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Normality and Quotient in the Category of Crossed Modules Within the Category of Groups with Operations

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ABSTRACT: In this paper we define the notions of normal subcrossed module and quotient crossed module within groups with operations; and then give some properties of such crossed modules in groups with operations.

Key Words: Group with operations, Quotient crossed module.

Contents

1. Introduction

Crossed modules as defined by Whitehead [29,30] have been widely used in homotopy theory [4], the theory of group representation (see [5] for a survey), in algebraic K-theory [16,17], and homological algebra [15,18]. Crossed modules over groups can be viewed as 2-dimensional groups [3].

The notions of subcrossed module and normal subcrossed module were defined in [25]. In [7] Brown and Spencer proved that the category of internal groupoids within the groups (which are also called $\mathcal{G}$-*groupoids* [7], *group-groupoids* [6] or 2-*groups* [2]) is equivalent to the category of crossed modules of groups. Using the equivalence in [7], recently in [22] normal and quotient objects in the category of group-groupoids have been obtained. In the light of this paper, Mucuk and Demir [20] and Temel [28] characterized normal and quotient objects in the category of crossed modules over groupoids independently using different approaches.

In [26] Porter proved a similar result to one in [7] holds for certain algebraic categories, introduced by Orzech [24], which definition was adapted by him and called category of groups with operations. Gürsoy et al. [13] generalized the result given in [7] in a different way. They introduced the notion of crossed module over generalized groups which they call it by generalized crossed module and of generalized group-groupoid. Also they proved that the category of generalized crossed

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modules is equivalent to that of generalized group-groupoids whose object sets are abelian generalized group. Applying Porter's result, the study of internal category theory was continued in the works of Datuashvili [11] and [12]. Moreover, she developed cohomology theory of internal categories, equivalently, crossed modules, in categories of groups with operations [9] and [10]. The equivalences of the categories in [7] and [26] enable us to generalize some results on group-groupoids which are internal categories within groups to the more general internal groupoids for a certain algebraic category C (see for example [1], [21], [23] and [19]).

In this paper for an algebraic category C we define normal subcrossed module and quotient crossed module for groups with operations.

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2. Preliminaries

We recall that a crossed module of groups originally defined by Whitehead [29,30], consists of two groups A and B, an action of B on A denoted by $b \cdot a$ for $a \in A$ and $b \in B$; and a homomorphism $\alpha\colon A \to B$ of groups satisfying the following conditions for all $a, a_1 \in A$ and $b \in B$

(i) $\alpha(b \cdot a) = b + \alpha(a) - b$,

(ii) $\alpha(a) \cdot a_1 = a + a_1 - a$.

We will denote such a crossed module by (A, B, α). Let (A, B, α) and (A', B', α') be two crossed modules. A morphism (f_1, f_2) from (A, B, α) to (A', B', α') is a pair of homomorphisms of groups $f_1\colon A \to A'$ and $f_2\colon B \to B'$ such that $f_2\alpha = \alpha' f_1$ and $f_1(b \cdot a) = f_2(b) \cdot f_1(a)$ for $a \in A$ and $b \in B$.

It was proved by Brown and Spencer in [7, Theorem 1] that the category $\mathsf{XMod}(\mathsf{Grp})$ of crossed modules over groups is equivalent to the category GrpGpd of group-groupoids.

3. Normality and quotient in the category of crossed modules within the category of groups with operations

The idea of the definition of categories of groups with operations comes from Higgins [14] and Orzech [24]; and the definition below is from Porter [26] and Datuashvili [8, pp. 21], which is adapted from Orzech [24].

Definition 3.1. From now on C will be a category of groups with a set of operations Ω and with a set E of identities such that E includes the group axioms, and the following conditions hold: If Ω_i is the set of i-ary operations in Ω, then

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) The group operations written additively $0, -$ and $+$ are respectively the elements of Ω_0, Ω_1 and Ω_2. Let $\Omega_2' = \Omega_2 \backslash \{+\}$, $\Omega_1' = \Omega_1 \backslash \{-\}$ and assume that if $\star \in \Omega_2'$, then $\star^\circ$ defined by $a \star^\circ b = b \star a$ is also in Ω_2'. Also assume that $\Omega_0 = \{0\}$;

(c) For each $\star \in \Omega'_2$, E includes the identity $a \star (b + c) = a \star b + a \star c$;

(d) For each $\omega \in \Omega'_1$ and $\star \in \Omega'_2$, E includes the identities $\omega(a+b) = \omega(a)+\omega(b)$ and $\omega(a) \star b = \omega(a \star b)$.

