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Applications of the Jack's Lemma for the Meromorphic Functions at the Boundary

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ABSTRACT: In this paper, a boundary version of the Schwarz lemma for classes $\mathcal{N}(\beta)$ is investigated. For the function $f(z)=\frac{1}{z}+a_0+a_1z+a_2z^2+...$ defined in the punctured disc E such that $f(z)\in\mathcal{N}(\beta)$, we estimate a modulus of the angular derivative of the function $\frac{zf'(z)}{f(z)}$ at the boundary point c with $\frac{cf'(c)}{f(c)}=\frac{1-2\beta}{\beta}$. Moreover, Schwarz lemma for class $\mathcal{N}(\beta)$ is given.

Key Words: Schwarz lemma, Analytic function, Jack's lemma, Angular derivative.

Contents

1. Introduction

Let $\mathcal{A}$ denote the class of functions

$$f(z)=\frac{1}{z}+a_0+a_1z+a_2z^2+..$$

which are analytic in the punctured disc $E=\{z:0<|z|<1\}$. Also, let $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right|<1-\beta, \tag{1.1}$$

where $\frac{1}{2}\leq\beta<1$.

To present our main results, we need the following lemma called Jack's Lemma [6] and Schwarz lemma [5].

Lemma 1.1 (Schwarz lemma)**.** *Let D be the unit disc in the complex plane $\mathbb{C}$. Let $f:D\to D$ be an analytic function with $f(0)=0$. Under these conditions, $|f(z)|\leq|z|$ for all $z\in D$ and $|f'(0)|\leq1$. In addition, if the equality $|f(z)|=|z|$ holds for any $z\neq0$, or $|f'(0)|=1$, then f is a rotation, which means $f(z)=ze^{i\gamma}$, where γ is real.*

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Lemma 1.2 (Jack's Lemma)**.** *Let $f(z)$ be a non-constant and analytic function in the unit disc D with $f(0)=0$. If $|f(z_0)| = \max\{|f(z)| : |z| \leq |z_0|\}$, then there exists a real number $k \geq 1$ such that*

$$\frac{z_0 f'(z_0)}{f(z_0)} = k.$$

Let $f(z) \in \mathcal{N}(\beta)$ and consider the function

$$\phi(z) = \frac{\beta\,(1+m(z))}{1-\beta}, \tag{1.2}$$

where $m(z) = \frac{zf'(z)}{f(z)}$. Clearly, $\phi(z)$ is analytic function in D and $\phi(0)=0$. Now let us show that the function $|\phi(z)|$ is less than 1 in the unit disc D. From (1.1) and (1.2), we have

$$\left|1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right| = (1-\beta)\left|\frac{z\phi'(z)}{(1-\beta)\,\phi(z)-\beta}\right| < (1-\beta)\,.$$

We suppose that there exists a point $z_0 \in D$ such that $\max\limits_{|z|\leq|z_0|} |\phi(z)| = |\phi(z_0)| = 1$. Thus, $\phi(z_0) = e^{i\theta}$. From the Jack's lemma, we obtain

$$\phi(z_0) = e^{i\theta} \quad \text{and} \quad \frac{z_0\phi'(z_0)}{\phi(z_0)} = k.$$

Using the last equality, we take by the elementary calculations

$$\begin{aligned}
\left|1+\frac{z_0 f''(z_0)}{f'(z_0)}-\frac{z_0 f'(z_0)}{f(z_0)}\right| &= (1-\beta)\left|\frac{z_0\phi'(z_0)}{(1-\beta)\,\phi(z_0)-\beta}\right| = (1-\beta)\left|\frac{k}{(1-\beta)\,e^{i\theta}-\beta}\right| \\
&= (1-\beta)\left|\frac{ke^{-i\theta}}{(1-\beta)-\beta e^{-i\theta}}\right| = (1-\beta)\left|\frac{k}{(1-\beta)-\beta e^{-i\theta}}\right| \\
&\geq (1-\beta)\,\frac{1}{|(1-\beta)-\beta e^{-i\theta}|}.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\left|1+\frac{z_0 f''(z_0)}{f'(z_0)}-\frac{z_0 f'(z_0)}{f(z_0)}\right| &\geq \frac{1-\beta}{|1-\beta-\beta\,(\cos\theta - i\sin\theta)|} \\
&= \frac{1-\beta}{\sqrt{(1-\beta-\beta\cos\theta)^2+\beta^2\sin^2\theta}}
\end{aligned}$$

and

$$\left|1+\frac{z_0 f''(z_0)}{f'(z_0)}-\frac{z_0 f'(z_0)}{f(z_0)}\right| \geq \frac{1-\beta}{\sqrt{(1-\beta)^2+\beta^2-2\beta\,(1-\beta)\cos\theta}}. \tag{1.3}$$

