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The (p, q)-Bernstein-Stancu Operator of Rough Statistical Convergence on Triple Sequence

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abstract: In the paper, we investigate rough statistical approximation properties of (p, q)-analogue of Bernstein-Stancu Operators. We study approximation properties based on rough statistical convergence. We also study error bound using modulus of continuity.

Key Words: (p, q)-Bernstein-Stancu operator, Rough statistical convergence, Natural density, Triple sequences, chi sequence, Korovkin type approximation theorem.

Contents

1. Introduction

First applied the concept of (p, q)-calculus in approximation theory and introduced the (p, q)-analogue of Bernstein operators. Later, based on (p, q)-integers, some approximation results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, (p, q)-Lorentz operators, Bleimann-Butzer and Hahn operators and Bernstein-Shurer operators etc.

Very recently, Khalid et al. have given a nice application in computer-aided geometric design and applied these Bernstein basis for construction of (p, q)-Bezier curves and surfaces based on (p, q)-integers which is further generalization of q-Bezier curves and surfaces.

Motivated by the above mentioned work on (p, q)-approximation and its application, in this paper we study statistical approximation properties of Bernstein-Stancu Operators based on (p, q)-integers.

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Now we recall some basic definitions about (p,q)-integers. For any $u,v,w \in \mathbb{N}$, the (p,q)-integer $[uvw]_{p,q}$ is defined by

$$[0]_{p,q} := 0 \text{ and } [uvw]_{p,q} = \frac{p^{uvw} - q^{uvw}}{p-q} \text{ if } u,v,w \geq 1,$$

where $0 < q < p \leq 1$. The (p,q)-factorial is defined by

$$[0]_{p,q}! := 1 \text{ and } [uvw]!_{p,q} = [1]_{p,q}[2]_{p,q} \cdots [uvw]_{p,q} \text{ if } u,v,w \geq 1.$$

Also the (p,q)-binomial coefficient is defined by

$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} = \frac{[u]!_{p,q}}{[m]!_{p,q}\,[u-m]!_{p,q}}\frac{[v]!_{p,q}}{[n]!_{p,q}\,[v-n]!_{p,q}}\frac{[w]!_{p,q}}{[k]!_{p,q}\,[w-k]!_{p,q}}$$

for all $u,v,w,m,n,k \in \mathbb{N}$ with $(u,v,w) \geq (m,n,k)$.

The formula for (p,q)-binomial expansion is as follows:

$$(ax+by)^{uvw}_{p,q} = \sum_{m=0}^{u}\sum_{n=0}^{v}\sum_{k=0}^{w} p^{\frac{(u-m)(u-m-1)+(v-n)(v-n-1)+(w-k)(w-k-1)}{2}} q^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}} \cdot$$

$$\binom{u}{m}\binom{v}{n}\binom{w}{k}_{p,q} a^{(u-m)+(v-n)+(w-k)} b^{m+n+k} x^{(u-m)+(v-n)+(w-k)} y^{m+n+k},$$

$$(x+y)^{uvw}_{p,q} = (x+y)(px+qy)\left(p^2x+q^2y\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x + q^{(u-1)+(v-1)+(w-1)}y\right),$$

$$(1-x)^{uvw}_{p,q} = (1-x)\,(p-qx)\,\left(p^2-q^2x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)} - q^{(u-1)+(v-1)+(w-1)}x\right),$$

and

$$(x)^{mnk}_{p,q} = x\,(px)\,\left(p^2x\right)\cdots\left(p^{(u-1)+(v-1)+(w-1)}x\right) = p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}.$$

The Bernstein operator of order (r,s,t) is given by

$$B_{rst}(f,x) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}\,(1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

(p,q)-Bernstein operators are defined as follows:

$$B_{rst,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{2}}} \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} \binom{r}{m}\binom{s}{n}\binom{t}{k} p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}} \cdot$$

$$x^{m+n+k} \prod_{u_1=0}^{(r-m-1)} \left(p^{u_1} - q^{u_1}x\right) \prod_{u_2=0}^{(s-n-1)} \left(p^{u_2} - q^{u_2}x\right) \prod_{u_3=0}^{(t-k-1)} \left(p^{u_3} - q^{u_3}x\right) \cdot \tag{1.1}$$

$$f\left(\frac{p^{(r-m)}\,[m]_{p,q} + p^{(s-n)}\,[n]_{p,q}\,p^{(t-k)}\,[k]_{p,q}}{[r]_{p,q} + [s]_{p,q} + [t]_{p,q}}\right), x \in [0,1]$$