A category satisfying the conditions (a)-(d) is called a *category of groups with operations*.

Remark 3.2. The set Ω_0 contains exactly one element, the group identity; hence for instance the category of associative rings with unit is not a category of groups with operations.

Example 3.3. The categories of groups, rings generally without identity, R-modules, associative, associative commutative, Lie, Leibniz, alternative algebras are examples of categories of groups with operations.

A *morphism* between any two objects of C is a group homomorphism, which preserves the operations in Ω'_1 and Ω'_2.

The topological version of this definition can be stated as follows:

Definition 3.4. Let X be an object in C. If X has a topology such that all operations in Ω are continuous, then X is called a *topological group with operations* in C.

In particular if C is the category of groups, then a topological group with operations just becomes a topological group and if C is the category of R-modules, then it becomes a topological R-module.

We will denote the category of topological groups with operations by $\mathsf{Top}^{\mathsf{C}}$.

For the objects A and B of C, the direct product $A \times B$ with the usual operations becomes a group with operations. Hence the category C has finite products.

The subobject in the category C can be defined as follows.

Definition 3.5. Let A be an object in C. A subset $B \subseteq A$ is called a *subgroup with operations of* A if the following conditions are satisfied:

(i) $b \star b_1 \in B$ for $b, b_1 \in B$ and $\star \in \Omega_2$;

(ii) $\omega(b) \in B$ for $b \in B$ and $\omega \in \Omega_1$.

The normal subobject in the category C of groups with operations is defined as follows.

Definition 3.6. [24, Definition 1.7] Let A be an object in C and N a subgroup with operations of A. N is called a *normal subgroup with operations* or an *ideal* of A and written $N \lhd A$ if the following conditions are satisfied:

(i) $(N, +)$ is a normal subgroup of $(A, +)$;

(ii) $a \star n \in N$ for $a \in A$, $n \in N$ and $\star \in \Omega'_2$.

For a morphism $f\colon A \to B$ in C, $\mathsf{Ker} f = \{a \in A \mid f(a) = 0\}$ is an ideal of A.

A quotient object in C is constructed as follows: Let A be an object in C and N an ideal of A. Then the relation on A defined by

$$a \sim a_1 \quad \text{iff} \quad a - a_1 \in N$$

is an equivalence relation. Then the quotient set A/N along with the operations defined by

$$\begin{aligned} [a]\star[a_1] &= [a \star a_1] \\ \omega([a]) &= [\omega(a)] \end{aligned}$$

for $\star \in \Omega_2$, $\omega \in \Omega_1$ becomes an object in C and called *quotient group with operations of A by N*.

In the following proposition we prove that the category C has kernels in categorical sense.

Proposition 3.7. *Let A be an object in C. Then N is an ideal of A if and only if it is a kernel of a morphism in C.*

Proof. We have already seen that the kernel of a morphism in C is an ideal of the domain.

Conversely if N is an ideal of A, then the quotient A/N becomes an object in C and quotient morphism $p\colon A \to A/N$ has N as kernel. □

Let A and B be two groups with operations in C. An *extension* of B via A is an exact sequence

$$0 \longrightarrow A \stackrel{\imath}{\longrightarrow} E \stackrel{p}{\longrightarrow} B \longrightarrow 0 \tag{3.1}$$

in which p is surjective and $\imath$ is the kernel of p. It is *split* if there is a morphism $s\colon B \to E$ such that $ps = \imath d_B$. A split extension of B via A is called a *B-structure* on A. Given such a B-structure on A we get actions of B on A corresponding to the operations in C. For any $b \in B$, $a \in A$ and $\star \in \Omega_2'$ we have the actions called *derived actions* by Orzech [24, pp.293]

$$\begin{aligned} b \cdot a &= s(b) + a - s(b) \\ b \star a &= s(b) \star a. \end{aligned} \tag{3.2}$$

Given a set of actions of B on A (one for each operation in Ω_2), let $A \rtimes B$ be a universal algebra whose underlying set is $A \times B$ and whose operations are

$$\begin{aligned} (a', b') + (a, b) &= (a' + b' \cdot a,\ b' + b), \\ (a', b') \star (a, b) &= (a' \star a + a' \star b + b' \star a,\ b' \star b). \end{aligned}$$

Theorem 3.8. [24, Theorem 2.4] *A set of actions of B on A is a set of derived actions if and only if $A \rtimes B$ is an object of C.*

We recall that for groups with operations A and B, in [9, Proposition 1.1] all necessary and sufficient conditions for the actions of B on A to be derived actions are determined.

