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Tauberian Theorems for the Product of Weighted and Cesàro Summability Methods for Double Sequences *

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ABSTRACT: In this paper, we obtain necessary and sufficient conditions, under which convergence of a double sequence in Pringsheim's sense follows from its weighted-Cesàro summability. These Tauberian conditions are one-sided or two-sided if it is a sequence of real or complex numbers, respectively.

Key Words: Tauberian theorems, Double sequences, Cesàro means, Weighted means, Convergence in Pringsheim's sense, Weighted-Cesàro summability.

Contents

1. Introduction

A double sequence $u = (u_{mn})$ is called convergent in Pringsheim's sense (in short P-convergent) to s [9], if for a given $\varepsilon > 0$ there exists a positive integer N_0 such that $|u_{mn} - s| < \varepsilon$ for all nonnegative integers $m, n \geq N_0$.

$(C, 1, 1)$ means of (u_{mn}) are defined by

$$\sigma_{mn}^{(11)}(u) := \frac{1}{(m+1)(n+1)} \sum_{i=0}^{m} \sum_{j=0}^{n} u_{ij}$$

for all nonnegative integers m and n. Similarly, $(C, 1, 0)$ and $(C, 0, 1)$ means of (u_{mn}) are defined respectively by

$$\sigma_{mn}^{(10)}(u) := \frac{1}{m+1} \sum_{i=0}^{m} u_{in}, \quad \sigma_{mn}^{(01)}(u) := \frac{1}{n+1} \sum_{j=0}^{n} u_{mj}$$

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for all nonnegative integers m and n.

A sequence (u_{mn}) is said to be (C,α,β) summable to s if $\lim_{m,n\to\infty}\sigma_{mn}^{(\alpha\beta)}(u)=s$, where $(\alpha,\beta)=(1,1)$, $(1,0)$ and $(0,1)$. In this case, we write $u_{mn}\to s\ (C,\alpha,\beta)$.

Let $p:=\{p_k\}_{k=0}^{\infty}$ and $q:=\{q_l\}_{l=0}^{\infty}$ be sequences of nonnegative real numbers with $p_0,q_0>0$ such that $P_m:=\sum_{k=0}^{m}p_k\neq 0$ for all $m\geq 0$ and $Q_n:=\sum_{l=0}^{n}q_l\neq 0$ for all $n\geq 0$. The weighted means $t_{mn}^{(\alpha\beta)}$ of a double sequence (u_{mn}), in short, the $\left(\overline{N},p,q;\alpha,\beta\right)$ means, are defined respectively by

$$t_{mn}(u)=t_{mn}^{(11)}(u):=\frac{1}{P_mQ_n}\sum_{k=0}^{m}\sum_{l=0}^{n}p_kq_lu_{kl}$$

$$t_{mn}^{(10)}(u)=\frac{1}{P_m}\sum_{k=0}^{m}p_ku_{kn},\quad t_{mn}^{(01)}(u)=\frac{1}{Q_n}\sum_{l=0}^{n}q_lu_{ml}$$

where $m,n\geq 0$. A sequence (u_{mn}) is said to be summable by the weighted mean method determined by the sequences p and q, in short, summable $(\overline{N},p,q;\alpha,\beta)$ where $(\alpha,\beta)=(1,1),(1,0),(0,1)$ if $\lim_{m,n\to\infty}t_{mn}^{(\alpha\beta)}=s$. In this case, we write $u_{mn}\to s\ \left(\overline{N},p,q;\alpha,\beta\right)$.

The product of $\left(\overline{N},p,q;1,1\right)$ and (C,α,β) summability is defined by $\left(\overline{N},p,q;1,1\right)(C,\alpha,\beta)$ summability, where $(\alpha,\beta)=(1,1),(1,0),(0,1)$.

The $\left(\overline{N},p,q;1,1\right)(C,\alpha,\beta)$ mean of (u_{mn}) is given by

$$t_{mn}^{(11)}(\sigma^{(\alpha\beta)}(u)):=\frac{1}{P_mQ_n}\sum_{k=0}^{m}\sum_{l=0}^{n}p_kq_l\sigma_{kl}^{(\alpha\beta)}(u).$$

A sequence (u_{mn}) is said to be $\left(\overline{N},p,q;1,1\right)(C,\alpha,\beta)$ summable to s if

$$\lim_{m,n\to\infty}t_{mn}^{(11)}(\sigma^{(\alpha\beta)}(u))=s. \tag{1.1}$$

In this case, we write $u_{mn}\to s\ \left(\overline{N},p,q;1,1\right)(C,\alpha,\beta)$.