Since the right hand side of (1.3) takes its minimum value for $\cos\theta = -1$, we take

$$\left|1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{z_0 f'(z_0)}{f(z_0)}\right| \geq \frac{1-\beta}{\sqrt{(1-\beta)^2 + \beta^2 + 2\beta(1-\beta)}} = 1-\beta.$$

This is contradictory to our condition (1.1). This means that there is no point $z_0 \in D$ such that $|\phi(z_0)| = 1$. Therefore, $|\phi(z)| < 1$ for $|z| < 1$. By the Schwarz Lemma, we obtain

$$|\phi'(0)| \leq 1$$

$$|a_0| \leq \frac{1-\beta}{\beta}.$$

Thus, the following lemma is obtained.

Lemma 1.3. *If $f(z) \in \mathcal{N}(\beta)$, then*

$$|a_0| \leq \frac{1-\beta}{\beta}. \tag{1.3}$$

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies, which is called the boundary version of Schwarz Lemma, are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows: if f extends continuously to some boundary point c with $|c| = 1$, and if $|f(c)| = 1$ and $f'(c)$ exists, then $|f'(c)| \geq 1$. R. Also, Osserman [13] has given the inequalities which are called the boundary Schwarz lemma. He has first shown that

$$|f'(c)| \geq \frac{2}{1 + |f'(0)|} \geq 1 \tag{1.4}$$

under the assumption $f(0) = 0$ where f is an analytic function mapping the unit disc into itself and c is a boundary point to which f extends continuously and $|f(c)| = 1$. Many studies have been carried out for the boundary Schwarz Lemma in the last 15 years (see, [1], [3], [4], [7], [8], [14], [15], [16], and references therein). Some of them are about the estimates from below for the modulus of the derivative of the function at the boundary points which satisfy the condition $|f(c)| = 1$.

For our main results we need the following lemma known as Julia-Wolff lemma [17].

Lemma 1.4 (Julia-Wolff lemma)**.** *Let f be an analytic function in D, $f(0) = 0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(c)$ at $c \in \partial D$, $|f(c)| = 1$, then the angular derivative $f'(c)$ exists and $1 \leq |f'(c)| \leq \infty$.*

Corollary 1.5. *Let f be an analytic function in D, $f(0) = 0$ and $f(D) \subset D$. Also, the analytic function f has a finite angular derivative $f'(c)$ if and only if f' has the finite angular limit $f'(c)$ at $c \in \partial D$.*

D. M. Burns and S. G. Krantz [9] and D. Chelst [2] studied the uniqueness part of the Schwarz Lemma. For more general results and relevant estimates, see also ([10], [11], [12] and [13]). Also, M. Jeong [7] got some inequalities at a boundary point for a different form of analytic functions and showed the sharpness of these inequalities. Also, M. Jeong found a necessary and sufficient condition for an analytic map having fixed points only on the boundary of the unit disc and compared its derivatives at fixed points to get some relations among them [8]. We refer to [1] for a more detailed explanation of the Schwarz Lemma and its applications on the boundary of the unit disc.

2. Main Results

In this section, for the function $f(z) = \frac{1}{z} + a_0 + a_1 z + a_2 z^2 + ...$ defined in the punctured disc E such that $f(z) \in \mathcal{N}(\beta)$, we estimate a modulus of the angular derivative $\frac{zf'(z)}{f(z)}$ function at the boundary point c with $\frac{cf'(c)}{f(c)} = \frac{1-2\beta}{\beta}$.

