsymbol{\varepsilon} : H \to \mathcal{H}$ and $Div : \mathcal{H} \to H$ are respectively operators of deformation and divergence defined by :

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\boldsymbol{\varepsilon}_{ij}(\boldsymbol{u})), \varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{ij} + u_{ji}) \text{ and } Div\boldsymbol{\sigma} = (\sigma_{ij,j}),$$

The tensors $\mathcal{E} = (e_{ijk})$ and its transpose $\mathcal{E}^* = (e_{kij})$ satisfy the equality $\mathcal{E}\boldsymbol{\sigma} \cdot \boldsymbol{v} = \boldsymbol{\sigma} \cdot \mathcal{E}^* \boldsymbol{v}$, where the index that follows a comma indicates a partial derivative with respect to the corresponding component of the independent variable. The space H, $\mathcal{H}$, H^1 and $\mathcal{H}^1$ are Hilbert spaces endowed with the inner products given by

$$\begin{array}{rcl} (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} & = & \int_\Omega \sigma_{ij}\tau_{ij}\, dx, \\ (\boldsymbol{u}, \boldsymbol{v})_{H_1} & = & (\boldsymbol{u}, \boldsymbol{v})_H + (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}}, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} & = & (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\boldsymbol{\sigma}, Div\boldsymbol{\tau})_H. \end{array}$$

For every vector $\boldsymbol{v} \in H_1$, we use the notation $\boldsymbol{v}$ for the trace of $\boldsymbol{v}$ on Γ and we denote by $\boldsymbol{v}_\nu$ and $\boldsymbol{v}_\tau$ the normal and tangential components of $\boldsymbol{v}$ on Γ, given by $v_\nu = \boldsymbol{v} \cdot \nu$ and $\boldsymbol{v}_\tau = \boldsymbol{v} - v_\nu \nu$. For regular stress field $\boldsymbol{\sigma}$ (say C^1), the application of its trace to ν is the Cauchy stress vector $\boldsymbol{\sigma}\nu$. We define the normal and tangential components of $\boldsymbol{\sigma}$ by $\sigma_\nu = (\boldsymbol{\sigma}\nu) \cdot \nu$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\nu - \boldsymbol{\sigma}_\nu \nu$, and recall that the Green's formula holds

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} + (Div\boldsymbol{\sigma}, \boldsymbol{v})_H = \int_\Gamma \boldsymbol{\sigma}\nu \cdot \boldsymbol{v}, \quad \forall \boldsymbol{v} \in H_1. \tag{4.13}$$

Let set now

$$V = \{\boldsymbol{v} \in H_1 \mid \boldsymbol{v} = \boldsymbol{0}, \text{ in } \Gamma_1\}, \ \ W = \left\{\xi \in H^1 \mid \xi = 0, \text{ in } \Gamma_a\right\},$$

for $\boldsymbol{u}$, $\boldsymbol{v} \in V$ we have $(\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}_1}$ and for $\varphi, \xi \in W$, $(\varphi, \xi)_W = (\nabla\varphi, \nabla\xi)_H$, and we have $|\boldsymbol{u}|_V = (\boldsymbol{u},\boldsymbol{u})_V^{1/2}$, $|\varphi|_V = (\varphi, \varphi)_W^{1/2}$. We give now the assumptions on the datas of the problem .The viscosity operator $\mathcal{A}$ and the elasticity one $\mathcal{G}$ satisfy the conditions

$$\left\{\begin{array}{l} \text{a) } \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d, \\ \text{b) there exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \left|\mathcal{A}\left(\boldsymbol{x}, \boldsymbol{\varepsilon}_1\right) - \mathcal{A}\left(\boldsymbol{x}, \boldsymbol{\varepsilon}_2\right)\right| \leq L_{\mathcal{A}} \left|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\right| \ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{, a.e. } \boldsymbol{x} \in \Omega, \\ \text{c) there exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \left(\mathcal{A}\left(\boldsymbol{x}, \boldsymbol{\varepsilon}_1\right) - \mathcal{A}\left(\boldsymbol{x}, \boldsymbol{\varepsilon}_2\right)\right) \cdot \left(\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\right) \geq m_{\mathcal{A}} \left|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\right|^2, \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \text{d) the mapping } \longmapsto \mathcal{A}\left(\boldsymbol{x}, \boldsymbol{\varepsilon}\right) \text{ is Lebesgue measurable on } \Omega, \\ \text{e) the mapping } \boldsymbol{x} \longmapsto \mathcal{A}\left(\boldsymbol{x}, \mathbf{0}\right) \text{ belongs to } \mathcal{H}. \end{array}\right. \tag{4.14}$$

$$\left\{\begin{array}{l} \text{a) } \mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d, \\ \text{b) there exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \left|\mathcal{G}\left(\boldsymbol{x}, \boldsymbol{\varepsilon}_1\right) - \mathcal{G}\left(\boldsymbol{x}, \boldsymbol{\varepsilon}_2\right)\right| \leq L_{\mathcal{G}} \left|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\right|, \\ \forall \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega, \text{ such that} \\ \text{c) the mapping } \boldsymbol{x} \longmapsto \mathcal{G}\left(\boldsymbol{x}, \boldsymbol{\varepsilon}\right) \text{ is Lebesgue measurable on } \Omega, \forall \boldsymbol{\varepsilon} \in \mathbb{S}^d, \\ \text{d) the mapping } \boldsymbol{x} \longmapsto \mathcal{G}\left(\boldsymbol{x}, \mathbf{0}, 0\right) \text{ belongs to } \mathcal{H}. \end{array}\right. \tag{4.15}$$

