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Liftings of Crossed Modules in the Category of Groups with Operations

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ABSTRACT: In this paper we define the notion of lifting of a crossed module via the morphism in groups with operations and give some properties of this type of liftings. Further we prove that the lifting crossed modules of a certain crossed module are categorically equivalent to the internal groupoid actions on groups with operations, where the internal groupoid corresponds to the crossed module.

Key Words: Internal groupoid, Covering groupoid, Group with operations, Lifting crossed module.

Contents

1. Introduction

Groupoids are mathematical structures which are known to be useful in many areas of science [5,16]. A groupoid is a small category in which each arrow has an inverse and a group-groupoid is an internal groupoid in the category of groups. For a group-groupoid G the covers of G in the category of group-groupoids are categorically equivalent to the group-groupoid actions of G on groups [7, Proposition 3.1]. A crossed module defined by Whitehead in [32,33] can be viewed as a 2-dimensional group [8] and has been widely used in homotopy theory [4], the theory of identities among relations for group presentations [9], algebraic K-theory [19], and homological algebra [18,20]. See [4] for a discussion of the relation of crossed modules to crossed squares and so to homotopy 3-types. We refer the readers to [6] for the structure of the actor 2-crossed module related to the automorphisms of a crossed module of groupoids.

In [10, Theorem 1] Brown and Spencer proved that group-groupoids are categorically equivalent to crossed modules. Then in [29, Section 3], Porter proved that a similar result holds for a certain algebraic category **C** introduced by Orzech [27]

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and called category of groups with operations including categories of groups, rings, associative algebras, associative commutative algebras, Lie algebras, Leibniz algebras, alternative algebras and others. In [14,15], Datuashvili continued the study of internal category in C by applying Porter's result. Moreover she introduced the cohomology theory of internal categories, which are equivalent to crossed modules in C [13,12]. Also, it is proved in [1] that for an internal groupoid G and the associated crossed module $\alpha : A \to B$ in C, the coverings of G and the covering crossed modules of $\alpha : A \to B$ are categorically equivalent.

On the other hand in [25], Mucuk and Şahan have recently defined the notion of lifting for crossed modules. If (A, B, α) is a crossed module and $\theta \colon X \to B$ is a morphism of groups, then a crossed module (A, X, φ) in which the action of X on A is defined via θ such that $\theta\varphi = \alpha$ is called a lifting of α over θ. Also they proved in [25] that the liftings of a certain crossed module are categorically equivalent to the actions of associated group-groupoid on groups. See also [30] for further works on lifting crossed modules.

The object of this paper is to extend the results given in [25] to a more general certain category C. First we give the notion of lifting crossed module of a crossed module in C. Then we observe some properties of such lifting crossed modules. Finally we prove that for an internal groupoid G in C, internal groupoid actions of G on groups with operations and liftings of the crossed module associated with G are categorically equivalent.

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2. Preliminaries

As it is defined in [5,21] a groupoid G has a set G of morphisms, which we call just *elements* of G, a set $\mathsf{Ob}(G)$ of *objects* together with maps $d_0, d_1 \colon G \to \mathsf{Ob}(G)$ and $\epsilon \colon \mathsf{Ob}(G) \to G$ such that $d_0\epsilon = d_1\epsilon = 1_{\mathsf{Ob}(G)}$. The maps d_0, d_1 are called *initial* and *final* point maps respectively and the map ϵ is called *object inclusion*. If $g, h \in G$ and $d_1(g) = d_0(h)$, then the *composition* $h \circ g$ exists such that $d_0(h \circ g) = d_0(g)$ and $d_1(h \circ g) = d_1(h)$. So there exists a partial composition defined by $G_{d_0} \times_{d_1} G \to G, (h, g) \mapsto h \circ g$, where $G_{d_0} \times_{d_1} G$ is the pullback of d_1 and d_0. Further, this partial composition is associative, for $x \in G_0$ the element $\epsilon(x)$ acts as the identity and is denoted by 1_x, and each element g has an inverse g^{-1} such that $d_0(g^{-1}) = d_1(g)$, $d_1(g^{-1}) = d_0(g)$, $g^{-1} \circ g = \epsilon d_0(g)$ and $g \circ g^{-1} = \epsilon d_1(g)$. The map $G \to G$, $g \mapsto g^{-1}$ is called the *inversion*.