Theorem 2.1. *Let $f(z) \in \mathcal{N}(\beta)$. Suppose that for some $c \in \partial D$, f' has an angular limit $f'(c)$ at c, $\frac{cf'(c)}{f(c)} = \frac{1-2\beta}{\beta}$. Then*

$$\left| \left(\frac{zf'(z)}{f(z)} \right)'_{z=c} \right| \geq \frac{1-\beta}{\beta}. \tag{2.1}$$

Proof. Let us consider the following function

$$\phi(z) = \frac{\beta\,(1 + m(z))}{1-\beta},$$

where $m(z) = \frac{zf'(z)}{f(z)}$.

Then $\phi(z)$ is an analytic function in the unit disc D and $\phi(0) = 0$. By the Jack's lemma and since $f(z) \in \mathcal{N}(\beta)$, we have $|\phi(z)| < 1$ for $|z| < 1$. Also, we have $|\phi(c)| = 1$ for $c \in \partial D$. That is,

$$|\phi(c)| = \frac{\beta}{1-\beta}\,|1 + m(c)| = \frac{\beta}{1-\beta}\,|1 + m(c)| = \frac{\beta}{1-\beta}\left|1 + \frac{1-2\beta}{\beta}\right| = 1.$$

It is clear that

$$\phi'(z) = \frac{\beta}{1-\beta} m'(z)$$

and

$$\phi'(c) = \frac{\beta}{1-\beta} m'(c).$$

From (1.4) we get

$$1 \leq |\phi'(c)| = \frac{\beta}{1-\beta}\,|m'(c)|$$

and

$$|m'(c)| \geq \frac{1-\beta}{\beta}.$$

□

The inequality (2.2) can be strengthened as below by taking into account a_0 which is first coefficient in the expansion of the function $f(z)$.

Theorem 2.2. *Under the hypothesis of the Theorem 1. Then*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=c}\right| \geq \frac{2\left(1-\beta\right)^2}{\beta\left(1-\beta+\beta\left|a_0\right|\right)}. \tag{2.3}$$

Proof. Let $\phi(z)$ be as in the above Theorem 1. Therefore, from (1.4),

$$\frac{2}{1+\left|\phi'(0)\right|} \leq \left|\phi'(c)\right| = \frac{\beta}{1-\beta}\left|m'(c)\right|.$$

Since

$$\begin{aligned}
\phi(z) &= \frac{\beta\left(1+\frac{zf'(z)}{f(z)}\right)}{1-\beta} = \frac{\beta}{1-\beta}\left(1+\frac{-\frac{1}{z}+a_1z+2az^2+...}{\frac{1}{z}+a_0+a_1z+a_2z^2+...}\right) \\
&= \frac{\beta}{1-\beta}\frac{a_0+2a_1z+3a_2z^2+...}{\frac{1}{z}+a_0+a_1z+a_2z^2+...} \\
&= \frac{\beta}{1-\beta}\left(a_0z+\left(2a_1-a_0^2\right)z^2+...\right)
\end{aligned}$$

and

$$\phi'(0) = \frac{\beta}{1-\beta}a_0,$$

it is clear that

$$\left|\phi'(0)\right| = \frac{\beta}{1-\beta}\left|a_0\right|.$$

Then

$$\frac{2}{1+\frac{\beta}{1-\beta}\left|a_0\right|} \leq \left|\phi'(c)\right| = \frac{\beta}{1-\beta}\left|m'(c)\right|$$

and

$$\left|m'(c)\right| \geq \frac{2\left(1-\beta\right)^2}{\beta\left(1-\beta+\beta\left|a_0\right|\right)}.$$

Hence, we get the desired inequality (2.3) □

The inequality (2.3) can be strengthened as below by taking into account a_1 which is second coefficient in the expansion of the function $f(z)$.

Theorem 2.3. *Let $f(z) \in \mathcal{N}(\beta)$. Suppose that for some $c \in \partial D$, f' has an angular limit $f'(c)$ at c, $\frac{cf'(c)}{f(c)} = \frac{1-2\beta}{\beta}$. Then*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=c}\right| \geq \frac{1-\beta}{\beta}\left(1+\frac{2\left(\left(1-\beta\right)-\beta\left|a_0\right|\right)^2}{\left(1-\beta\right)^2-\beta^2\left|a_0\right|^2+\beta\left(1-\beta\right)\left|2a_1-a_0^2\right|}\right). \tag{2.4}$$

