Bol. Soc. Paran. Mat. (3s.) **v. 38** 7 (2020): 203–218.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v38i7.44284

Boundary Behaviour of Holomorphic Functions on the Cardioid Domain with some Applications

Shatha S. Alhily and Deepmala

ABSTRACT: The objective of this research paper is to show how the *Brennan's* conjecture becomes a useful tool to construct a holomorphic function on the cardioid domain Ω, where $\phi'(0) = 0$, for $0 < \mu = \frac{1}{n^2} \leq 1$, $n \in \mathbb{N}$ and another belongs to *Hardy space* $H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D})$, $n \in \mathbb{N}$, on the boundary of unit disk. Moreover, we have addressed some applications on the existance of cusp on the boundary of arising from integrability of conformal maps through one of the polar functions in the general solution of *Laplace* equation.

Key Words: Brennan's conjecture, Conformal mapping, Cardioid domain, Laplace equation.

Contents

1. Introduction

Let ϕ be a conformal mapping defined on the unit disk $\mathbb{D}$ onto non-empty open subset of $\mathbb{C}$(simply connected domain Ω). It is obvious and known that we may suppose condition in order to constract a conformal map ϕ, but sometime we have to be aware when the boundary $\partial\Omega$ of Ω is deformated (irregular) with expecting to have rough behaviour even if ϕ can be extended continously to $\partial\mathbb{D}$.

The radial limit is a good tool to study of the function behaviour on the boundary of given domain that is why *P. L. Duren* [6] shown that the radial limit in *Hardy* spaces theory

$$\lim_{r\to 1} \phi(re^{i\theta}) = \phi(e^{i\theta}).$$

of any function $\phi \in S$ [1] exists finitely almost everywhere in θ. *G. Piranian A. J. Lohwater and W. Rudin* [8], proved that the radial limit of a meromorphic function

2010 *Mathematics Subject Classification:* 30C20, 30C35.
Submitted August 26, 2018. Published February 01, 2019

[1] The set of conformal maps of the unit disk normalized under two conditions $\phi(0) = 0$ and $\phi'(0) = 1$, is denoted by S and is called class of *univalent* functions.

and bounded in $\mathbb{D}$ exists and is finite, for almost all points on $\partial\mathbb{D}$, but the radial limit of the derivative of a meromorphic function and bounded in $\mathbb{D}$ fails to exist and be finite.

On the other hand, the conformal image of a (measurable) subset A has the area

$$\text{area } \phi(A) = \int\int_A |\phi'(z)|^2 \mathrm{d}x\mathrm{d}y,$$

so, it is natural to address also the integral means of $|\phi'|^p$ along circles $|z| = r$, where ϕ is the conformal mapping and p is a real parameter to obtain measure of the overall size of $|\phi|$, as follows:

$$I_p(r, \phi') = \frac{1}{2\pi} \int_{|z|=r} |\phi'(z)|^p \mathrm{d}\theta \ (0 < r < 1), \tag{1.1}$$

where $\mathrm{d}\theta$ is the angular measure $\frac{|\mathrm{d}z|}{r}$.

The growth of the integral means of the derivative $\phi'(re^{i\theta})$, where $z = re^{i\theta}$, is related with Hardy spaces in case $\phi(z)$ is an holomorphic function in the unit disc $\{z = re^{i\theta} : |z| < 1\}$, univalent or not.

Now, it has become necessary to address *Brennan's conjecture* in our work which is formulated as an estimate for conformal maps $\psi : \Omega \to \mathbb{D}$,

$$\int\int_\Omega |\psi'|^p \mathrm{d}x\mathrm{d}y < \infty. \tag{1.2}$$

for $\frac{4}{3} < p < 4$. Changing the variables will offer us the possibility to write (1.2) in terms of $\psi^{-1} = \phi$:

$$\int\int_{\mathbb{D}} |\phi'|^{2-p} \mathrm{d}x\mathrm{d}y < \infty. \tag{1.3}$$

Brennan [2], introduces an interesting result about increasing of upper bound to $3+\tau$ by using a harmonic measure argument of *Carleson*. Pommerenke [13] shows that (1.3) holds for $\frac{3}{4} < p < 3.399$. In this aspect, we refer to read [12], [4], [5] .

Our work, is in part an elaboration of *P. Hajlasz*'s idea,when he constructed a bounded domain $\Omega \subset \mathbb{R}^2$ with the cone property and proved that $u(z) = \Im(z^{-1/2} + \frac{i}{2})$ is a harmonic function in Ω and belong to $\mathbb{W}_0^{1,p}(\Omega)$ for all $1 \le p < \frac{4}{3}$, and shown that the Dirichlet problem for the Laplace equation cannot be in general solved with the boundary data in $\mathbb{W}^{1,p}(\Omega)$, for all $p < 4$.

Therefore, we consider *Laplace's* equation with *Dirichlet* conditions to exist of cusp on the boundary of *cardioid* domain arising from integrability of conformal maps through one of the polar functions in the general solution of *Laplace* equation. Moreover, we proved that there is an holomorphic function defined on the boundary of cardioid domain when $\phi'(0) = 0$, for $0 < \mu = \frac{1}{n^2} \le 1$, $n \in \mathbb{N}$ and another belongs to *Hardy space* $H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D})$, $n \in \mathbb{N}$, on the boundary of unit disk, which is largely related to some results obtained by *D. Khavinson* [3,7].

In this stage, we must refer reference [17], [18],and [10], [11], which cover basic concept about this topic.