In a groupoid G for $x, y \in \mathsf{Ob}(G)$ we write $G(x, y)$ for the set of all morphisms with initial point x and final point y. According to [5] G is *transitive (simply transitive, 1-transitive and totally intransitive)* if for all $x, y \in \mathsf{Ob}(G)$, the set $G(x, y)$ is not empty (has not more than one element, has exactly one element and is empty for $x \neq y$). For $x \in \mathsf{Ob}(G)$ the *star* of x is defined as $\{g \in G \mid d_0(g) = x\}$ and denoted as $\mathsf{St}_G x$; and the *object group* at x is defined as $G(x, x)$ and denoted as $G(x)$.

Let G and H be groupoids. A *morphism* from H to G is a pair of maps $f\colon H \to G$ and $\mathsf{Ob}(f)\colon \mathsf{Ob}(H) \to \mathsf{Ob}(G)$ such that $d_0 f = \mathsf{Ob}(f)d_0$, $d_1 f = \mathsf{Ob}(f)d_1$, $f\epsilon = \epsilon f_0$ and $f(h \circ g) = f(h) \circ f(g)$ for all $(h,g) \in H_{d_0} \times_{d_1} H$. For such a morphism we simply write $f\colon H \to G$.

Let $p\colon \widetilde{G} \to G$ be a morphism of groupoids. Then p is called a *covering morphism* and $\widetilde{G}$ a *covering groupoid* of G if for each $\tilde{x} \in \mathsf{Ob}(\widetilde{G})$ the restriction $\mathsf{St}_{\widetilde{G}}\tilde{x} \to \mathsf{St}_G p(\tilde{x})$ is bijective. A covering morphism $p\colon \widetilde{G} \to G$ is called *transitive* if both $\widetilde{G}$ and G are transitive. A transitive covering morphism $p\colon \widetilde{G} \to G$ is called *universal* if $\widetilde{G}$ covers every cover of G, i.e., if for every covering morphism $q\colon \widetilde{H} \to G$ there is a unique morphism of groupoids $\tilde{p}\colon \widetilde{G} \to \widetilde{H}$ such that $q\tilde{p} = p$ (and hence $\tilde{p}$ is also a covering morphism), this is equivalent to that for $\tilde{x}, \tilde{y} \in \mathsf{Ob}(\widetilde{G})$ the set $\widetilde{G}(\tilde{x}, \tilde{y})$ has not more than one element.

A morphism $p\colon (\widetilde{G}, \tilde{x}) \to (G, x)$ of pointed groupoids is called a *covering morphism* if the morphism $p\colon \widetilde{G} \to G$ is a covering morphism. Let $p\colon (\widetilde{G}, \tilde{x}) \to (G, x)$ be a covering morphism of pointed groupoids and $f\colon (H, z) \to (G, x)$ a morphism of pointed groupoids. We say f lifts to p if there exists a unique morphism $\tilde{f}\colon (H, z) \to (\widetilde{G}, \tilde{x})$ such that $f = p\tilde{f}$. For any groupoid morphism $p\colon \widetilde{G} \to G$ and an object $\tilde{x}$ of $\widetilde{G}$ we call the subgroup $p(\widetilde{G}(\tilde{x}))$ of $G(p\tilde{x})$ the *characteristic group* of p at $\tilde{x}$. The characteristic group determines a necessary and sufficient condition for a morphism $f\colon (H, z) \to (G, x)$ lifts to a covering morphism $p\colon (\widetilde{G}, \tilde{x}) \to (G, x)$ [5, 10.3.3].

The action of a groupoid on a set is defined in [5, pp.374] as follows.

Definition 2.1. Let G be a groupoid. An *action of G on a set* consists of a set X, a function $\theta\colon X \to \mathsf{Ob}(G)$ and a function $G_{d_0} \times_\theta X \to X, (g, x) \mapsto g \bullet x$ defined on the pullback $G_{d_0} \times_\theta X$ of θ and d_0 such that

(i) $\theta(g \bullet x) = d_1(g)$ for $(g, x) \in G_{d_0} \times_\theta X$;

(ii) $(h \circ g) \bullet x = h \bullet (g \bullet x)$ for $(h, g) \in G_{d_0} \times_{d_1} G$ and $(g, x) \in G_{d_0} \times_\theta X$;

(iii) $\epsilon(\theta(x)) \bullet x = x$ for $x \in X$.