The $\left(\overline{N},p,q;1,1\right)(C,\alpha,\beta)$ summability method is regular under boundness condition that is to say if

$$\lim_{m,n\to\infty}u_{mn}=s \tag{1.2}$$

exists and (u_{mn}) is bounded, then (u_{mn}) is $\left(\overline{N},p,q;1,1\right)(C,\alpha,\beta)$ summable to s.

The product of $\left(\overline{N},p,q;1,0\right)$ and (C,α,β) summability is defined by $\left(\overline{N},p,q;1,0\right)(C,\alpha,\beta)$ summability, where $(\alpha,\beta)=(1,1),(1,0),(0,1)$.

The $\left(\overline{N},p,q;1,0\right)(C,\alpha,\beta)$ mean of (u_{mn}) is given by

$$t_{mn}^{(10)}(\sigma^{(\alpha\beta)}(u)):=\frac{1}{P_m}\sum_{k=0}^{m}p_k\sigma_{kn}^{(\alpha\beta)}(u).$$

A sequence (u_{mn}) is said to be $\left(\overline{N},p,q;1,0\right)(C,\alpha,\beta)$ summable to s if

$$\lim_{m,n\to\infty} t_{mn}^{(10)}(\sigma^{(\alpha\beta)}(u)) = s. \tag{1.3}$$

In this case, we write $u_{mn} \to s \left(\overline{N},p,q;1,0\right)(C,\alpha,\beta)$.

The notion of summability $\left(\overline{N},p,q;0,1\right)(C,\alpha,\beta)$ is defined analogously.

We also note that $\left(\overline{N},p,q;1,0\right)(C,\alpha,\beta)$ and $\left(\overline{N},p,q;0,1\right)(C,\alpha,\beta)$ summability method are regular under boundness condition.

If a sequence (u_{mn}) is convergent in Pringsheim's sense, then it is summable $\left(\overline{N},p,q;1,1\right)(C,\alpha,\beta)$, $\left(\overline{N},p,q;1,0\right)(C,\alpha,\beta)$ and $\left(\overline{N},p,q;0,1\right)(C,\alpha,\beta)$ to the same number under boundedness condition. However, the converse statement is not true in general. That the converse of this statement holds true is possible under some suitable condition which is so-called a Tauberian condition. Any theorem which states that convergence of a double sequence follows from its weighted-Cesàro summability and some Tauberian condition is said to be a Tauberian theorem for the weighted-Cesàro summability.

Recently, there has been an increasing interest on Tauberian theorems for Cesàro and weighted mean methods for single and double sequences. For some new interesting Tauberian theorems for single sequences which have been recently published, we refer to [2] and [10]. In [3], Chen and Hsu established necessary and sufficient conditions under which $u_{mn} \to s$ follows from $u_{mn} \to s\,(\overline{N},p,q;\alpha,\beta)$, where $(\alpha,\beta) = (1,1),(1,0)$ and $(0,1)$. Móricz [7], and Baron and Stadtmüller [1] and Stadtmüller [11] have investigated several particular cases of the weighted mean methods of double sequences. Móricz [7] obtained necessary and sufficient conditions under which convergence of (u_{mn}) follows from (C,α,β) summability of (u_{mn}), where $(\alpha,\beta) = (1,1),(1,0)$ and $(0,1)$. Baron and Stadtmüller [1] studied the relations between power series methods, weighted mean methods and ordinary convergence for double sequences. In particular, they proved that analogues of Landau's two-sided conditions for double sequences are Tauberian conditions for the weighted mean method $(\overline{N},p,q;1,1)$, where P and Q are regularly varying sequences.

In [11], Stadtmüller established the relations between weighted mean methods and ordinary convergence for double sequences and he obtained a Tauberian theorem for $(C,1,1)$ summability method which includes a classical Tauberian theorem of Knopp [5] as a special case of his results and generalized theorems due to Móricz [7].