Also, we have

$$(1-x)_{p,q}^{rst} = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}(-1)^{m+n+k}\,p^{\frac{(r-m)(r-m-1)+(s-n)(s-n-1)+(t-k)(t-k-1)}{2}}\cdot q^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}\binom{r}{m}\binom{s}{n}\binom{t}{k}x^{m+n+k}.$$

$$S_{rst,p,q}(f,x) = \frac{1}{p^{\frac{r(r-1)+s(s-1)+t(t-1)}{2}}}\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{k}p^{\frac{m(m-1)+n(n-1)+k(k-1)}{2}}\cdot x^{m+n+k}\prod_{u_1=0}^{(r-m-1)}(p^{u_1}-q^{u_1}x)\prod_{u_2=0}^{(s-n-1)}(p^{u_2}-q^{u_2}x)\prod_{u_3=0}^{(t-k-1)}(p^{u_3}-q^{u_3}x)\cdot f\left(\frac{p^{(r-m)}[m]_{p,q}+p^{(s-n)}[n]_{p,q}\,p^{(t-k)}[k]_{p,q}+\eta}{[r]_{p,q}+[s]_{p,q}+[t]_{p,q}+\mu}\right), x\in[0,1] \tag{1.2}$$

Note that for $\eta=\mu=0$, (p,q)-Bernstein-Stancu operators given by (1.2) reduces into (p,q)-Bernstein operators. Also for $p=1$, (p,q)-Bernstein-Stancu operators given by (1.1) turn out to be q-Bernstein-Stancu operators.

Lemma 1.1. *For $x\in[0,1]$, $0<q<p\le 1$, we have*

(i) $S_{rst,p,q}(1,x)=1$

(ii) $S_{rst,p,q}(t,x)=\frac{[rst]_{pq}}{[rst]_{pq}+\mu}x+\frac{\eta}{[rst]_{pq}+\mu}$

(iii)

$$S_{rst,p,q}(t^2,x) = \frac{q[rst]_{pq}[(r-1)+(s-1)+(t-1)]}{\left([rst]_{pq}+\mu\right)^2}x^2 + \frac{q[rst]_{pq}\left(2\eta+p^{(r-1)+(s-1)+(t-1)}\right)}{\left([rst]_{pq}+\mu\right)^2}x+\frac{\eta^2}{\left([rst]_{pq}+\mu\right)^2}.$$

Proof. The proof can be made in a similar way as in [8, Lemma 2.1]. □

Lemma 1.2. *For $x\in[0,1]$, $0<q<p\le 1$, we have*

(i) $S_{rst,p,q}((t-x),x)=\frac{\eta-\mu x}{[rst]_{pq}+\mu}$,

(ii)

$$S_{rst,p,q}\left((t-x)^2,x\right) = \left\{\frac{q[rst]_{pq}[(r-1)+(s-1)+(t-1)]_{pq}-[rst]_{pq}^2+\mu^2}{\left([rst]_{pq}+\mu\right)^2}\right\}x^2 + \left\{\frac{p^{(r-1)+(s-1)+(t-1)}[rst]_{pq}-2\eta\mu}{\left([rst]_{pq}+\mu\right)^2}\right\}x+\frac{\eta^2}{\left([rst]_{pq}+\mu\right)^2}.$$

The idea of statistical convergence was introduced by Steinhaus and also independently by Fast for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, and let us denote the set $\{(m, n, k) \in K : m \leq u, n \leq v, k \leq w\}$ by K_{uvw}. Then the natural density of K is given by $\delta(K) = \lim_{uvw \to \infty} \frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, a finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$ where $K^c = \mathbb{N} \backslash K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X, d). Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}$, $m, n, k \in \mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0, 1]$. A triple sequence of Bernstein-Stancu polynomials $(B_{rst}(f, x))$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st - \lim x = 0$, provided that the set

$$K_\epsilon := \left\{(m, n, k) \in \mathbb{N}^3 : |B_{mnk}(f, x) - f(x)| \geq \epsilon\right\}$$

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein-Stancu polynomials. i.e., $\delta(K_\epsilon) = 0$. That is,

$$\lim_{rst \to \infty} \frac{1}{rst} \left|\{(m, n, k) \leq (r, s, t) : |B_{mnk}(f, x) - f(x)| \geq \epsilon\}\right| = 0.$$

In this case, we write $\delta - \lim B_{mnk}(f, x) = f(x)$ or $B_{mnk}(f, x) \to^{S_B} f(x)$.