□

According to [5, pp.374] for given such an action, *semidirect product groupoid* $G \ltimes X$ is defined to be the groupoid with object set X and elements of $(G \ltimes X)(x, y)$ the pairs (g, x) such that $g \in G(\theta(x), \theta(y))$ and $g \bullet x = y$. The groupoid composition is defined to be

$$(h, y) \circ (g, x) = (h \circ g, x).$$

Mucuk and Şahan in [25] recently defined the notion of lifting for crossed modules in the category of groups and proved that the liftings of a certain crossed module are categorically equivalent to the actions of associated group-groupoid on groups.

3. Crossed modules in groups with operations

The idea of the definition of categories of groups with operations comes from Higgins [17] and Orzech [27,28]; and the definition below is from Porter [29] and Datuashvili [11, p.21], which is adapted from Orzech [27].

Definition 3.1. The notion of a group with a set of operations consists of a pair (Ω, E) where E is a set of identities including the group laws and Ω of operations which includes the group operations, and the following conditions hold: If Ω_i is the set of i-ary operations in Ω, then

1. $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;
2. The group operations written additively $0, -$ and $+$ are the elements of Ω_0, Ω_1 and Ω_2 respectively. Let $\Omega_2' = \Omega_2 \backslash \{+\}$, $\Omega_1' = \Omega_1 \backslash \{-\}$ and assume that if $\star \in \Omega_2'$, then $\star^\circ$ defined by $a \star^\circ b = b \star a$ is also in Ω_2'. Also assume that $\Omega_0 = \{0\}$;
3. For each $\star \in \Omega_2'$, E includes the identity $a \star (b + c) = a \star b + a \star c$;
4. For each $\omega \in \Omega_1'$ and $\star \in \Omega_2'$, E includes the identities $\omega(a + b) = \omega(a) + \omega(b)$ and $\omega(a) \star b = \omega(a \star b)$.

Then the category C satisfying the conditions (1)-(4) is called a *category of groups with operations.* □

From now on C will be a category of groups with operations.

A *morphism* between any two objects of C is a group homomorphism, which preserves the operations of Ω_1' and Ω_2'.

Remark 3.2. The set Ω_0 contains exactly one element, the group identity; hence for instance the category of associative rings with unit is not a category of groups with operations.

Example 3.3. The categories of groups, rings generally without identity, R-modules, associative, associative commutative, Lie, Leibniz, alternative algebras are examples of categories of groups with operations. □

If A and B are objects of C an *extension* of B by A is an exact sequence

$$0 \longrightarrow A \stackrel{\imath}{\longrightarrow} E \stackrel{p}{\longrightarrow} B \longrightarrow 0$$

in which p is surjective and $\imath$ is the kernel of p. It is *split* if there is a morphism $s\colon B \to E$ such that $ps = \imath d_B$. A split extension of B by A is called a *B-structure* on A. Given such a B-structure on A we get actions of B on A corresponding to the operations in Ω_2. For any $b \in B$, $a \in A$ and $\star \in \Omega_2'$ we have the actions called *derived actions* by Orzech [27, p.293]

$$b \cdot a = s(b) + a - s(b)$$
$$b \star a = s(b) \star a.$$

Theorem 3.4. *[27, Theorem 2.4] A set of actions (one for each operation in Ω_2) is a set of derived actions if and only if the semidirect product $A \rtimes B$ with underlying set $A \times B$ and operations*

$$\begin{array}{rcl}(a,b)+(a',b') & = & (a+b\cdot a+a', b+b') \\ (a,b)*(a',b') & = & (a*a'+a*b'+b*a', b*b')\end{array}$$

is an object in C.

Definition 3.5. An *internal groupoid* G in C is a groupoid in which the initial and final point maps $s,t\colon G \rightrightarrows \mathsf{Ob}(G)$, the object inclusion map $\epsilon\colon \mathsf{Ob}(G) \to G$ and the partial composition $\circ\colon G_{d_0} \times_{d_1} G \to G, (h,g) \mapsto h\circ g$ are the morphisms in the category C.