In this paper, we obtain necessary and sufficient conditions, under which convergence of a double sequence in Pringsheim's sense follows from its weighted-Cesàro summability. These Tauberian conditions are one-sided or two-sided if it is a sequence of real or complex numbers, respectively.

2. Tauberian theorems for $(\overline{N},p,q;1,1)(C,\alpha,\beta)$ summability method

Theorem 2.1. *Let*

$$\limsup_{m\to\infty} \frac{P_m}{P_{\lambda_m}} < 1, \quad \limsup_{n\to\infty} \frac{Q_n}{Q_{\lambda_n}} < 1, \quad \lambda > 1 \tag{2.1}$$

where λ_m and λ_n denote the integer parts of λm and λn for every $m,n \in \mathbb{N}$, respectively. Assume that (u_{mn}) is a sequence of real numbers which is $\left(\overline{N},p,q;1,1\right)(C,\alpha,\beta)$ summable to s. Then (u_{mn}) is convergent to s if and only if the following two conditions are satisfied:

$$\sup_{\lambda>1}\liminf_{m,n\to\infty}\frac{1}{\left(P_{\lambda_m}-P_m\right)\left(Q_{\lambda_n}-Q_n\right)}\sum_{k=m+1}^{\lambda_m}\sum_{l=n+1}^{\lambda_n}p_kq_l\left(\sigma_{kl}^{(\alpha\beta)}(u)-u_{mn}\right)\geq 0 \tag{2.2}$$

and

$$\sup_{0<\lambda<1}\liminf_{m,n\to\infty}\frac{1}{\left(P_m-P_{\lambda_m}\right)\left(Q_n-Q_{\lambda_n}\right)}\sum_{k=\lambda_{m+1}}^{m}\sum_{l=\lambda_{n+1}}^{n}p_kq_l\left(u_{mn}-\sigma_{kl}^{(\alpha\beta)}(u)\right)\geq 0. \tag{2.3}$$

2.1. Auxiliary results

In what follows we prove some auxiliary lemmas which are needed in the sequel.

Lemma 2.2. *([8]) The following assertions are equivalent:*

$$\limsup_{n\to\infty}\frac{P_m}{P_{\lambda_m}}<1 \quad (\lambda>1),$$

$$\limsup_{n\to\infty}\frac{P_{\lambda_m}}{P_m}<1 \quad (0<\lambda<1)$$

Lemma 2.3. *Assume that relations (2.1) are satisfied and let (u_{mn}) be a sequence of complex numbers which is $\left(\overline{N},p,q;1,1\right)(C,\alpha,\beta)$ summable to s. Then*

$$\lim_{m,n\to\infty}\frac{1}{\left(P_{\lambda_m}-P_m\right)\left(Q_{\lambda_n}-Q_n\right)}\sum_{k=m+1}^{\lambda_m}\sum_{l=n+1}^{\lambda_n}p_kq_l\sigma_{kl}^{(\alpha\beta)}(u)=s \tag{2.4}$$

for $\lambda>1$ and

$$\lim_{m,n\to\infty}\frac{1}{\left(P_m-P_{\lambda_m}\right)\left(Q_n-Q_{\lambda_n}\right)}\sum_{k=\lambda_{m+1}}^{m}\sum_{l=\lambda_{n+1}}^{n}p_kq_l\sigma_{kl}^{(\alpha\beta)}(u)=s \tag{2.5}$$

for $0<\lambda<1$.