2. Main Results on the Existence of a cusp on the boundary of cardioid domain

Here are some specific concepts and techniques would be uesful in the following:

Definition 2.1. *[1], [9] Let $\Omega \subset \mathbb{R}^n$ be a simply connected domain. Fix point $z = 0$ in Ω and let $\partial\Omega$ be the boundary of Ω, let γ in Ω be defined as a simple Jordan arc which divides Ω into two subdomains.*

Let $K = (\gamma_n)_{n=1}^{\infty}$, a sequence of γ_n in the given domain Ω, be called a chain, if it satisfies all the following conditions :

i) The diameter of γ_n tends to zero as $n \to \infty$.

ii) for each n the intersection $\gamma_n \cap \gamma_{n+1}$ is empty.

iii) any path connecting $z = 0$ in Ω with arc γ_n for all $n > 1$ intersects with arc γ_{n-1}.

Moreover, any two chains $K = (\gamma_n)$ and $K' = (\gamma_n')$ in Ω are equivalent if the arc γ_n separates the point $z = 0$ from all arcs γ_n' except for a finite number of them. An equivalence class of chains in Ω is called a prime end.

Remark 2.2. *[14] Let ϕ map unit disk $\mathbb{D}$ conformally onto simply connected domain $\Omega \subset \mathbb{C}$ with locally connected boundary $\partial\Omega$. Let $\zeta = e^{i\theta} \in \partial\mathbb{D}$. Then $\partial\Omega$ has a corner of opening $\pi\alpha (0 \leq \alpha \leq 2)$ at $\phi(\zeta) \neq \infty$ if*

$$arg[\phi(e^{it}) - \phi(e^{i\theta})] \to \begin{cases} \gamma & as\ t \to \theta_+, \\ \gamma + \pi\alpha & as\ t \to \theta_-. \end{cases} \tag{2.1}$$

Hence, if ϕ maps the unit disk onto the domain Ω, this will induce a one-to-one mapping between the points on the unit circle and the prime ends of Ω. That is, there may exist another point $\zeta' \in \partial\mathbb{D}$ with $\phi(\zeta') = \phi(\zeta)$ where there may be a corner of opening $\pi\alpha'$ or none at all. Also, if $\alpha = 1$ then we obtain a tangent of direction angle γ. If $\alpha = 0$, then we will obtain an outward-pointing cusp, and if $\alpha = 2$, we will get an inward-pointing cusp.

Theorem 2.3. *[14]*
Let ϕ maps $\mathbb{D}$ conformally onto the bounded domain $\Omega \subset \mathbb{C}$. Then the following four conditions are equivalent:

i) ϕ has a continuous extension to $\overline{\mathbb{D}}$;

ii) $\partial\Omega$ is a curve, that is $\partial\Omega = \{\varphi(\zeta) : \zeta \in \partial\mathbb{D}\}$;

iii) $\partial\Omega$ is locally connected;

iv) $\mathbb{C} \backslash \Omega$ is locally connected.

2.1. The main result reads as follows:

Consider the two-dimentional Laplace equation in polar coordinates (r, θ)

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0, \tag{2.2}$$

in the circular sector $\{(r, \theta) : \ 0 < \theta < \alpha, \ 0 < r < a\} = \Omega$.

Let us start with the general solution of the *Laplace* equation under polar coordinates (r, θ) [15] as follows:

$$U(r,\theta) = \underbrace{A_1 r^{\sqrt{\mu}} cos\sqrt{\mu}\theta}_{T_1} + \underbrace{A_2 r^{\sqrt{\mu}} \sin\sqrt{\mu}\theta}_{T_2} + \underbrace{A_3 r^{-\sqrt{\mu}} cos\sqrt{\mu}\theta}_{T_3} + \underbrace{A_4 r^{-\sqrt{\mu}} \sin\sqrt{\mu}\theta}_{T_4}. \tag{2.3}$$

Choose $u(r, \theta) = r^{-\sqrt{\mu}} \sin\sqrt{\mu}\theta$, a harmonic function on Ω, such that

$$u(r, \theta) = r^{-\sqrt{\mu}} \sin\sqrt{\mu}\theta - 1 \tag{2.4}$$

to be zero on the boundary of Ω, where $r = (\sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$, then the integrability $\int\int_\Omega |\nabla u|^p \mathrm{d}x\mathrm{d}y$ depends on the local behaviour at a point $e^{i\theta} \in \partial\Omega$ as follows:

$$\begin{aligned}
|\nabla u|^2 &= \left[(\frac{\partial u}{\partial r})^2 + (\frac{1}{r}\frac{\partial u}{\partial \theta})^2 \right], \\
|\nabla u|^2 &= \mu r^{2(-\sqrt{\mu}-1)} \sin^2\sqrt{\mu}\theta + \mu r^{2(-\sqrt{\mu}-1)} cos^2\sqrt{\mu}\theta, \\
|\nabla u|^p &= \mu^{\frac{p}{2}} r^{(-\sqrt{\mu}-1)p}.
\end{aligned}$$

Then,

$$\int\int_\Omega |\nabla u|^p \mathrm{d}x\mathrm{d}y = \int\int_\Omega \mu^{\frac{p}{2}} r^{(-\sqrt{\mu}-1)p+1} \mathrm{d}r\mathrm{d}\theta \tag{2.5}$$

$$= \mu^{\frac{p}{2}} \int_0^{n\pi} \mathrm{d}\theta \left[\frac{r^{(-\sqrt{\mu}-1)p+2}}{(-\sqrt{\mu}-1)p+2} \right|_0^a \tag{2.6}$$

such that $(-\sqrt{\mu} - 1)p + 2 > 0 \Rightarrow p < \frac{2}{\sqrt{\mu}+1}$.