Note that since ϵ is a morphism in C, $\epsilon(0)=0$ and that the operation $\circ$ being a morphism in C implies that for all $g,h,k,l \in G$ and $\star \in \Omega_2$,

$$(k\star h)\circ(l\star g) = (k\circ l)\star(h\circ g) \tag{3.1}$$

whenever one side makes sense. This is called the *interchange law* [29].

For the category of internal groupoids in C we use the same notation $\mathbf{Cat}(\mathsf{C})$ as in [29].

In particular if C is the category of groups, then an internal groupoid G in C becomes a groupoid object in the category of groups, which is quite often called 2-group [3], *group-groupoid* or $\mathcal{G}$-groupoid [10]. Recently the notion of monodromy for topological group-groupoids was developed in [24] and the normality and quotient in group-groupoids were developed in [26]. More recently however, Mucuk and Demir [23] and Temel [31] characterized, independently, normal and quotient objects in the category of crossed modules over groupoids via double groupoids and via 2-groupoids, respectively which extend the results of the paper [26]. In the case where C is the category of rings, an internal groupoid is a ring object in the category of groupoids [22] (see also [2] for topological R-module case).

Definition 3.6. Let $p\colon \widetilde{G} \to G$ be a morphism of internal groupoids in C. Then p is called a *covering morphism* of internal groupoids if it is a covering morphism of underlying groupoids.

Definition 3.7. Let G be an internal groupoid in C and X an object of C. If the underlying groupoid of G acts on the underlying set of X in the sense of Definition 2.1 such that the maps $\theta\colon X \to \mathsf{Ob}(G)$ and $G_{d_0} \times_\theta X \to X, (g,x) \mapsto g\bullet x$ are morphisms in C, then we say that the internal groupoid G *acts* on the group with operations X via θ. □

We write (X,θ) for an action. Here note that $G_{d_0} \times_\theta X \to X, (g,x) \mapsto g\bullet x$ is a morphism in C if and only if

$$(g\star h)\bullet(x\star y) = (g\bullet x)\star(h\bullet y) \tag{3.2}$$

for $x, y \in X$; $g, h \in G$ and $\star \in \Omega_2$ whenever one side is defined.

A morphism $f\colon (X,\theta) \to (X',\theta')$ of such actions is a morphism $f\colon X \to X'$ of groups with operations and underlying operations of G. Then we have the category $\mathsf{Act}_{\mathsf{Cat}(\mathsf{C})}(G)$ of actions of G in C.

Example 3.8. Let G and $\widetilde{G}$ be internal groupoids in C and let $p\colon \widetilde{G} \to G$ be a covering morphism of internal groupoids. Then the internal groupoid G acts on the group with operations $X = \mathsf{Ob}(\widetilde{G})$ via $\mathsf{Ob}(p)\colon X \to \mathsf{Ob}(G)$ assigning to $x \in X$ and $g \in \mathsf{St}_G p(x)$ the target of the unique lifting $\tilde{g}$ in $\widetilde{G}$ of g with source x. Clearly the underlying groupoid of G acts on the underlying set and by evaluating the uniqueness of the lifting, the equation (3.2) is satisfied for $x, y \in X$ and $g, h \in G$ whenever one side is defined. □

It is given in [24] that the categories of actions and coverings of an internal groupoid in a category of groups with operations are equivalent.

The conditions of a crossed module in groups with operations are formulated in [29, Proposition 2] as follows.

Definition 3.9. A crossed module $\alpha\colon A \to B$ in C is a morphism in C, where B acts on A (i.e. we have a derived action in C) with the conditions for any $b \in B$, $a, a' \in A$, and $\star \in \Omega_2'$:

CM1 $\alpha(b \cdot a) = b + \alpha(a) - b$;

CM2 $\alpha(a) \cdot a' = a + a' - a$;

CM3 $\alpha(a) \star a' = a \star a'$;

CM4 $\alpha(b \star a) = b \star \alpha(a)$ and $\alpha(a \star b) = \alpha(a) \star b$.