Lemma 3.9. *Let A, B, S and T be objects in C. Suppose that we have a set of derived actions of B on A and a set of derived actions of T on S. If $S \rtimes T$ is an ideal of $A \rtimes B$ then the followings are satisfied.*

(a) S and T are ideals of A and B, respectively,

(b) $b \cdot s \in S$ for all $b \in B$, $s \in S$,

(c) $(t \cdot a) - a \in S$ for all $t \in T$, $a \in A$,

(d) $b \star s \in S$ for all $b \in B$, $s \in S$,

(e) $t \star a \in S$ for all $t \in T$, $a \in A$.

Proof. It can be shown by an easy calculation. □

Definition 3.10. [26] A crossed module in C is a triple (A, B, α), where A and B are the objects of C, B acts on A, i.e., we have a derived action in C, and $\alpha\colon A \to B$ is a morphism in C with the conditions:

CM1. $\alpha(b \cdot a) = b + \alpha(a) - b$;

CM2. $\alpha(a) \cdot a' = a + a' - a$;

CM3. $\alpha(a) \star a' = a \star a'$;

CM4. $\alpha(b \star a) = b \star \alpha(a)$, $\alpha(a \star b) = \alpha(a) \star b$

for any $b \in B$, $a, a' \in A$, and $\star \in \Omega_2'$.

A morphism $(A, B, \alpha) \to (A', B', \alpha')$ between two crossed modules is a pair $f\colon A \to A'$ and $g\colon B \to B'$ of the morphisms in C such that

(i) $g\alpha(a) = \alpha' f(a)$,

(ii) $f(b \cdot a) = g(b) \cdot f(a)$,

(iii) $f(b \star a) = g(b) \star f(a)$

for any $b \in B$, $a \in A$ and $\star \in \Omega_2'$.

Definition 3.11. We call a crossed module (S, T, σ) in C as a *subcrossed module* of a crossed module (A, B, α) in C if

SCM1. S is a subobject of A and T is a subobject of B;

SCM2. σ is the restriction of α to S;

SCM3. the action of T on S is induced by the action of B on A.

Definition 3.12. A subcrossed module (S,T,σ) of (A,B,α) is called *normal* if

NCM1. T is an ideal of B,

NCM2. $b\cdot s\in S$ for all $b\in B$, $s\in S$,

NCM3. $(t\cdot a)-a\in S$ for all $t\in T$, $a\in A$,

NCM4. $b\star s\in S$ for all $b\in B$, $s\in S$,

NCM5. $t\star a\in S$ for all $t\in T$, $a\in A$.

Remark 3.13. Here we note that if (S,T,σ) is a normal subcrossed module of (A,B,α) then by the conditions [CM2] of Definition 3.10; and [NCM2] and [NCM4] of Definition 3.12, S becomes an ideal of A.

As an example if $(f,g)\colon (A,B,\alpha)\to(A',B',\alpha')$ is a morphism of crossed modules in C, then $(\mathsf{Ker}f,\mathsf{Ker}g,\alpha_{|\mathsf{Ker}f})$, the kernel of (f,g), is a normal subcrossed module of (A,B,α).

As a corollary of Lemma 3.9, a normal subcrossed module in C can be characterized as follow.

Corollary 3.14. *A subcrossed module (S,T,σ) of (A,B,α) is normal if and only if $S\rtimes T$ is an ideal of $A\rtimes B$.*

We now obtain quotient crossed module in C as follows:

Theorem 3.15. *Let (S,T,σ) be a normal subcrossed module of (A,B,α) in C. Then we have a crossed module $(A/S,B/T,\alpha^*)$ called* quotient crossed module, *where A/S and B/T are quotient groups with operations.*