Hence,

$$\int\int_\Omega |\nabla u|^p \mathrm{d}x\mathrm{d}y < \infty \ for \ p < \frac{2}{\sqrt{\mu}+1}. \tag{2.7}$$

It is clear that all the values of p depend on μ, which in turn depend on the condition

$$r = \phi(\theta) = (\sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}, \tag{2.8}$$

such that if $0 \le \theta \le \pi$ then $\phi(\theta) = 0$ when $\theta = \frac{\pi}{\sqrt{\mu}}$.

Calculate the tangent vector for the function $r = \phi(\theta)$ in equation (2.8), by using the formula below:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\phi(\theta)\cos\theta + \sin\theta\phi'(\theta)}{-\phi(\theta)\sin\theta + \cos\theta\phi'(\theta)}$$

such that when $\theta = \frac{\pi}{\sqrt{\mu}}$ this implies that $\phi(\theta = \frac{\pi}{\sqrt{\mu}}) = 0$
Hence we will get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin\theta\phi'(\theta)}{\cos\theta\phi'(\theta)} = \tan\theta \text{ at } \theta = \frac{\pi}{\sqrt{\mu}}.$$

Suppose, $\tan\theta = 0$ at $\theta = \frac{\pi}{\sqrt{\mu}}$, then $\frac{\pi}{\sqrt{\mu}} = 0, \pi, 2\pi, \ldots = n\pi$, $n \in \mathbb{N}$, and it gives that

$$\mu = \frac{1}{n^2},\ n \in \mathbb{N}. \tag{2.9}$$

Consequently, we deduce that

$$\int\int_\Omega |\nabla u|^p \mathrm{d}x\mathrm{d}y < \infty \ for\ p < \frac{2}{\sqrt{\mu}+1},\ \text{where } \mu = \frac{1}{n^2},\ n \in \mathbb{N}.$$

Hence, we can classify this result, depending on n as follows:-

i) In case, $n = 2$ then $|\nabla u| \in L_p(\Omega)$ for all $p < \frac{4}{3}$ and $u(r, \theta)$ vanishes on $\partial\Omega$ except the discontinuity point $z = 0$.

ii) In case, $n \geq 3$ then $|\nabla u| \in L_p(\Omega)$ for all $p < \frac{2}{\sqrt{\mu}+1}$, and u vanishes on $\partial\Omega$ except some inward cusps at the neighborhoods of $z = 0$.

iii) According to the above, the harmonic function $u = r^{-\sqrt{\mu}} \sin\sqrt{\mu}\theta - 1$ in Ω, belongs to $\mathbb{W}^{1,p}(\Omega)$ for all $p < \frac{2}{\sqrt{\mu}+1}$, where $\mu = \frac{1}{n^2}$, $n \in \mathbb{N}$, that is, $u = 0$ on $\partial\Omega$ for $r = (\sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$ and we deduce that when we approach the vertex of the cusp $z = 0$ along the boundary of Ω we will have zero limit, whereas if we properly approach the vertex of the cusp $z = 0$ from the interior of Ω we will have

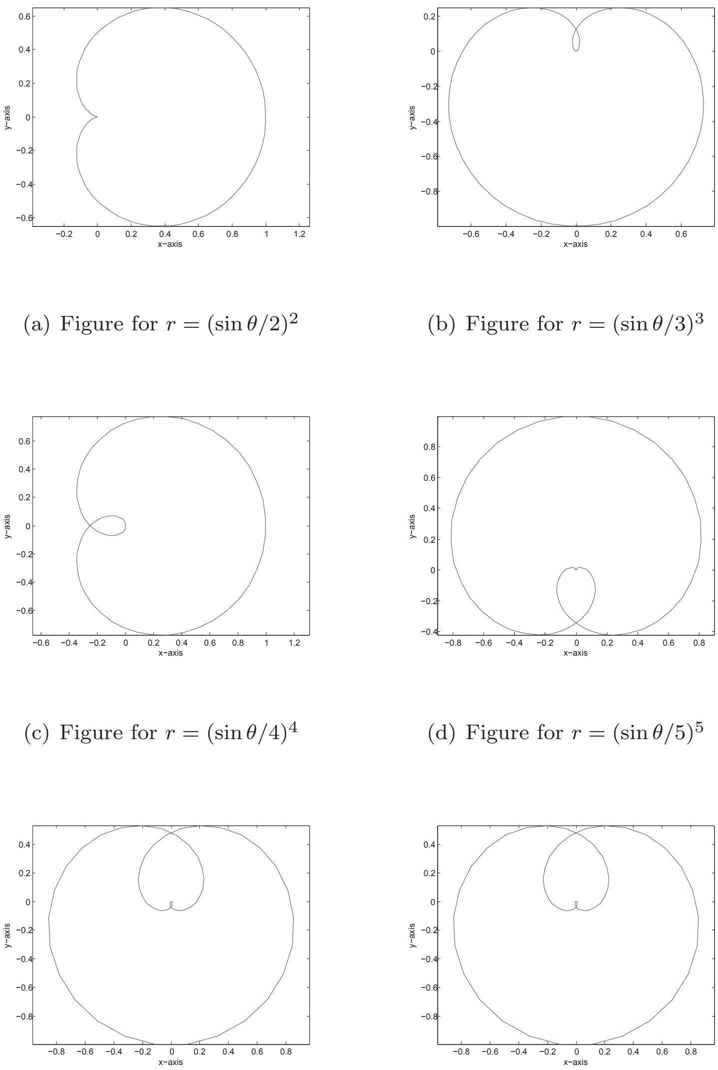
$$\lim_{z\to 0} u = r^{-\sqrt{\mu}} \sin\sqrt{\mu}\theta - 1 = -\infty.$$

For this we can say that $u = 0$ on the boundary of a given domain except at discontinuity point $z = 0$.