If a triple sequence is statistically convergent, then for every $\epsilon > 0$, infinitely many terms of the sequence may remain outside the ϵ- neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence satisfies some property P for all m, n, k except a set of natural density zero, then we say that the triple sequence satisfies P for almost all (m, n, k) and we abbreviate this by a.a. (m, n, k).

Let $\left(x_{m_i n_j k_\ell}\right)$ be a sub sequence of $x = (x_{mnk})$. If the natural density of the set $K = \left\{(m_i, n_j, k_\ell) \in \mathbb{N}^3 : (i, j, \ell) \in \mathbb{N}^3\right\}$ is different from zero, then $\left(x_{m_i n_j k_\ell}\right)$ is called a non thin sub sequence of a triple sequence x.

$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence $x = (x_{mnk})$ provided that the natural density of the set

$$\left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - c| < \epsilon\right\}$$

is different from zero for every $\epsilon > 0$. We denote the set of all statistical cluster points of the sequence x by Γ_x.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein-Stancu polynomials $(B_{rst}(f,x))$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x) - f(x)|^{1/(m+n+k)} \geq M\right\}\right) = 0$$

That is,

$$\lim_{rst \to \infty} \frac{1}{rst}\left|\left\{(m,n,k) \leq (r,s,t) : |B_{mnk}(f,x) - f(x)|^{1/(m+n+k)} \geq M\right\}\right| = 0.$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [10], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [1] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [9] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

In this paper, we introduce the notion of Bernstein-Stancu operator of rough statistical convergence of triple sequences. Defining the set of Bernstein-Stancu polynomials of rough statistical limit points of a triple sequence, we obtain to Bernstein-Stancu operator of statistical convergence criteria associated with this set. Later, we prove that this set of Bernstein-Stancu operator of statistical cluster points and the set of rough statistical limit points of a triple sequence. The sum of this paper was presented in International Conference of Mathematical Sciences (ICMS, 2018), [14].

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N}^3 \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [11,12], Esi et al. [2,3,4,5], Datta et al. [6], Subramanian et al. [13], Debnath et al. [7] and many others.

Throughout the paper let r be a nonnegative real number.

2. Definitions and Preliminaries

Definition 2.1. *Let $C[a,b]$ be a continuous function f defined on $[a,b]$ and let B be a linear operator which maps $C[a,b]$ into itself. We say that B is positive if for every non-negative $f \in C[a,b]$, we have $B(f,x) \geq 0$ for all $x \in [a,b]$. A triple sequence of Bernstein-Stancu polynomials $B_{rst}(f,x) : C[a,b] \to C[a,b]$ be a linear operators. Then $\lim |B_{rst}(f,x) - (f,x)| = 0$, for all $f \in C[a,b]$ if and only if $\lim_{rst\to\infty} |B_{rst}(f_{ij\ell},x) - (f_{ij\ell},x)| = 0$, for each $i,j,\ell = 0,1,2,\ldots$, where the test function $f_{ij\ell}(x) = x^{i+j+\ell}$.*

3. Main Results

3.1. Korovkin type approximation theorem

Theorem 3.1. *Let A_{rst} be the triple sequence of linear positive operators from $C[0,1]$ to $C[0,1]$ satisfying three conditions*

$$st - \lim_{rst} \left| S_{rst,p,q}\left(\left(t^{\gamma}, x \right) \right), (x) \right|_{C[0,1]} = 0 \text{ for } \gamma = 0,1,2.$$

then for any function $f \in C[0,1]$

$$st - \lim_{rst} \left| S_{rst,p,q}\left(f, x \right), (f, x) \right|_{C[0,1]} = 0.$$

To obtain the statistical convergence of the operators (1.2) we need the following results.