Proof: For $\lambda>1$, we have

$$\begin{aligned}
&\frac{1}{\left(P_{\lambda_m}-P_m\right)\left(Q_{\lambda_n}-Q_n\right)}\sum_{k=m+1}^{\lambda_m}\sum_{l=n+1}^{\lambda_n}p_kq_l\sigma_{kl}^{(\alpha\beta)}(u)\\
&=\frac{P_{\lambda_m}Q_{\lambda_n}}{\left(P_{\lambda_m}-P_m\right)\left(Q_{\lambda_n}-Q_n\right)}\Big(t_{\lambda_m,\lambda_n}(\sigma^{(\alpha\beta)}(u))-t_{\lambda_m,n}(\sigma^{(\alpha\beta)}(u))\\
&-t_{m,\lambda_n}(\sigma^{(\alpha\beta)}(u))+t_{m,n}(\sigma^{(\alpha\beta)}(u))\Big)\\
&+\frac{P_{\lambda_m}}{P_{\lambda_m}-P_m}\Big(t_{\lambda_m,n}(\sigma^{(\alpha\beta)}(u))-t_{m,n}(\sigma^{(\alpha\beta)}(u))\Big)\\
&+\frac{Q_{\lambda_n}}{Q_{\lambda_n}-Q_n}\Big(t_{m,\lambda_n}(\sigma^{(\alpha\beta)}(u))-t_{m,n}(\sigma^{(\alpha\beta)}(u))\Big)+t_{m,n}(\sigma^{(\alpha\beta)}(u))
\end{aligned} \tag{2.6}$$

Since

$$\frac{P_{\lambda_m}}{P_{\lambda_m}-P_m}=\frac{1}{\inf\limits_{k\geq m}\left(1-\dfrac{P_k}{P_{\lambda_k}}\right)}=\frac{1}{1-\lim\limits_{m\to\infty}\sup\limits_{k\geq m}\left(\dfrac{P_k}{P_{\lambda_k}}\right)}=\frac{1}{1-\limsup\limits_{m\to\infty}\dfrac{P_m}{P_{\lambda_m}}}, \quad (2.7)$$

$$\frac{Q_{\lambda_n}}{Q_{\lambda_n}-Q_n}=\frac{1}{\inf\limits_{k\geq n}\left(1-\dfrac{Q_k}{Q_{\lambda_k}}\right)}=\frac{1}{1-\lim\limits_{n\to\infty}\sup\limits_{k\geq n}\left(\dfrac{Q_k}{Q_{\lambda_k}}\right)}=\frac{1}{1-\limsup\limits_{n\to\infty}\dfrac{Q_n}{Q_{\lambda_n}}} \quad (2.8)$$

and (u_{mn}) is $(\overline{N},p,q;1,1)(C,1,1)$ summable to s, (2.4) follows from (2.1), (2.6), (2.7) and (2.8).

Proof of (2.5) can be proved in a similar way. □

Proof of Theorem 2.1 *Necessity.* Supppose that (1.1) and (1.2) are satisfied. For $\lambda>1$, from Lemma 2.3, we obtain

$$\begin{aligned}
&\lim_{m,n\to\infty}\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{k=m+1}^{\lambda_m}\sum_{l=n+1}^{\lambda_n}p_kq_l(\sigma_{kl}^{(\alpha\beta)}(u)-u_{mn})\\
&=\lim_{m,n\to\infty}\left\{\left(\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{k=m+1}^{\lambda_m}\sum_{l=n+1}^{\lambda_n}p_kq_l\sigma_{kl}^{(\alpha\beta)}(u)\right)-u_{mn}\right\}\\
&=0
\end{aligned}$$

and thus inequality (2.2) is satisfied. The same is true for all $0<\lambda<1$.
Sufficiency. Assume that conditions (2.2) and (2.3) are satisfied. From (2.2) it follows that given any $\varepsilon>0$, there exists $\lambda_1>0$ such that

$$\liminf_{m,n\to\infty}\frac{1}{\left(P_{\lambda_{m_1}}-P_m\right)\left(Q_{\lambda_{n_1}}-Q_n\right)}\sum_{k=m+1}^{\lambda_{m_1}}\sum_{l=n+1}^{\lambda_{n_1}}p_kq_l(\sigma_{kl}^{(\alpha\beta)}(u)-u_{mn})\geq-\varepsilon, \quad (2.9)$$

where $\lambda_{m_1}=[\lambda_1 m]$ and $\lambda_{n_1}=[\lambda_1 n]$ for $m,n=1,2,...$ Taking into account that (u_{mn}) is summable $\left(\overline{N},p,q;1,1\right)(C,\alpha,\beta)$ to s, (2.4) and (2.9), we hence get,

$$\limsup_{m,n\to\infty}u_{mn}\leq s+\varepsilon. \quad (2.10)$$

From (2.3) it follows that given any $\varepsilon>0$ there exists $0<\lambda_2<1$ such that