Proof. The actions of B/T on A/S are defined by

$$[b]\cdot[a]=[b\cdot a]$$
$$[b]\star[a]=[b\star a].$$

These actions are well defined. If $[b]=[b_1]\in B/T$ and $[a]=[a_1]\in A/S$, then $b-b_1\in T$ and $a-a_1\in S$. Hence

$$\begin{aligned}b\cdot a-b_1\cdot a_1&=b\cdot(a-a_1+a_1)-b_1\cdot a_1\\&=b\cdot(a-a_1)+b\cdot a_1-b_1\cdot a_1\\&=b\cdot(a-a_1)+(b-b_1+b_1)\cdot a_1-b_1\cdot a_1\\&=b\cdot(a-a_1)+(b-b_1)\cdot(b_1\cdot a_1)-b_1\cdot a_1\end{aligned}$$

and by the substitutions $s=a-a_1\in S$, $t=b-b_1\in T$ and $a_2=b_1\cdot a_1\in A$, we get that

$$\begin{aligned}b\cdot a-b_1\cdot a_1&=b\cdot(a-a_1)+(b-b_1)\cdot(b_1\cdot a_1)-b_1\cdot a_1\\&=b\cdot s+(t\cdot a_2)-a_2.\end{aligned}$$

Since $b \cdot s, (t \cdot a_2) - a_2 \in S$ by the conditions [NCM2] and [NCM3], we have that $b \cdot s + (t \cdot a_2) - a_2 = b \cdot a - b_1 \cdot a_1 \in S$. This means $[b \cdot a] = [b_1 \cdot a_1]$ in B/T. Moreover

$$\begin{aligned} b \star a - b_1 \star a_1 &= b \star a - b \star a_1 + b \star a_1 - b_1 \star a_1 \\ &= b \star (a - a_1) + (b - b_1) \star a_1 \end{aligned}$$

and by the same substitutions above we get

$$b \star a - b_1 \star a_1 = b \star s + t \star a_1.$$

and so by the conditions [NCM4] and [NCM5] $b \star a - b_1 \star a_1 \in S$. Hence $[b \star a] = [b_1 \star a_1]$ and therefore these actions are well defined.

These are derived actions since the conditions of [9, Proposition 1.1] are satisfied.

On the other hand it is clear that the boundary map, $\alpha^* \colon A/S \to B/T$ defined by $\alpha^*([a]) = [\alpha(a)]$, is well defined and the conditions [CM1]-[CM4] of Definition 3.10 are satisfied. □

The following result is useful for some proofs and since the proof is clear it is omitted.

Proposition 3.16. *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S & \longrightarrow & A & \longrightarrow & P & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \pi} & & \\ 1 & \longrightarrow & T & \longrightarrow & B & \longrightarrow & Q & \longrightarrow & 1 \end{array}$$

be a short exact sequence of crossed modules in C. *Then* (S, T, σ) *is a normal subcrossed module of* (A, B, α) *and we have the short exact sequence of groups with operations*

$$1 \longrightarrow S \rtimes T \longrightarrow A \rtimes B \longrightarrow P \rtimes Q \longrightarrow 1$$

and so $S \rtimes T$ *is an ideal of* $A \rtimes B$.

In the following result we prove that a normal subcrossed module in C is categorically a normal object in the category of crossed modules in C.

Theorem 3.17. (S, T, σ) *is a normal subcrossed module of* (A, B, α) *if and only if it is a kernel of a morphism* $(f, g) \colon (A, B, \alpha) \to (C, D, \gamma)$ *of crossed modules in* C.

Proof. We know that the kernel of a crossed module morphism $(f, g) \colon (A, B, \alpha) \to (C, D, \gamma)$ is a normal subcrossed module of (A, B, α).

Conversely let (S, T, σ) be a normal subcrossed module of (A, B, α). Then we have a morphism $(p_A, p_B) \colon (A, B, \alpha) \to (A/S, B/T, \alpha^*)$ of crossed modules whose kernel is the crossed module (S, T, σ), where $p_A \colon A \to A/S$ and $p_B \colon B \to B/T$ are the natural projections. □

Proposition 3.18. *Let* (S,T,σ) *be a normal subcrossed module of* (A,B,α) *in* C. *Then the semi-direct product groups with operations* $A/S \rtimes B/T$ *and* $(A \rtimes B)/(S \rtimes T)$ *are isomorph in* C.