3.2. Triple sequence of Korovkin type statistical approximation properties

Now, we consider two triple sequences $p = (p_{rst})$ and $q = (q_{rst})$ satisfying the following expression $0 < q_{rst} < p_{rst} \leq 1$

$$st - \lim_{rst} q_{rst} = 1$$

and

$$st - \lim_{rst} p_{rst} = 1$$

Theorem 3.2. *Let $S_{rst,p,q}$ be the triple sequence of Bernstein-Stancu operators and the sequence $p = p_{rst}$ and $q = q_{rst}$ satisfy for $0 < q_{rst} < p_{rst} \leq 1$ then for any function $f \in [0,l]$*

$$st - \lim_{rst} \left| s_{rst,p_{rst},q_{rst}}\left(f, x \right) - (f, x) \right| = 0.$$

Proof. $s_{rst,p_{rst},q_{rst}}(1,x) = 1 = (1,x)$. We can write

$$st - \lim_{rst} \left| s_{rst,p_{rst},q_{rst}}\left(1, x \right) - (1, x) \right| = \lim_{rst} |1 - 1| = 0.$$

that is

$$st - \lim_{rst} \left| s_{rst,p_{rst},q_{rst}}\left(1, x \right) - 1 \right| = 0.$$

For $\gamma = 1$

$$\begin{aligned}
\left| s_{rst,p,q}\left(t, x \right) - x \right| &\leq \left| \frac{[rst]_{pq}}{[rst]_{pq} + \mu} x + \frac{\eta}{[rst]_{pq} + \mu} - x \right| \\
&\leq \left| \left(\frac{[rst]_{pq}}{[rst]_{pq} + \mu} - 1 \right) x + \frac{\eta}{[rst]_{pq} + \mu} \right| \\
&\leq \left| \frac{[rst]_{pq}}{[rst]_{pq} + \mu} - 1 \right| + \left| \frac{\eta}{[rst]_{pq} + \mu} \right|.
\end{aligned}$$

For a given $\epsilon > 0$ then there exists a real number $\beta > 0$, let us define the following sets.

$$U = \{(rst) : |s_{rst,p,q}(t,x) - x| \geq \beta + \epsilon\}$$

$$U^{'} = \left\{(rst) : 1 - \frac{[rst]_{pq}}{[rst]_{pq} + \mu} \geq \beta + \epsilon\right\}$$

$$U^{''} = \left\{(rst) : \frac{\eta}{[rst]_{pq} + \mu} \geq \beta + \epsilon\right\}.$$

It is obvious that $U \cup U^{'} \cup U^{''}$, it is clear that

$$st - \lim_{rst}\left(1 - \frac{[rst]_{pq}}{[rst]_{pq} + \mu}\right) = 0$$

So

$$\delta\left(\left\{(m,n,k) \leq (r,s,t) : \left|1 - \frac{[rst]_{pq}}{[rst]_{pq} + \mu}\right| \geq M\right\}\right) = 0.$$

Then

$$st - \lim_{rst}|s_{rst,p,q}(t,x) - x| = 0.$$

Now consider for $\gamma = 2$ we have

$$\begin{aligned} S_{rst,p,q}\left((t-x)^2 - x^2\right) \quad &\leq \quad \left\{\frac{q\,[rst]_{pq}\,[(r-1)+(s-1)+(t-1)]_{pq} - [rst]_{pq}^2 + \mu^2}{\left([rst]_{pq} + \mu\right)^2}\right\} x^2 \\ &+ \left\{\frac{p^{(r-1)+(s-1)+(t-1)}\,[rst]_{pq} - 2\eta\mu}{\left([rst]_{pq} + \mu\right)^2}\right\} x + \frac{\eta^2}{\left([rst]_{pq} + \mu\right)^2}. \end{aligned}$$

If we choose

$$\eta_{rst} = \left\{\frac{q\,[rst]_{pq}\,[(r-1)+(s-1)+(t-1)]_{pq} - [rst]_{pq}^2 + \mu^2}{\left([rst]_{pq} + \mu\right)^2}\right\},$$

$$\mu_{rst} = \left\{\frac{p^{(r-1)+(s-1)+(t-1)}\,[rst]_{pq} - 2\eta\mu}{\left([rst]_{pq} + \mu\right)^2}\right\},$$

and

$$\gamma_{rst} = \frac{\eta^2}{\left([rst]_{pq} + \mu\right)^2}.$$

We have

$$st - \lim_{rst} \eta_{rst} = st - \lim_{rst} \mu_{rst} = st - \lim_{rst} \gamma_{rst} = 0.$$