$$\liminf_{m,n\to\infty}\frac{1}{\left(P_m-P_{\lambda_{m_2}}\right)\left(Q_n-Q_{\lambda_{n_2}}\right)}\sum_{k=\lambda_{m_2}+1}^{m}\sum_{l=\lambda_{n_2}+1}^{n}p_kq_l(u_{mn}-\sigma_{kl}^{(\alpha\beta)}(u))\geq-\varepsilon$$

where $\lambda_{m_2}=[\lambda_2 m]$ and $\lambda_{n_2}=[\lambda_2 n]$ for $m,n=1,2,...$ whence in a similar way, we obtain

$$\liminf_{m,n\to\infty}u_{mn}\geq s-\varepsilon. \quad (2.11)$$

Since $\varepsilon>0$ is arbitrary, combining (2.10) and (2.11) yields $\lim_{m,n\to\infty}u_{mn}=s$.

Theorem 2.4. *Let* (2.1) *be satisfied and* (u_{mn}) *be a sequence of complex numbers, which is* $\left(\overline{N},p,q;1,1\right)(C,\alpha,\beta)$ *summable to* s. *Then* (u_{mn}) *is convergent to* s *if and only if one of the following two conditions is satisfied:*

$$\inf_{\lambda>1}\limsup_{m,n\to\infty}\left|\frac{1}{(P_{\lambda_m}-P_m)(Q_{\lambda_n}-Q_n)}\sum_{k=m+1}^{\lambda_m}\sum_{l=n+1}^{\lambda_n}p_kq_l(\sigma_{kl}^{(\alpha\beta)}(u)-u_{mn})\right|=0 \tag{2.12}$$

$$\inf_{0<\lambda<1}\limsup_{m,n\to\infty}\left|\frac{1}{(P_m-P_{\lambda_m})(Q_n-Q_{\lambda_n})}\sum_{k=\lambda_{m+1}}^{m}\sum_{l=\lambda_{n+1}}^{n}p_kq_l(u_{mn}-\sigma_{kl}^{(\alpha\beta)}(u))\right|=0 \tag{2.13}$$

Proof: *Necessity*. Assume that (1.1) and (1.2) are satisfied. Then by Lemma 2.3, we obtain (2.12) for $\lambda>1$ and (2.13) for $0<\lambda<1$. This part of proof can be done easily by the similar technique as in the proof of Theorem 2.1.
Sufficiency. Suppose that (1.1), (2.1) and (2.12) are satisfied. Then for any given $\varepsilon>0$, there exists $\lambda_3>1$ such that

$$\limsup_{m,n\to\infty}\left|\frac{1}{(P_{\lambda_{m_3}}-P_m)(Q_{\lambda_{n_3}}-Q_n)}\sum_{k=m+1}^{\lambda_{m_3}}\sum_{l=n+1}^{\lambda_{n_3}}p_kq_l(\sigma_{kl}^{(\alpha\beta)}(u)-u_{mn})\right|\le\varepsilon,$$

where $\lambda_{m_3}=[\lambda_3 m]$ and $\lambda_{n_3}=[\lambda_3 n]$ for $m,n=1,2...$ Taking into account the fact that (u_{mn}) is $(\overline{N},p,q;1,1)(C,\alpha,\beta)$ summable to s, by Lemma 2.3 and (1.1), we obtain

$$\begin{aligned}
&\limsup_{m,n\to\infty}|s-u_{mn}|\\
\le\;&\limsup_{m,n\to\infty}\left|s-\frac{1}{(P_{\lambda_{m_3}}-P_m)(Q_{\lambda_{n_3}}-Q_n)}\sum_{k=m+1}^{\lambda_{m_3}}\sum_{l=n+1}^{\lambda_{n_3}}p_kq_l\sigma_{kl}^{(\alpha\beta)}(u)\right|\\
+\;&\limsup_{m,n\to\infty}\left|\frac{1}{(P_{\lambda_{m_3}}-P_m)(Q_{\lambda_{n_3}}-Q_n)}\sum_{k=m+1}^{\lambda_{m_3}}\sum_{l=n+1}^{\lambda_{n_3}}p_kq_l(\sigma_{kl}^{(\alpha\beta)}(u)-u_{mn})\right|\le\varepsilon.
\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, we have $\lim_{m,n\to\infty}u_{mn}=s$.