□

A *morphism* from $\alpha\colon A \to B$ to $\alpha'\colon A' \to B'$ is a pair $f_1\colon A \to A'$ and $f_2\colon B \to B'$ of morphisms in C such that

1. $f_2\alpha(a) = \alpha' f_1(a)$,
2. $f_1(b \cdot a) = f_2(b) \cdot f_1(a)$,
3. $f_1(b \star a) = f_2(b) \star f_1(a)$

for any $b \in B$, $a \in A$ and $\star \in \Omega_2'$. So we have a category $\mathsf{XMod}(\mathsf{C})$ of crossed modules in C.

A morphism $(f_1, f_2)\colon (\widetilde{A}, \widetilde{B}, \widetilde{\alpha}) \to (A, B, \alpha)$ of crossed modules in C such that $f_1\colon \widetilde{A} \to A$ an isomorphism is called a *covering morphism*. Then we have the category of coverings of the crossed module (A, B, α) in C denoted by $\mathsf{Cov}_{\mathsf{XMod}(\mathsf{C})}/(A, B, \alpha)$.

The following theorem was proved in [29, Theorem1].

Theorem 3.10. *The category* $\mathsf{XMod}(\mathbf{C})$ *of crossed modules and the category* $\mathbf{Cat}(\mathbf{C})$ *of internal groupoids in* $\mathbf{C}$ *are equivalent.*

Proposition 3.11. *Let G be an internal groupoid in* $\mathbf{C}$ *and (A, B, α) the crossed module corresponding to G. If G is transitive (resp. simply transitive, 1-transitive and totally intransitive), then α is surjective (resp. injective, bijective; and a zero morphism such that A is singular).*

Proof. The proof follows from the Theorem 3.10. □

Definition 3.12. Let (A, B, α) be a crossed module in $\mathbf{C}$. Then (A, B, α) is called *transitive* (resp. simply transitive, *1-transitive* and *totally intransitive* if α is surjective (resp. injective, bijective; and zero morphism such that A is singular).

Example 3.13. If X is a topological group with operations whose underlying topology is path-connected (resp. totally disconnected), then the crossed module $(\mathsf{St}_{\pi X}0, X, d_1)$ is transitive (resp. totally intransitive).

4. Liftings of crossed modules in groups with operations

In this section we give the notion of lifting crossed module of a crossed module associated with an internal groupoid in $\mathbf{C}$ and clarify the properties of lifting crossed modules.

Let G be an internal groupoid in $\mathbf{C}$ acting on a group with operations X by an action $G_{d_0} \times_\theta X \to X, (g, x) \mapsto g \bullet x$, via a morphism of groups with operations $\theta\colon X \to \mathsf{Ob}(G)$ and let (A, B, α) be the crossed module corresponding to G in $\mathbf{C}$. Since $B = \mathsf{Ob}(G)$ then we have a morphism $\theta\colon X \to B$ of groups with operations and derived actions of X on $A = \mathsf{St}_G 0$ defined by

$$x \cdot a = 1_{\theta(x)} + a - 1_{\theta(x)}$$

and

$$x \star a = 1_{\theta(x)} \star a$$

for each $\star \in \Omega_2'$.

By the internal groupoid action of G on X we have a morphism of groups with operations

$$\varphi\colon A \to X, a \mapsto \varphi(a) = a \bullet 0_X$$

such that $\theta\varphi = \alpha$, where 0_X is the identity element of the group with operations X.

Now we prove the following theorems.

Theorem 4.1. *By the action of X on A defined above, (A, X, φ) becomes a crossed module in* $\mathbf{C}$.

Proof. By [25, Theorem 4.1.] the conditions [CM1] and [CM2] are satisfied. Then we need to prove that the conditions [CM3] and [CM4] are also satisfied.