Hence given $\epsilon > 0$ and there exists a real number $\beta > 0$, we define the sets:

$$U = \left| S_{rst,p,q} \left((t-x)^2 - x^2 \right) \right| \geq \beta + \epsilon$$

$$U' = \left\{ (rst) : \eta_{rst} \geq \beta + \frac{\epsilon}{3} \right\}$$

$$U'' = \left\{ (rst) : \mu_{rst} \geq \beta + \frac{\epsilon}{3} \right\}$$

$$U''' = \left\{ (rst) : \gamma_{rst} \geq \beta + \frac{\epsilon}{3} \right\}.$$

It is obviuous that $U \bigcup U' \bigcup U'' \bigcup U'''$. Thus we obtain

$$\delta \left\{ M \leq (r,s,t) : \left| S_{rst,p,q} \left((t-x)^2 - x^2 \right) \right| \geq \beta + \epsilon \right\}$$
$$\leq \delta \left\{ M \leq (rst) : \eta_{rst} \geq \beta + \frac{\epsilon}{3} \right\} + \delta \left\{ M \leq (rst) : \mu_{rst} \geq \beta + \frac{\epsilon}{3} \right\}$$
$$+ \delta \left\{ M \leq (rst) : \gamma_{rst} \geq \beta + \frac{\epsilon}{3} \right\}.$$

So the right hand side of the inequalities is zero. Then

$$st - \lim_{rst} \left| s_{rst,p,q} (t-x)^2 - x^2 \right| = 0.$$

holds. □

4. Rate of Stastical Convergence

In this part, rates of statistical convergence of the Triple sequence of Bernstein-Stancu operator (1.2) by means of modulus of continuity and maximal functions are introduced. The modulus of continuity for the space of function $f \in C[0,1]$ is defined by

$$w(f, \delta) = \sup |(f, t) - (f, x)|$$

where $w(f, \delta)$ satisfies the following condition for all $f \in C[0,1]$.

(i) $\lim_{\delta \to 0} w(f, \delta) = 0$,

(ii) $|(f,t) - (f,x)| \leq w(f, \delta) \left(\frac{|t-x|}{\delta} + 1 \right)$.

Theorem 4.1. *Let the triple sequence $p = (p_{rst})$ and $q = (q_{rst})$ satisfy for $0 < q < p \leq 1$, so we have*

$$|S_{rst,p,q}(t,x) - (f,x)| \leq w \left(f, \sqrt{\delta_{rst}(x)} \right) (1 + q_{rst})$$

where

$$\delta_{rst}(x) = \frac{1}{\left([rst]_{pq} + \beta\right)^2} \left[\left(q\,[rst]_{pq}\,[(r-1)+(s-1)+(t-1)]_{pq} - [rst]^2 + \mu^2\right) x^2 + \left([rst]_{pq}\, p^{r(r-1)+s(s-1)+t(t-1)} - 2\eta\mu\right) x + \eta^2\right].$$

Proof.

$$\left|S_{rst,p,q}(t,x) - (f,x)\right| \leq S_{rst,p,q}\left(\left|(f,t)-(f,x)\right| : x\right)$$

We get

$$\left|S_{rst,p,q}(t,x) - (f,x)\right| \leq w(f,\delta)\left\{S_{rst,p,q}(1,x) + \frac{1}{\delta} S_{rst,p,q}((t-x),x)\right\}$$

by using Cauchy Schwarz inequality, we have

$$\left|S_{rst,p,q}(t,x) - (f,x)\right| \leq w(f,\delta)\left(1 + \frac{1}{\delta}\left[\left(S_{rst,p,q}(t-x)^2, x\right)\right]^{\frac{1}{2}}\left[S_{rst,p,q}(1,x)\right]^{\frac{1}{2}}\right) \leq$$

$$w(f,\delta)\left(1 + \frac{1}{\delta}\left\{\frac{1}{\left([rst]_{pq}+\mu\right)^2}\left[\left(q\,[rst]_{pq}\,[(r-1)+(s-1)+(t-1)]_{pq} - [rst]^2 + \mu^2\right)x^2 + \left([rst]_{pq}\,p^{r(r-1)+s(s-1)+t(t-1)} - 2\eta\mu\right)x + \eta^2\right]\right\}\right).$$