If we suppose that (1.1), (2.1) and (2.13) are satisfied, then we similarly recover convergence of (u_{mn}) as in the first part of the proof. So, we omit the proof of it. □

3. Tauberian theorems for $(\overline{N},p,q;1,0)(C,\alpha,\beta)$ summability method

The $(\overline{N},p,q;1,0)(C,\alpha,\beta)$ summability method is independent of q. Hence, we write $(\overline{N},p,*;1,0)(C,\alpha,\beta)$ in the place of $(\overline{N},p,q;1,0)(C,\alpha,\beta)$.

Theorem 3.1. *Let*

$$\limsup_{m\to\infty} \frac{P_m}{P_{\lambda_m}} < 1, \quad \lambda > 1. \tag{3.1}$$

Assume that (u_{mn}) is a sequence of real numbers which is $\left(\overline{N}, p, *; 1, 0\right)(C, \alpha, \beta)$ summable to s. Then (u_{mn}) is convergent to s if and only if the following two conditions are satisfied:

$$\sup_{\lambda>1} \liminf_{m,n\to\infty} \frac{1}{P_{\lambda_m} - P_m} \sum_{k=m+1}^{\lambda_m} p_k(\sigma_{kn}^{(\alpha\beta)}(u) - u_{mn}) \geq 0 \tag{3.2}$$

and

$$\sup_{0<\lambda<1} \liminf_{m,n\to\infty} \frac{1}{P_m - P_{\lambda_m}} \sum_{k=\lambda_{m+1}}^{m} p_k(u_{mn} - \sigma_{kn}^{(\alpha\beta)}(u)) \geq 0. \tag{3.3}$$

Lemma 3.2. *Assume that $\limsup_{m\to\infty} \frac{P_m}{P_{\lambda_m}} < 1$ for $\lambda > 1$ is satisfied and let (u_{mn}) be a sequence of complex numbers which is $\left(\overline{N}, p, *; 1, 0\right)(C, \alpha, \beta)$ summable to s. Then*

$$\lim_{m,n\to\infty} \frac{1}{P_{\lambda_m} - P_m} \sum_{k=m+1}^{\lambda_m} p_k \sigma_{kn}^{(\alpha\beta)}(u) = s \tag{3.4}$$

for $\lambda > 1$ and

$$\lim_{m,n\to\infty} \frac{1}{P_m - P_{\lambda_m}} \sum_{k=\lambda_{m+1}}^{m} p_k \sigma_{kn}^{(\alpha\beta)}(u) = s \tag{3.5}$$

for $0 < \lambda < 1$.

We note that we have

$$\begin{aligned} \frac{1}{P_{\lambda_m} - P_m} \sum_{k=m+1}^{\lambda_m} p_k \sigma_{kn}^{(\alpha\beta)}(u) &= t_{m,n}^{(10)}(\sigma^{(\alpha\beta)}(u)) \\ &+ \frac{P_{\lambda_m}}{P_{\lambda_m} - P_m}(t_{\lambda_m,n}^{(10)}(\sigma^{(\alpha\beta)}(u)) - t_{mn}^{(10)}(\sigma^{(\alpha\beta)}(u))) \end{aligned}$$

for all $\lambda > 1$ and

$$\begin{aligned} \frac{1}{P_m - P_{\lambda_m}} \sum_{k=\lambda_m+1}^{m} p_k \sigma_{kn}^{(\alpha\beta)}(u) &= t_{mn}^{(10)}(\sigma^{(\alpha\beta)}(u)) \\ &+ \frac{P_{\lambda_m}}{P_m - P_{\lambda_m}}(t_{mn}^{(10)}(\sigma^{\alpha\beta}(u)) - t_{\lambda_m,n}^{(10)}(\sigma^{(\alpha\beta)}(u))) \end{aligned}$$

for all $0 < \lambda < 1$.
By using above equalities, Lemma 3.2 can be proved as in the proof of the corresponding Lemma for single sequences in [6].
Proof of Theorem 3.1 *Necessity.* Supppose that (1.1) and (1.3) are satisfied. For $\lambda > 1$, from Lemma 3.2, we obtain