CM3. For $a, a' \in A$ we have the following equality

$$\begin{aligned} \varphi(a) \star a' &= (a \bullet 0_X) \star a' \\ &= 1_{\theta(a \bullet 0_X)} \star a' \\ &= 1_{d_1(a)} \star a' \\ &= a \star a'. \end{aligned}$$

CM4. For all $a \in A$ and $x \in X$, we have

$$\begin{aligned} \varphi(x \star a) &= \varphi(1_{\theta(x)} \star a) \\ &= (1_{\theta(x)} \star a) \bullet 0_X \\ &= (1_{\theta(x)} \star a) \bullet (0_X \star 0_X) \\ &= (1_{\theta(x)} \bullet 0_X) \star (a \bullet 0_X) \\ &= x \star \varphi(a). \end{aligned}$$

□

Theorem 4.2. *Let (A, B, α) be a crossed module in C and $\theta \colon X \to B$ a morphism in C. Then any morphism $\varphi \colon A \to X$ such that $\theta\varphi = \alpha$ is a crossed module in C with the action defined via θ if and only if the map $\overline{\varphi} \colon A \rtimes X \to X$ defined by*

$$\overline{\varphi}(a, x) = \varphi(a) + x$$

is a morphism in C.

Proof. Assume that $\varphi \colon A \to X$ is a crossed module in C. Since by [25, Theorem 4.2.] we have the following equality,

$$\overline{\varphi}((a, x) + (a', x')) = \overline{\varphi}(a, x) + \overline{\varphi}(a', x')$$

for $a, a' \in A$ and $x, x' \in X$, we only need to prove that $\overline{\varphi} \colon A \rtimes X \to X$ preserves the operations of Ω_2' and Ω_1'.

For $a, a' \in A$, $x, x' \in X$ and $\star \in \Omega_2'$,

$$\begin{aligned} \overline{\varphi}((a, x) \star (a', x')) &= \overline{\varphi}(a \star x' + x \star a' + a \star a', x \star x') \\ &= \varphi(a \star x' + x \star a' + a \star a') + x \star x' \\ &= \varphi(a \star x') + \varphi(x \star a') + \varphi(a \star a') + x + x' \\ &= (\varphi(a) \star x') + (\varphi(a') \star x) + (\varphi(a) \star \varphi(a')) + x \star x' \\ &= (\varphi(a) + x) \star (\varphi(a') + x') + (\varphi(a) + x) \star (\varphi(a') + x') \\ &= (\varphi(a) + x) \star (\varphi(a') + x') \\ &= \overline{\varphi}((a, x)) \star \overline{\varphi}((a', x')). \end{aligned}$$

On the other hand for $a \in A$, $x \in X$ and $w \in \Omega_1'$, we have that

$$\begin{aligned}\overline{\varphi}(\omega(a,x)) &= \overline{\varphi}(\omega(a), w(x)) \\ &= \varphi(\omega(a)) + \omega(x) \\ &= \omega(\varphi(a)) + \omega(x) \\ &= \omega(\varphi(a) + x) \\ &= \omega(\overline{\varphi}(a,x)).\end{aligned}$$

Conversely, suppose that the map $\varphi\colon A \to X$ defined by $\overline{\varphi}(a,x) = \varphi(a) + x$ is a morphism of groups with operations. Then we have to prove that φ satisfies the conditions of Definition 3.9.

We know from [25] that φ satisfies the conditions [CM1] and [CM2]. It is sufficient to show that conditions [CM3] and [CM4] are satisfied.

CM3. For $a, a' \in A$,

$$\begin{aligned}\varphi(x \star a) = \overline{\varphi}(x \star a, x) &= \overline{\varphi}((0,x) \star (a,0)) \\ &= \overline{\varphi}(0,x) \star \overline{\varphi}(a,0) \\ &= (\varphi(0) + x) \star (\varphi(a) + 0) \\ &= x \star \varphi(a)\end{aligned}$$

CM4. For $a, a' \in A$, $\varphi(a) \star a' = a \star a'$.

□

Now we define the notion of lifting of a crossed module as follows.

Definition 4.3. Let (A, B, α) be a crossed module and $\theta\colon X \to B$ a morphism in C. Then a crossed module (A, X, φ) in which the action of X on A is defined via θ, is a called a *lifting of* α *over* θ and denoted by (φ, X, θ) whenever $\theta\varphi = \alpha$.

Remark 4.4. If (φ, X, θ) is a lifting of (A, B, α), then $\mathsf{Ker}\varphi \subseteq \mathsf{Ker}\alpha$ and $(1_A, \theta)$ is morphism of crossed modules in C.