Hence $st - \lim_{rst} \delta_{rst} = 0$, we have

$$st - \lim_{rst} w(f,\delta) = 0.$$

This gives us the pointwise rate of statistical convergence of the Bernstein-Stancu Operators $S_{rst,p,q}(f,x) \to (f,x)$. □

Theorem 4.2. *If $S_{rst,p,q}$ is defined by (1.2), then for all $f \in W_\eta \cdot E$*

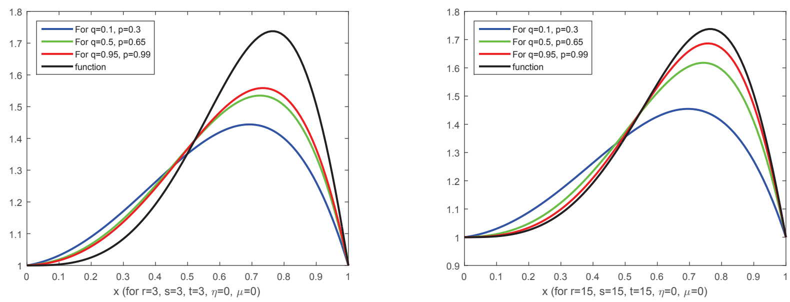
$$\left|S_{rst,p,q}(t,x) - (f,x)\right| \leq M\left(\rho_{rst}\frac{\eta}{6} q_{rst}\frac{6-\eta}{6} + 6 q_{rst} d(x,E)\right)$$

where

$$\rho_{rst} = \frac{1}{\left([rst]_{pq}+\mu\right)^2}\left[\left(q\,[rst]_{pq}\,[(r-1)+(s-1)+(t-1)]_{pq} - [rst]^2 + \mu^2\right)x^2 + \left([rst]_{pq}\,p^{r(r-1)+s(s-1)+t(t-1)} - 2\eta\mu\right)x + \eta^2\right].$$

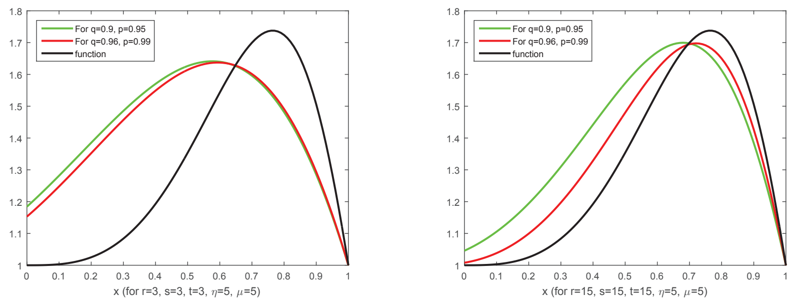
Example 4.3. *With the help of Matlab, we show comparisons and some illustrative graphics for the convergence of operators (1.2) to the function $f(x) = 1 + x\sin\left(\pi x^2\right)$ under different parameters.*

From figure 1(a), it can be observed that as the value the q and p approaches towards 1 provided $0 < q < p \leq 1$, (p, q)-Bernstein-Stancu operators given by (1.2) converges towards the function $f(x) = 1 + x \sin\left(\pi x^2\right)$. From figure 1(a) and (b), it can be observed that for $\eta = \mu = 0$, as the value the (r, s, t) increases, (p, q)-Bernstein-Stancu operators given by (1.2) converges towards the function. Similarly from figure 2(a), it can be observed that for $\eta = \mu = 3$, as the value the q and p approaches towards 1 or some thing else provided $0 < q < p \leq 1$, (p, q)-Bernstein-Stancu operators given by (1.2) converges towards the function. From figure 2(a) and (b), it can be observed that as the value the $[r, s, t]$ increases, (p, q)-Bernstein-Stancu operators given by $f(x) = 1 + x \sin\left(\pi x^2\right)$ converges towards the function.



(a) (b)

Figure 1: (p, q)-Bernstein-Stancu operators



(a) (b)

Figure 2: (p, q)-Bernstein-Stancu operators

Competing Interests

The authors declare that there is not any conflict of interests regarding the publication of this manuscript.

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