Likewise, it can be carried out along the same lines for another harmonic function in the general solution of *Laplace equation* which is:

$$u(r, \theta) = r^{-\sqrt{\mu}} \cos\sqrt{\mu}\theta - 1,\ \text{where } r = (\cos\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}},\ \text{and } \mu \text{ is an eigenvalue.}$$

In order to derive more information about the existence of *inward-pointing cusp* on the boundary of Ω at the point $z = 0$ and its neighborhoods, see figure (2.1) and table (1) below.



(a) Figure for $r = (\sin\theta/2)^2$

(b) Figure for $r = (\sin\theta/3)^3$

(c) Figure for $r = (\sin\theta/4)^4$

(d) Figure for $r = (\sin\theta/5)^5$

(e) Figure for $r = (\sin\theta/6)^6$

(f) Figure for $r = (\sin\theta/7)^7$

Figure 1: Existence of inward cusp on the boundary Ω where $r = (sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$.

Table 1: Self intersection points for the polar function $r = \phi(\theta) = (\sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$.

n	$\mu = \frac{1}{n^2}$	$r = (\sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$	(θ_1, θ_2)	inward-pointing cusp at the neighborhood of $z = (r, \theta) = 0$
3	$\frac{1}{9}$	$\sin^3\frac{\theta}{3}$	$(\frac{\pi}{2}, \frac{5\pi}{2})$	$(0.125, \frac{\pi}{2})$ $(0.125, \frac{5\pi}{2})$
4	$\frac{1}{16}$	$\sin^4\frac{\theta}{4}$	$(\pi, 3\pi)$	$(0.2500, \pi)$ $(0.2500, 3\pi)$
5	$\frac{1}{25}$	$\sin^5\frac{\theta}{5}$	$(\frac{3\pi}{2}, \frac{7\pi}{2})$ $(\frac{\pi}{2}, \frac{9\pi}{2})$	$(0.0028, \frac{\pi}{2})$ $(0.0028, \frac{9\pi}{2})$
6	$\frac{1}{36}$	$\sin^6\frac{\theta}{6}$	$(2\pi, 4\pi)$ $(\pi, 5\pi)$	$(0.0156, \pi)$ $(0.0156, 5\pi)$
7	$\frac{1}{49}$	$\sin^7\frac{\theta}{7}$	$(\frac{5\pi}{2}, \frac{9\pi}{2})$ $(\frac{3\pi}{2}, \frac{11\pi}{2})$ $(\frac{\pi}{2}, \frac{13\pi}{2})$	$(0.4819, \frac{5\pi}{2})$ $(0.4819, \frac{9\pi}{2})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

At this stage, we consider $n \notin \mathbb{N}$ for example, then

$$\int\int_{\Omega} |\nabla u|^p \mathrm{d}x\mathrm{d}y \not< \infty \text{ for } p < \frac{2}{\sqrt{\mu}+1}, \text{ where } \mu = \frac{1}{n^2}.$$

However there is no inward-pointing cusp on the boundary of Ω, that is, we have outward-pointing cusp on the boundary of Ω.

For instance, let $n = \sqrt{2} \notin \mathbb{N}$, then $\mu = \frac{1}{2} \Rightarrow \sqrt{\mu} = \frac{1}{\sqrt{2}}$, such that

$$u(r,\theta) = \left(r^{\frac{-1}{\sqrt{2}}} \sin\frac{\theta}{\sqrt{2}} \right) - 1, \; 0 \le \theta \le \sqrt{2}\pi.$$

and $u = 0$ on $\partial\Omega$ where $r = \phi(\theta) = (\sin\frac{\theta}{\sqrt{2}})^{\sqrt{2}}$.

Calculating the tangent vector for the function $r = \phi(\theta)$ as follows:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\phi(\theta)\cos\theta + \sin\theta\phi'(\theta)}{-\phi(\theta)\sin\theta + \cos\theta\phi'(\theta)}$$

such that in case, $\theta = 0 \Rightarrow \phi(\theta = 0) = 0$, which implies to

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin\theta\phi'(\theta)}{\cos\theta\phi'(\theta)} = \tan\theta$$

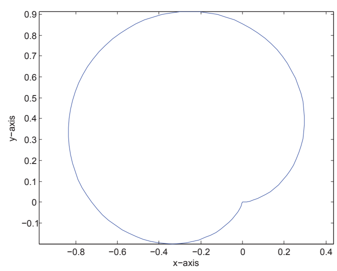
and $\frac{dy}{dx} = 0$, however in case $\theta = \sqrt{2}\pi \Rightarrow f(\theta = \sqrt{2}\pi) = 0$, then

$$\frac{dy}{dx} = \tan(\sqrt{2}\pi) = 3.6202$$

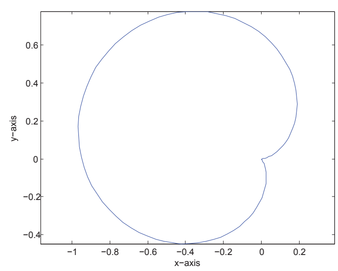
is a straight-line equation

$$y = 3.6202\ x + c,\ \ c \text{ is a constant.}$$

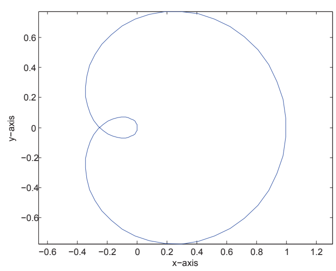
Some figures in (2) which plotted in Matlab program help to locate more information about the existence of *outward-pointing cusp* on the boundary of Ω at the point $z = 0$ and its neighborhoods for the function $r = \phi(\theta) = (sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$.