Therefore, if (A, B, α) is a simply transitive crossed module in C, then $\mathsf{Ker}\alpha$ is trivial so is $\mathsf{Ker}\varphi$. Hence the crossed module (A, X, φ) is also simply transitive.

Lemma 4.5. *Let* (A, B, α) *be a crossed module in* C *and* φ *a lifting of* α *over* $\theta\colon X \to B$*. If there are isomorphism* $f\colon B \to B'$ *and* $g\colon X' \to X$ *in* C*, then* φ' *is a lifting of* α' *over* $\theta'\colon X' \to B'$ *where* $\varphi' = g^{-1}\varphi$*,* $\alpha' = f\alpha$ *and* $\theta' = f\theta g$*.*

Proof. We need to prove that $\theta'\varphi' = \alpha'$. We know that (A, B, α) be a crossed module in C and φ a lifting of α over $\theta\colon X \to B$, so the equation $\theta\varphi = \alpha$ exists. Since $\alpha' = f\alpha$ and $\varphi' = g^{-1}\varphi$ are crossed modules and the morphisms f, g are isomorphism in C, then we have

$$\begin{aligned}\theta'\varphi' &= (f\theta g)(g^{-1}\varphi)\\ &= f\theta\varphi\\ &= f\alpha\\ &= \alpha'.\end{aligned}$$

□

Proposition 4.6. *Let (A, B, α) be a crossed module in C and (φ, X, θ) a lifting of (A, B, α). If (φ', X', θ') is a lifting of (A, X, φ), then $(\varphi', X', \theta\theta')$ is also a lifting of (A, B, α).*

Proof. The proof is straightforward. □

Let (φ, X, θ) and (φ', X', θ') be two liftings of (A, B, α). A morphism f from (φ, X, θ) to (φ', X', θ') is a morphism of groups with operations $f\colon X \to X'$ such that $f\varphi = \varphi'$ and $\theta' f = \theta$. Hence lifting crossed modules of (A, B, α) and morphisms between them form a category which we denote by $\mathbf{LXMod(C)/(A, B, \boldsymbol{\alpha})}$. By the Proposition 4.6 it follows that if φ is a lifting of (A, B, α) over $\theta\colon X \to B$, then $\mathbf{LXMod(C)/(A, X, \boldsymbol{\varphi})}$ is a full subcategory of $\mathbf{LXMod(C)/(A, B, \boldsymbol{\alpha})}$.

Let (A, B, α) be a transitive crossed module in C. Then a lifting (φ, X, θ) of (A, B, α) is called an *n-lifting* when $|\mathsf{Ker}\theta| = n$.

Corollary 4.7. *If (φ, X, θ) is a 1-lifting of (A, B, α), then θ is an isomorphism. Hence $(A, X, \varphi) \cong (A, B, \alpha)$.*

Proof. If (φ, X, θ) is a 1-lifting of (A, B, α), then θ becomes surjective and $|\mathsf{Ker}\theta| = 1$, i.e., θ is injective. Hence θ is an isomorphism. □

Theorem 4.8. *Let $(f, g)\colon (\widetilde{A}, \widetilde{B}, \widetilde{\alpha}) \to (A, B, \alpha)$ be a morphism of crossed modules in C where (A, B, α) is transitive and let (φ, X, θ) be a lifting of (A, B, α). Then there is a unique morphism of crossed modules $(f, \widetilde{g})\colon (\widetilde{A}, \widetilde{B}, \widetilde{\alpha}) \to (A, X, \varphi)$ such that $\theta\widetilde{g} = g$ if and only if $f(\mathsf{Ker}\widetilde{\alpha}) \subseteq \mathsf{Ker}\varphi$.*

Proof. The proof follows from [25, Theorem 4.12]. □

Corollary 4.9. *Let (A, B, α) be a crossed module in C. Assume that (φ, X, θ) and $(\widetilde{\varphi}, \widetilde{X}, \widetilde{\theta})$ are two liftings of (A, B, α) such that $(A, \widetilde{X}, \widetilde{\varphi})$ is transitive. Then $\widetilde{\varphi}$ is a lifting of φ if and only if $\mathsf{Ker}\widetilde{\varphi} \subseteq \mathsf{Ker}\varphi$.*