The permeability tensor γ satisfy

$$\left\{\begin{array}{l} \text{a) } \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d, \\ \text{b) } \mathcal{E}\left(\boldsymbol{x}, \zeta\right) = \left(e_{ijk}(\boldsymbol{x})\zeta_{jk}\right), \quad \forall \zeta = \left(\zeta_{ij}\right) \in \mathbb{S}^d \text{ a.e. } \boldsymbol{x} \in \Omega, \\ \text{c) } e_{ijk} = e_{ikj} \in L^\infty\left(\Omega\right) \end{array}\right. \tag{4.16}$$

$$\left\{\begin{array}{l} \text{a) } \gamma : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \\ \text{b) } \gamma\left(\boldsymbol{x}, \mathbf{E}\right) = \left(\gamma_{ij}(\boldsymbol{x}) E_j\right), \quad \forall \mathbf{E} = \left(E_i\right) \in \mathbb{R}^d \text{a.e. } \boldsymbol{x} \in \Omega, \\ \text{c) } \gamma_{ik} = \gamma_{ji} \in L^\infty\left(\Omega\right), \\ \text{d) there exists } m_\gamma > 0 \text{ such that } \gamma_{ij}(\boldsymbol{x}) E_i E_j \geq m_\gamma \left\|\mathbf{E}\right\|^2, \\ \forall \mathbf{E} = \left(E_i\right) \in \mathbb{R}^d \text{a.e. } \boldsymbol{x} \in \Omega. \end{array}\right. \tag{4.17}$$

We also assume that the normal compliance function p satisfies

$$\left\{\begin{array}{l} \text{a) } p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+, \\ \text{b) there exists } L_p > \text{ such that } \left|p(\boldsymbol{x}, r_1) - p(\boldsymbol{x}, r_2)\right| \leq L_p \left|r_1 - r_2\right|, \\ \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma, \\ \text{c) the mapping } \boldsymbol{x} \longmapsto p(\boldsymbol{x}, r) \text{ is Lebesgue mesurable in } \Gamma_3, \forall r \in \mathbb{R}, \\ \text{d) } r \leq 0, \quad p(\boldsymbol{x}, r) = 0, \text{ a.e. } \boldsymbol{x} \in \Gamma_3. \end{array}\right. \tag{4.18}$$

As example of normal compliance functions which satisfy (4.18), we may consider $p(\boldsymbol{x}, r) = c r_+$, where $c > 0$ and $r_+ = \max\{0, r\}$. This condition (4.18) means that the reaction of the obstacle is proportional to the penetration $(u_\nu)_+$. The gap function g satisfies the initial potential φ_0, the friction coefficient the volume of forces $\boldsymbol{f}_0$ and $\boldsymbol{f}_2$ and the charges densities q_a, q_b satisfy

$$g \in L^2(\Gamma_3), g \geq 0 \text{ a.e. on } \Gamma_3, \ \varphi_0 \in L^2(\Gamma_3), \tag{4.19}$$

$$\mu \in L^\infty(\Gamma_3), \mu \geq 0 \text{ a.e. on } \Gamma_3, \ |\mu|_{L^\infty(\Gamma_3)} \leq \mu_0, \tag{4.20}$$

$$\boldsymbol{f}_0 \in L^2\left(0, T; L^2\left(\Omega\right)^d\right), \ \boldsymbol{f}_2 \in L^2\left(0, T; L^2\left(\Gamma_2\right)^d\right), \tag{4.21}$$

$$q_0 \in W^{1,p}\left(0, T; L^2\left(\Omega\right)\right), q_b \in W^{1,p}\left(0, T; L^2\left(\Gamma_b\right)\right). \tag{4.22}$$