Proof. Let $\phi(z)$ be the same as in proof of Theorem 1. Let us consider the function

$$h(z)=\frac{\phi(z)}{\upsilon(z)},$$

where $\upsilon(z)=z$. The function $h(z)$ is analytic in D. According to the maximum modulus princible, we have $|h(z)|<1$ for each $z\in D$. From equality of $h(z)$, we have

$$\begin{aligned}
h(z) &= \frac{\phi(z)}{z}=\frac{\frac{\beta(1+m(z))}{1-\beta}}{z}=\frac{\frac{1}{\alpha}\left(\frac{1}{1+c_2z+(2c_3-c_2^2)z^2+\dots}-1\right)}{z}\\
&= \frac{\frac{\beta}{1-\beta}\left(a_0z+\left(2a_1-a_0^2\right)z^2+\dots\right)}{z}\\
&= \frac{\beta}{1-\beta}\left(a_0+\left(2a_1-a_0^2\right)z+\dots\right)
\end{aligned}$$

Thus, we have

$$|h(0)|=\frac{\beta}{1-\beta}|a_0|\leq 1$$

and

$$|h'(0)|=\frac{\beta}{1-\beta}\left|2a_1-a_0^2\right|.$$

Moreover, it can be seen that

$$\frac{c\phi'(c)}{\phi(c)}=|\phi'(c)|\geq|\upsilon'(c)|=\frac{c\upsilon'(c)}{\upsilon(c)}.$$

Let

$$\psi(z)=\frac{h(z)-h(0)}{1-\overline{h(0)}h(z)}.$$

This function is analytic in D, $|\psi(z)|\leq 1$ for $|z|<1$, $\psi(0)=0$, and $|\psi(b)|=1$ for $c\in\partial D$. From (1.4),

$$\begin{aligned}
\frac{2}{1+|\psi'(0)|} &\leq |\psi'(b)|=\frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(c)\right|^2}|h'(c)|\\
&\leq \frac{1+|h(0)|}{1-|h(0)|}\left\{|\phi'(c)|-|\upsilon'(c)|\right\}.
\end{aligned}$$

Since

$$\psi'(z)=\frac{1-|h(0)|^2}{\left(1-\overline{h(0)}h(z)\right)^2}h'(z),$$

$$\psi'(0)=\frac{h'(0)}{1-|h(0)|^2},$$

and

$$\left|\psi'(0)\right| = \frac{\frac{\beta}{1-\beta}\left|2a_1 - a_0^2\right|}{1-\left(\frac{\beta}{1-\beta}\left|a_0\right|\right)^2} = \beta\left(1-\beta\right)\frac{\left|2a_1 - a_0^2\right|}{\left(1-\beta\right)^2 - \beta^2\left|a_0\right|^2},$$

we get

$$\begin{aligned}
\frac{2}{1+\beta\left(1-\beta\right)\frac{\left|2a_1-a_0^2\right|}{(1-\beta)^2-\beta^2|a_0|^2}} &\leq \frac{1+\frac{\beta}{1-\beta}\left|a_0\right|}{1-\frac{\beta}{1-\beta}\left|a_0\right|}\left\{\frac{\beta}{1-\beta}\left|m'(c)\right|-1\right\},\\
\frac{2\left(\left(1-\beta\right)^2-\beta^2\left|a_0\right|^2\right)}{\left(1-\beta\right)^2-\beta^2\left|a_0\right|^2+\beta\left(1-\beta\right)\left|2a_1-a_0^2\right|} &\leq \frac{1-\beta+\beta\left|a_0\right|}{1-\beta-\beta\left|a_0\right|}\left\{\frac{\beta}{1-\beta}\left|m'(c)\right|-1\right\},\\
\frac{2\left(\left(1-\beta\right)-\beta\left|a_0\right|\right)^2}{\left(1-\beta\right)^2-\beta^2\left|a_0\right|^2+\beta\left(1-\beta\right)\left|2a_1-a_0^2\right|} &\leq \frac{\beta}{1-\beta}\left|m'(c)\right|-1
\end{aligned}$$

and

$$\left|m'(c)\right| \geq \frac{1-\beta}{\beta}\left(1+\frac{2\left(\left(1-\beta\right)-\beta\left|a_0\right|\right)^2}{\left(1-\beta\right)^2-\beta^2\left|a_0\right|^2+\beta\left(1-\beta\right)\left|2a_1-a_0^2\right|}\right).$$

The last inequality is the desired inequality. □

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