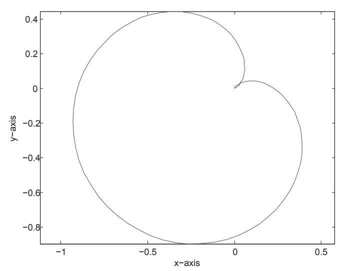
(a) Figure for $r = (\sin\theta/\sqrt{2})^{\sqrt{2}}$



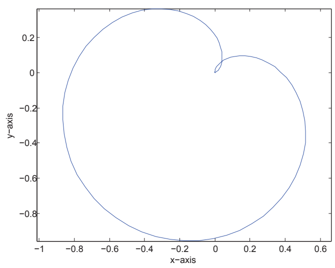
(b) Figure for $r = (\sin\theta/\sqrt{3})^{\sqrt{3}}$



(c) Figure for $r = (\sin\theta/\sqrt{5})^{\sqrt{5}}$



(d) Figure for $r = (\sin\theta/\sqrt{6})^{\sqrt{6}}$



(e) Figure for $r = (\sin\theta/\sqrt{7})^{\sqrt{7}}$

Figure 2: Existence of outward cusp on the boundary Ω where $r = (sin\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$.

3. Generating an holomorphic function on cardioid domain, unit disk

In this part in our work we are so keen to show how can be generated a holomorphic function on the cardioid domain Ω by a harmonic function defined on Ω, and vanishes on the boundary of Ω, and another on the unit disk by holomorphic function belongs to *Smirnov domain (cardioid type)*, which is settled in Theorems 3.1 and 3.4.

Theorem 3.1. *Let* $u(r,\theta) = \left(r^{-\sqrt{\mu}} \cos\sqrt{\mu}\theta\right) - 1$ *be a harmonic function on cardioid domain* Ω *and* $u(r,\theta) = 0$ *on* $\partial\Omega$ *where* $r = (\cos\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$, *where* μ *is an eigenvalue. Then the polar function* $r = (\cos\sqrt{\mu}\theta)^{\frac{1}{\sqrt{\mu}}}$ *generates holomorphic function on* Ω *for all* $0 < n \leq 1$.

Proof. Let

$$z = \psi(\zeta) = c(1+\zeta)^n \tag{3.1}$$

be a conformal mapping defined on the simply conneected domain Ω onto unit disk $\{\zeta : |\zeta| < 1\}$cf. [16] To derive polar function we shall define $\zeta = e^{i\alpha}$ on the boundary of the unit circle such that

$$z = \psi(\zeta) = c(1+e^{i\alpha})^n = c(1+\cos\alpha + i\sin\alpha)^n. \tag{3.2}$$

Since $z = re^{i\theta}$ is a point on the curve C in the interior of Ω then equation (3.2) becomes

$$\begin{aligned} z &= c(1+\cos\alpha + i\sin\alpha)^n \\ re^{i\theta} &= c\left[2\cos^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right]^n \\ &= c\left[2\cos\frac{\alpha}{2}(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2})\right]^n \\ re^{i\theta} &= c\left[2^n\cos^n\frac{n\alpha}{2}\right]e^{\frac{in\alpha}{2}}. \end{aligned}$$

Therefore $\theta = \frac{n\alpha}{2}$ and

$$r = c\left[2\cos\frac{\alpha}{2}\right]^n. \tag{3.3}$$

Substituting $\alpha = \frac{2\theta}{n}$ into equation (3.3) we obtain

$$r = c_1\left[\cos\frac{\theta}{2}\right]^n \quad \text{where } c_1 = 2^n c \text{ is a constant.} \tag{3.4}$$

This polar function will generate a cardioid domain when $n = 2$. Apply *Fouier* expanson to the function

$$f(\theta) = \cos^{2n}\frac{\theta}{n}, \ (-L \leq \theta \leq L) \text{ where } L = \frac{n\pi}{2}. \tag{3.5}$$

We obtain that

$$f(\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi\theta}{L}.$$

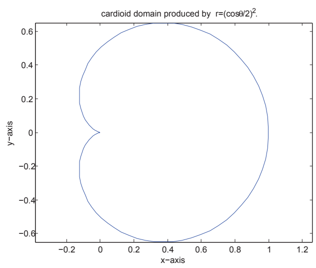


Figure 3: Cardioid domain, $r = \left[\cos\frac{\theta}{2}\right]^2$.