Corollary 4.10. *Let (A, B, α) be a crossed module in C. Assume that (φ, X, θ) and $(\widetilde{\varphi}, \widetilde{X}, \widetilde{\theta})$ are two liftings of (A, B, α) such that (A, X, φ) and $(A, \widetilde{X}, \widetilde{\varphi})$ are both transitive. Then $(\varphi, X, \theta) \cong (\widetilde{\varphi}, \widetilde{X}, \widetilde{\theta})$ if and only if $\mathsf{Ker}\varphi = \mathsf{Ker}\widetilde{\varphi}$.*

Theorem 4.11. *Let (A, B, α) be a crossed module, X an object of $\mathbf{C}$ and let $\theta\colon X \to B$ be an injective morphism in $\mathbf{C}$. Then any morphism $\varphi\colon A \to X$ such that $\theta\varphi = \alpha$ becomes a lifting of α over θ.*

Proof. According to Theorem 4.2 it is sufficient to show that $\overline{\varphi}(a, x) = \varphi(a) + x$ is a morphism in $\mathbf{C}$. Since in [25, Theorem 4.15], it is proved that the operation $+$ is preserved under the morphism $\overline{\varphi}$, then we only need to show that the operations $\star \in \Omega_2'$ is also preserved under $\overline{\varphi}$, i.e., $\varphi(a \star x) = \varphi(a) \star x$ and $\varphi(x \star a) = x \star \varphi(a)$ for all $a \in A$ and $x \in X$.

$$\begin{aligned} \theta(\varphi(x \star a)) &= \theta(\varphi(\theta(x) \star a)) \\ &= \alpha(\theta(x) \star a) && \text{(since } \theta\varphi = \alpha) \\ &= \theta(x) \star \alpha(a) && \text{(by CM4)} \end{aligned}$$

and on the other hand

$$\begin{aligned} \theta(x \star \varphi(a)) &= \theta(x) \star \theta(\varphi(a)) \\ &= \theta(x) \star \alpha(a). && \text{(since } \theta\varphi = \alpha) \end{aligned}$$

Since θ is injective and $\theta(\varphi(x \star a)) = \theta(x \star \varphi(a))$, we have $\varphi(x \star a) = x \star \varphi(a)$ and similarly $\varphi(a \star x) = \varphi(a) \star x$ for all $a \in A$ and $x \in X$. □

5. Equivalences of the categories

In this section for a certain internal groupoid G in $\mathbf{C}$, the category of internal groupoid actions of G and the lifting crossed modules of the crossed module corresponding to G are equivalent.

Theorem 5.1. *Let G be an internal groupoid in $\mathbf{C}$ and (A, B, α) the crossed module corresponding to G. Then the category $\mathbf{Act}_{\mathbf{Cat(C)}}(G)$ of internal groupoid actions of G in $\mathbf{C}$ and the category $\mathbf{LXMod(C)/(A, B, \alpha)}$ of lifting crossed modules of (A, B, α) are equivalent.*

Proof. A functor $\delta\colon \mathbf{Act}_{\mathbf{Cat(C)}}(G) \to \mathbf{LXMod(C)/(A, B, \alpha)}$ is defined as follows: For each object (X, θ) of $\mathbf{Act}_{\mathbf{Cat(C)}}(G)$, $\delta(X, \theta)$ defines a lifting of (φ, X, θ) of (A, B, α) where

$$\varphi\colon A \to X, a \mapsto a \bullet 0_X$$

such that $\theta\varphi = \alpha$, where $0_X \in \Omega_0$. Then by way of the action of X on A, (A, X, φ) becomes a crossed module in $\mathbf{C}$.

Conversely define a functor $\eta\colon \mathbf{LXMod(C)/(A, B, \alpha)} \to \mathbf{Act}_{\mathbf{Cat(C)}}(G)$ assigning each lifting (φ, X, θ) of the crossed module (A, B, α) to an internal groupoid action (X, θ) of G on the group with operations X via an action map defined by

$$G_{d_0} \times_\theta X \to X, (g, x) \mapsto g \bullet x = \varphi(g - 1_{d_0(g)}) + x.$$

The required equivalences $\delta\eta \simeq 1$ and $\eta\delta \simeq 1$ follows from the details of the proof of [25, Theorem 5.1]. □

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