By means of a Riesz representation theorem and the assumptions on the problem's data, let define the mappings

$$(\boldsymbol{f}(t),\boldsymbol{v})_{V',V} = (\boldsymbol{f}_0(t),\boldsymbol{v})_H + (\boldsymbol{f}_2(t),\boldsymbol{v})_{L^2(\Gamma_2)^d}\,, \quad \forall \boldsymbol{v}\in V, \text{ a.e. } t\in(0,T)\,, \tag{4.23}$$

$$(q(t),\xi)_W = -\left(q_0(t),\xi\right)_{L^2(\Omega)} - \left(q_b(t),\xi\right)_{L^2(\Gamma_b)}\,, \quad \forall \xi\in W,\ t\in(0,T)\,, \tag{4.24}$$

$$j(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_3} p(u_\nu - g)v_\nu\ da + \int_{\Gamma_3} \mu p(u_\nu - g)\left\|\mathbf{v}_\tau\right\|\ da,\ \forall \boldsymbol{u},\boldsymbol{v}\in V, \tag{4.25}$$

$$\left(h\left(\boldsymbol{u},\varphi\right),\xi\right) = \int_{\Gamma_3} \psi\left(u_\nu - g\right)\phi_L\left(\varphi-\varphi_0\right)\xi\ da, \forall \boldsymbol{u}\in V,\quad \forall \xi,\varphi\in W,$$

The variational formulation of the problem P is given by

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}\,(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}){-}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}\,(t)\,))_{\mathcal{H}} + (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}){-}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}\,(t)\,))_{\mathcal{H}} +\\ &(\mathcal{E}^*\nabla\varphi(t),\boldsymbol{\varepsilon}(\boldsymbol{v}){-}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}\,(t)\,))_{\mathcal{H}} + j\left(\boldsymbol{u}(t),\boldsymbol{v}\right) - j\left(\boldsymbol{u}(t),\dot{\boldsymbol{u}}\,(t)\right)\\ &\geq (\boldsymbol{f}(t),\boldsymbol{\varepsilon}(\boldsymbol{v}){-}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}\,(t)\,))_{V',V}\,,\ \forall \boldsymbol{v}\in V, \forall t\in[0,T]\,,\end{aligned} \tag{4.26}$$

$$(\gamma\nabla\varphi(t),\nabla\xi)_{\mathcal{H}} - (\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t),\nabla\xi)_{\mathcal{H}} + \left(h\left(\boldsymbol{u}(t),\varphi(t)\right),\xi\right)_W \tag{4.27}$$

$$= (q(t),\xi)_W\,, \forall \xi\in W,\ \forall t\in[0,T]\,, \tag{4.28}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0. \tag{4.29}$$

Note on one hand that the continuous imbeddings $V\subset\mathcal{H}\subset V'$ resp $(W\subset H\subset W')$ and the Riesz representation theorem allow us to define the operators $A,B: V\to V$, $C:W\to W$, $E^*:W\to W$ and $E:V\to W$ such that

$$(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}\,(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (A\dot{u}(t),v)_V,\quad (\mathcal{G}(\boldsymbol{\varepsilon}(\boldsymbol{u}(t)),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (Bu(t),v)_V, \tag{4.30}$$

$$(\mathcal{E}^*\nabla\varphi(t),\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{H}} = (E^*\varphi(t),v)_V\,,\quad (\gamma\nabla\varphi(t),\nabla\xi)_{\mathcal{H}} = (C\varphi(t),\xi)_W\,, \tag{4.31}$$

$$(\mathcal{E}\boldsymbol{\varepsilon}(\boldsymbol{u}(t),\nabla\xi)_{\mathcal{H}} = (E\varphi(t),\xi)_W\,,\ \text{for all } v\in V,\ \zeta\in W, \tag{4.32}$$

and on the other hand by assumptions (4.18), (4.20) and the Sobolev's trace theorem (2.2), the functional j given by (4.25) is convex, l.s.c. and satisfies

$$\begin{aligned}&j(u_1,v_2) - j(u_2,v_1) + j(u_2,v_1) - j(u_2,v_2)\\ &\leq m\left|u_1-u_2\right|\left|v_1-v_2\right|_V,\ \forall u_i,v_i\in V,\ i=1,2,\end{aligned} \tag{4.33}$$

where $m=\widetilde{c}_0^2\mu_0 L_\nu$. In addition to the assumptions data of the problem P, we deduce that all the assumptions of Theorem 2.1 hold, hence we conclude that there exists a unique weak solution of the problem P satisfing (4.26)-(4.29) with the regularity $(u,\varphi)\in W^{2,p}(0,T;V)\times W^{1,p}(0,T;W)$.

5. Conclusion

We studied a class of abstract evolutionary variational problema modelling a the quasistatic process of frictional contact between a deformable body made of an electro-viscoelastic material, and a conductive deformable foundation. An example of a contact modeled with the normal compliance condition and the associated Coulomb's law of dry friction.

Our interest is to show that the abstract problem with a regularised electric boundary conditions has a unique weak solution. The existence of the unique weak solution for the problem was established by using arguments from the theory of evolutionary variational inequalities involving nonlinear strongly monotone Lipschitz continuous operators, and a fixed-point theorem. The novelly in this work is that we established the result of existence of solution without a smallness assumption given in [2,7], from now on, this will not represents a physical obstacle to study a contact problems with a deformable and conductive foundation, under the electric boundary condition (4.11) which combining the penetration of the body and it's potential drop.

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