We know that $f(\theta)$ being even implies $b_k = 0$. Substituting equation (3.5) and $L = \frac{n\pi}{2}$ into a_k-Formula, $a_k = \frac{2}{L}\int_0^L f(\theta)\cos\frac{k\pi\theta}{L}\mathrm{d}\theta$, we obtain

$$a_k = \frac{4}{n\pi}\int_0^{\frac{n\pi}{2}} \cos^{2n}\frac{\theta}{n}\cos\frac{2k\theta}{n}\mathrm{d}\theta. \tag{3.6}$$

Assuming $\frac{\theta}{n} = \theta_1$ in equation (3.6), for convenince, gives

$$a_k = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos^{2n}\theta_1 \cos(2k\theta_1)\mathrm{d}\theta. \tag{3.7}$$

Now, we need to apply *Cauchy's* formula which is related to *Gmma* function [2]

Put $\rho = 2n$ and $\beta = 2k$, to obtain

$$a_k = \frac{n}{2^{2n-2}}\frac{\Gamma(2n)}{\Gamma(1+n+k)\Gamma(1+n-k)}, \text{ for } k = 0, 1, 2, \dots \tag{3.8}$$

Substituting equation (3.8) into the *Fourier* series expansion as follows

$$f(\theta) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi\theta}{L}.$$

$$\cos^{2n}\frac{\theta}{n} = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} \frac{n}{2^{2n-2}}\frac{\Gamma(2n)}{\Gamma(1+n+k)\Gamma(1+n-k)} \cos\frac{2k\theta}{n}.$$

$$\cos^{2n}\frac{\theta}{n} = \frac{n\Gamma(2n)}{2^{2n-1}}\left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty}\frac{\cos\frac{2k\theta}{n}}{\Gamma(1+n+k)\Gamma(1+n-k)}\right]. \tag{3.9}$$

We notice that $\cos^{2n}\frac{\theta}{n} = r^2$ by equation (3.4) and

$$\left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty}\frac{\cos\frac{2k\theta}{n}}{\Gamma(1+n+k)\Gamma(1+n-k)}\right]$$

$$=$$

$$Re\left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty}\frac{e^{ik\alpha}}{\Gamma(1+n+k)\Gamma(1+n-k)}\right]$$

where $\alpha = \frac{2\theta}{n}$. And hence, there exists function defined on the boundary of unit disk $\partial\mathbb{D}$ in ζ-plane, which is

$$\Phi(\zeta) = \frac{n\Gamma(2n)}{2^{2n-1}}\left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty}\frac{e^{ik\alpha}}{\Gamma(1+n+k)\Gamma(1+n-k)}\right] \tag{3.10}$$

such that $e^{i\alpha} = \cos(\alpha) + i\sin(\alpha) = \eta + i\xi = \zeta.$, that is

$$\Phi(\zeta) = \frac{n\Gamma(2n)}{2^{2n-1}}\left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty}\frac{\zeta^k}{\Gamma(1+n+k)\Gamma(1+n-k)}\right] \tag{3.11}$$

[2] we refer to *Cauchy*'s formula which is related with *Gammaa* function

$$\int_0^{\frac{\pi}{2}} (\cos t)^{\rho}\cos(\beta t)\mathrm{d}t = \frac{\pi\Gamma(1+\rho)2^{-\rho-1}}{\Gamma(1+\frac{1}{2}\rho+\frac{1}{2}\beta)\Gamma(1+\frac{1}{2}\rho-\frac{1}{2}\beta)}.$$

By equation (3.1) we obtain $\zeta = (\frac{z}{c})^{\frac{1}{n}} - 1$.

Consequently, we deduce

$$\begin{aligned} &\phi(z) = \phi_1(x, y) + i\phi_2(x, y) \\ \Rightarrow \ &\phi(z) = \phi(\psi(\zeta)) = \phi_1(\eta, \xi) + i\phi_2(\eta, \xi) = \Phi(\zeta), \text{ where } z = \psi(\zeta) \end{aligned}$$

such that,

$$\phi(z) = \frac{n\Gamma(2n)}{2^{2n-1}} \left[\frac{1}{n^2\Gamma^2(n)} + 2\sum_{k=1}^{\infty} \frac{(\frac{z}{c})^{\frac{1}{n}} - 1}{\Gamma(1+n+k)\Gamma(1+n-k)} \right]. \tag{3.12}$$

It is clear that ϕ is holomorphic function at $z = 0$ whose derivative exists and is continuous at $z = 0$, such that $\phi'(0) = 0$ for $0 < n \le 1$. However on the other hand, $\phi(z)$ is not holomorphic at $z = 0$ for $1 < n \le 2$, because $\phi'(z)$ at $z = 0$ does not exist. □

Before moving on to a new theorem, we need just a little more background about *Smirnov* classes and *M.Keldysh, M. Lavrentyev* Theorem.

Definition 3.2. ***(Smirnov classes)*** *[6]*
Any holomorphic function defined on Ω is said to be of class $E^p(\Omega)$ for $0 < p \le \infty$ if there exists a sequence of rectifiable Jordan curves $\gamma_1, \gamma_2, \dots$ in Ω, approaching the boundary Ω (in the sense of γ) such that

$$\int_{\gamma_i} |f(z)|^p |\mathrm{d}z| \le const < \infty.$$

Theorem 3.3. ***(M.Keldysh, M. Lavrentyev)***
$f(z) \in E^p(\Omega)$ if and only if $F(w) = f(\phi(w))[\phi'(w)]^{1/p} \in H^p(\mathbb{D})$ for some conformal mapping $\phi(w)$ of the unit disk onto Ω.