Proof. It can be proved that the function

$$\begin{array}{cccc} \varphi & : & A/S \rtimes B/T & \longrightarrow & (A \rtimes B)/(S \rtimes T) \\ & & ([a],[b]) & \longmapsto & [(a,b)] \end{array}$$

is an isomorphism in C. First we need to show that φ is well defined. Let $[a]=[a_1]$ and $[b]=[b_1]$ for some $a,a_1 \in A$ and $b,b_1 \in B$. Then $a-a_1 \in S$ and $b-b_1 \in T$. Here

$$\begin{aligned} (a,b)-(a_1,b_1) &= (a,b)+((-b_1)\cdot(-a_1),-b_1) \\ &= (a+b\cdot((-b_1)\cdot(-a_1)),b-b_1) \\ &= (a+(b-b_1)\cdot(-a_1),b-b_1) \end{aligned}$$

and since $b-b_1 \in T$ by [NCM3.] of Definition 3.12 $(a,b)-(a_1,b_1) \in S \rtimes T$. This implies that φ is well defined. It can easily be seen that φ is a bijection, that is one-to-one and onto. Now we only need to show that φ is a morphism in C. Let $a,a_1 \in A$, $b,b_1 \in B$, $\omega \in \Omega_1$ and $\star \in \Omega_2'$. Then

$$\begin{aligned} \varphi\left(([a],[b])+([a_1],[b_1])\right) &= \varphi\left(([a]+[b]\cdot[a_1],[b]+[b_1])\right) \\ &= \varphi\left(([a+b\cdot a_1],[b+b_1])\right) \\ &= [(a+b\cdot a_1,b+b_1)] \\ &= [(a,b)+(a_1,b_1)] \\ &= [(a,b)]+[(a_1,b_1)] \\ &= \varphi\left(([a],[b])\right)+\varphi\left(([a_1],[b_1])\right) \end{aligned}$$

So φ is a group homomorphism. Also

$$\begin{aligned} \varphi\left(([a],[b])\star([a_1],[b_1])\right) &= \varphi\left(([a]\star[a_1]+[a]\star[b_1]+[b]\star[a_1],[b]\star[b_1])\right) \\ &= \varphi\left(([a\star a_1+a\star b_1+b\star a_1],[b\star b_1])\right) \\ &= [(a\star a_1+a\star b_1+b\star a_1,b\star b_1)] \\ &= [(a,b)\star(a_1,b_1)] \\ &= [(a,b)]\star[(a_1,b_1)] \\ &= \varphi\left(([a],[b])\right)\star\varphi\left(([a_1],[b_1])\right) \end{aligned}$$

and

$$\begin{aligned} \varphi\omega\left(([a],[b])\right) &= \varphi\left((\omega[a],\omega[b])\right) \\ &= \varphi\left(([\omega(a)],[\omega(b)])\right) \\ &= [(\omega(a),\omega(b))] \\ &= [\omega\,(a,b)] \\ &= \omega[(a,b)] \\ &= \omega\varphi\left(([a],[b])\right). \end{aligned}$$

Hence φ is a morphism in C. This completes the proof. □

The isomorphism theorem for crossed modules in C can be given as follows.

Theorem 3.19. *Let $(f,g)\colon (A,B,\alpha)\to(A',B',\alpha')$ be a morphism of crossed modules; and let $\mathsf{Ker}f = S$ and $\mathsf{Ker}g = T$. Then the image $(f(A),g(B),\alpha')$ is a subcrossed module and isomorph to the quotient crossed module $(A/S,B/T,\rho)$.*

Proof. It is easy to see that $(f(A),g(B),\alpha')$ is a subcrossed module and

$$(\widetilde{f},\widetilde{g})\colon (A/S,B/T,\rho)\to(f(A),g(B),\alpha')$$

is an isomorphism of crossed modules, where $\widetilde{f}(aS)=f(a)$ and $\widetilde{g}(bT)=g(b)$ for $aS\in A/S$ and $bT\in B/T$. □

Corollary 3.20. *Let $(f,g)\colon (A,B,\alpha)\to(A',B',\alpha')$ be an epimorphism of crossed modules; and let $\mathsf{Ker}f = S$ and $\mathsf{Ker}g = T$. Then the image (A',B',α') is isomorph to the quotient crossed module $(A/S,B/T,\rho)$.*

4. Conclusion

In this paper for an algebraic category C we have defined normal subcrossed module and quotient crossed module for groups with operations. In [26] Porter proved that the crossed modules within groups with operation are categorically equivalent to the internal groupoids in C. Therefore it could be possible to obtain normal and quotient terms in the internal categories in C.

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