Theorem 3.4. *Let Ω be a domain bounded by a curve that is real holomorphic except at the point z_0 where it has a corner with interior angle θ. If $n\pi < \theta \le (n+1)\pi$, $n \in \mathbb{N}$, then for all $p \ge 2 - \frac{\theta}{n\pi}$, $n \in \mathbb{N}$ every $f(z) \in E^p(\Omega)$, generates an holomorphic function on unit disk $\mathbb{D}$ does not have poles on $\partial\mathbb{D} = \mathbb{T}$ such that $H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D}) \subseteq E^p(\Omega)$.*

Proof. Consider a simply connected domain $\Omega \subset \mathbb{C}$, bounded by a Jordan rectifiable curve γ, let $\phi(w) = (1-w)^2 : \mathbb{D} \to \Omega$ be a conformal mapping of the unit disk $\mathbb{D}$ onto Ω (cardioid type). Assume that there is a function $F(w) = i\frac{1-w}{1+w}$ which maps unit disk onto the upper- half plane H^+.

So, there is a function $f(\phi(w)) = F(\phi^{-1}(z))$ which maps Ω(cardioid type) onto upper- half plane H^+, where $\phi^{-1}(z) = w$.

Given $n\pi < \theta \le (n+1)\pi$, that is, $1 < \frac{\theta}{n\pi} \le 1 + \frac{1}{n}$ where $n \in \mathbb{N}$.

We will start to show that in such Ω there exists a function $f(z) \in E^p(\Omega)$ with real boundary values for some values of p.

For this purpose, consider $F(w) = i\frac{1-w}{1+w} : \mathbb{D} \to \mathbb{H}^+$ and acording to *M.Keldysh, M. Lavrentyev* Theorem 3.3 and (cf. [14], Theorem 3.9 pp.52), we obtain

$$\begin{aligned} f(\phi(w))[\phi'(w)]^{\frac{1}{p}} &= f(\phi(w)) \left[(1-w)^{\frac{\theta}{n\pi}-1} g(w)\right]^{\frac{1}{p}} \\ &= F(w) \left[(1-w)^{\frac{\theta}{n\pi p}-\frac{1}{p}} g(w)^{\frac{1}{p}}\right]; \text{ since } f(\phi(w)) = F(w). \\ &= i\frac{(1-w)^{1+\frac{\theta}{n\pi p}-\frac{1}{p}}}{(1+w)} g(w)^{\frac{1}{p}}. \end{aligned}$$

So, $f(\phi(w))[\phi'(w)]^{\frac{1}{p}} \in H^p(\mathbb{D})$ for $1 < p < 1+\frac{1}{n}$, since $p(1+\frac{\theta}{n\pi p}-\frac{1}{p}) = p+\frac{\theta}{n\pi}-1 > 1$, which follows from the fact that, $\frac{\theta}{n\pi} < p+\frac{\theta}{n\pi}-1 < \frac{1}{n}+\frac{\theta}{n\pi}$, and in addition; $\frac{\theta}{n\pi} > 1$ implies $p \geq 2 - \frac{\theta}{n\pi}$, $n \in \mathbb{N}$.

Hence, there exists a function such $f(z) \in E^p(\Omega)$, for $p \geq 2 - \frac{\theta}{n\pi}$.

Here, we shall apply Theorem 3.3 and ([14],Theorem 3.9, pp.52) once again to prove that, there exists an holomorphic function with pole at $w = 1$ of order greater than 1 on $\partial\mathbb{D} = \mathbb{T}$ as follows:

$$f(\phi(w))[\phi'(w)]^{\frac{1}{2n\pi-\theta/n\pi}} = f(\phi(w)) \left[(1-w)^{\frac{\theta}{n\pi}-1} g(w)\right]^{\frac{n\pi}{2n\pi-\theta}}. \tag{3.13}$$

$$= f(\phi(w)) \left[(1-w)^{\frac{\theta-n\pi}{2n\pi-\theta}} g(w)^{\frac{n\pi}{2n\pi-\theta}}\right] \in H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D}). \tag{3.14}$$

Let

$$G(w) = f(\phi(w))[\phi'(w)]^{\frac{n\pi}{2n\pi-\theta}}.$$

$$\Rightarrow G(w) = f(\phi(w))(1-w)^{1-\frac{3n\pi-2\theta}{2n\pi-\theta}} g(w)^{\frac{n\pi}{2n\pi-\theta}} \tag{3.15}$$

such that

$$\Rightarrow \frac{G(w)(1-w)^{\frac{3n\pi-2\theta}{2n\pi-\theta}}}{g(w)^{\frac{n\pi}{2n\pi-\theta}}} = f(\phi(w))(1-w) \in H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D}).$$

Set,

$$G^*(w) = \frac{G(w)(1-w)^{\frac{3n\pi-2\theta}{2n\pi-\theta}}}{g(w)^{\frac{n\pi}{2n\pi-\theta}}} = f(\phi(w))(1-w).$$

The last equation can be written as follows:

$$\begin{aligned} G^*(w)(1-\overline{w}) &= f(\phi(w))(1-w)(1-\overline{w}). \\ \Rightarrow G^*(w)(1-\overline{w}) &= f(\phi(w))|1-w|^2. \end{aligned}$$

Set again,

$$K(w) = G^*(w)(1-\overline{w})$$

such that

$$K(w) = f(\phi(w))|1-w|^2 \in H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D}),\ n \in \mathbb{N}. \tag{3.16}$$

As we obtained $f(z) \in E^p(\Omega)$, for $p \geq 2 - \frac{\theta}{n\pi}$, it can be set,

$$f(z) = u(x,y) + iv(x,y).$$

Now, we need to rewrite equation (3.16) as follows:

$$K(w) = f(\phi(w))\frac{w-\alpha_1}{1-w}\ (1-\overline{\alpha_2}w)(w-\alpha_2),\ \text{where } \alpha_1 \in \partial\mathbb{D} = \mathbb{T}\ \&\ \alpha_2 \in \overline{\mathbb{D}}. \tag{3.17}$$

Hence, $\frac{w-\alpha_1}{1-w} \in \mathbb{R}$ on $\partial\mathbb{D} = \mathbb{T}$.

Let us assume $w = x+iy$, $\alpha_1 = a+ib$ and $(1-\overline{\alpha_2}w)(w-\alpha_2) = t$, where $t \in \mathbb{R}$, since it gives real values. Then

$$\begin{aligned} K(w) &= f(\phi(w))\frac{w-\alpha_1}{1-w}\ (1-\overline{\alpha_2}w)(w-\alpha_2) \\ &= t\ [u(x,y)+iv(x,y)]\ \frac{(x-a)+i(y-b)}{(1-x)-iy} \\ &= t\ [u(x,y)+iv(x,y)]\ \frac{[(x-a)(1-x)-y(y-b)]+i[(y-b)(1-x)+y(x-a)]}{(1-x)^2+y^2}. \end{aligned}$$

Since $\frac{w-\alpha_1}{1-w}$ is a real value on $\partial\mathbb{D} = \mathbb{T}$ when $a = 1, b = 0$, then $\alpha_1 = 1$, so that

$$K(w) = t\ [u(x,y)+iv(x,y)]\frac{w-1}{1-w} = tf(\phi(w)), \tag{3.18}$$

that is, $K(w)$ does not have poles on $\partial\mathbb{D} = \mathbb{T}$.

Hence, by equations (3.16), (3.18) we obtain that, $H^{\frac{2n\pi-\theta}{n\pi}}(\mathbb{D}) \subseteq E^p(\Omega)$. □

Acknowledgments

The author would like to express a deep thanks and gratitude to Department of Mathematics , College of Science, Mustansiriyah University for deep supporting to appear this research paper as it is now.

Also, would like to convey a gratitude and thankful for the Maltepe University for geting a good opportunity for attending and presenting this research paper at the International conference of Mathematical Sciences(ICMS 2018),Maltepe University,Istanbul,Turkey. I enjoyed, Learned and liked every session of the ICMS-Turkey Conference.

References

1. L.V.Ahlfors,*Conformal Invariants; topics in geometric function theory.*McGraw-Hill, New York, 1973.

2. J.E.Brennan, *The integrability of the derivative in conformal mapping*,J.London Math.Soc. 2,261-272, 1978.

3. L. DeCastro and D. Khavinson, *Analytic Function in Smirnov Classes E^p with Real Boundary Values*. 1-6, Received December 13, 2011.

4. Deepmala, *A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications.*PhD thesis, Pt. Ravishankar Shukla University,Raipur 492 010, Chhatisgarh, India, 2014.

5. Deepmala and H. K. Pathak, *A study on some problems on existence of solutions for nonlinear functional-integral equations.*Acta Mathematica Scientia 33, 1305-1313,2013.

6. P.L. Duren, *Theory of H^p Spaces*, Academic Press, New York and London,1970.

7. D. Khavinson, *Remarks Concerning Boundary Properties of Analytic Functions of E^p-classes.* Indiana Univ. Math 31,779-787, 1982.

8. A. Lohwater and G. Piranian and W. Rudin, *The derivative of a schlicht function.*Math.Scand 3,103-106, 1955.

9. A .I. Markushevich, *Theory of functions of a complex variables.*Vol3,Prentice-hall, INC,United States of America,1967.

10. L.N. Mishra, *On existence and behavior of solutions to some nonlinear integral equations with Applications.*PhD thesis,National Institute of Technology, Silchar 788 010, Assam, India, 2017.

11. V.N. Mishra,*Some Problems on Approximations of Functions in Banach Spaces.*PhD thesis, Indian Institute of Technology, Roorkee 24, 667, Uttarakhand, India,2007.

12. V.N. Mishra, L.N. Mishra, *Trigonometric Approximation of Signals Functions in Lp (p = 1)- norm.* .International Journal of Contemporary Mathematical Sciences 7, 909-918, 2012.

13. Ch. Pommerenke,*On the integral means of the derivative of a univalent function.* J.London .Math.Soc 32, 254-258, 1985.

14. Ch. Pommerenke,*Boundary Behaviour of Conformal Mappings.*Springer, United States of America, 1992.

15. M. Renardy and R. C. Rogers,*An Introduction to Partial Differential Equations.* Springer-Verlag ,New York, 1986.

16. Shatha.S.Alhily,*Integrability under an Inner Boundary Complex Domain.*International Journal of Advance Research 3,287-293, 2015.

17. Shatha.S.Alhily, *Higher Integrability of the Gradient of Conforaml Maps.*, PhD thesis, Uk, 104-120, 2013.

18. Shatha.S.Alhily,*Boundary Behaviour of Holomorphic Functions on the Cardioid Domain with some Applications.* Oral presentation was done at the 2nd International Conference of Mathematical Sciences (ICMS) Maltepe University, Istanbul, Turkey, 2018 August.

Shatha S. Alhily,
Department of Mathematics,
Mustansiriyah University ,
Baghdad, Iraq.
E-mail address: shathamaths@ uomustansiriyah.edu.iq and shathamaths@yahoo.co.uk.

and

Deepmala,
Mathematics Discipline,
PDPM Indian Institute of Information Technology, Design and Manufacturing,
Madhya Pradesh, India.
E-mail address: dmrai23@gmail.com