$$\lim_{m,n\to\infty} \frac{1}{P_{\lambda_m} - P_m} \sum_{k=m+1}^{\lambda_m} p_k(\sigma_{kn}^{(\alpha\beta)}(u) - u_{mn})$$
$$= \lim_{m,n\to\infty} \left\{ \left(\frac{1}{P_{\lambda_m} - P_m} \sum_{k=m+1}^{\lambda_m} p_k \sigma_{kn}^{(\alpha\beta)}(u) \right) - u_{mn} \right\} = 0$$

and thus inequality (3.2) is satisfied. The same is true for all $0 < \lambda < 1$.
Sufficiency. Assume that conditions (3.2) and (3.3) are satisfied. From (3.2) it follows that given any $\varepsilon > 0$, there exists $\lambda_1 > 0$ such that

$$\liminf_{m,n\to\infty} \frac{1}{P_{\lambda_{m_1}} - P_m} \sum_{k=m+1}^{\lambda_{m_1}} p_k(\sigma_{kn}^{(\alpha\beta)}(u) - u_{mn}) \geq -\varepsilon \tag{3.6}$$

where $\lambda_{m_1} = [\lambda_1 m]$ and for $m = 1, 2, \ldots$. Taking into consideration that (u_{mn}) is summable $\left(\overline{N}, p, *; 1, 0\right)(C, \alpha, \beta)$ to s, (3.2) and (3.6), we hence get,

$$\limsup_{m,n\to\infty} u_{mn} \leq s + \varepsilon. \tag{3.7}$$

From (3.3) it follows that given any $\varepsilon > 0$ there exists $0 < \lambda_2 < 1$ such that

$$\liminf_{m,n\to\infty} \frac{1}{P_m - P_{\lambda_{m_2}}} \sum_{k=\lambda_{m_2}+1}^{m} p_k(u_{mn} - \sigma_{kn}^{(\alpha\beta)}(u)) \geq -\varepsilon$$

whence in a similar way, we obtain

$$\liminf_{m,n\to\infty} u_{mn} \geq s - \varepsilon. \tag{3.8}$$

Since $\varepsilon > 0$ is arbitrary, combining (3.7) and (3.8) yields $\lim_{m,n\to\infty} u_{mn} = s$.

Theorem 3.3. *Let*

$$\limsup_{m\to\infty} \frac{P_m}{P_{\lambda_m}} < 1, \quad \lambda > 1. \tag{3.9}$$

Assume that (u_{mn}) *is a sequence of complex numbers which is* $\left(\overline{N}, p, *; 1, 0\right)(C, \alpha, \beta)$ *summable to* s*. Then* (u_{mn}) *is convergent to* s *if and only if one the following two conditions is satisfied:*

$$\sup_{\lambda>1} \liminf_{m,n\to\infty} \left| \frac{1}{P_{\lambda_m} - P_m} \sum_{k=m+1}^{\lambda_m} p_k(\sigma_{kn}^{(\alpha\beta)}(u) - u_{mn}) \right| = 0 \tag{3.10}$$

$$\sup_{0<\lambda<1} \liminf_{m,n\to\infty} \left| \frac{1}{P_m - P_{\lambda_m}} \sum_{k=\lambda_{m+1}}^{m} p_k(u_{mn} - \sigma_{kn}^{(\alpha\beta)}(u)) \right| = 0. \tag{3.11}$$

Proof: *Necessity*. Assume that (1.2) and (1.3) are satisfied. Then by Lemma 3.2, we obtain condition (3.10) for $\lambda > 1$ and condition (3.11) for $0 < \lambda < 1$. This part of proof can be done easily by the similar technique as in the proof of Theorem 3.1. *Sufficiency*. Suppose that (1.3), (3.9) and (3.10) are satisfied. Then for any given $\varepsilon > 0$, there exists $\lambda_3 > 1$ such that

$$\limsup_{m,n\to\infty} \left| \frac{1}{(P_{\lambda_{m_3}} - P_m)} \sum_{k=m+1}^{\lambda_{m_3}} p_k(\sigma_{kn}^{(\alpha\beta)}(u) - u_{mn}) \right| \le \varepsilon,$$

where $\lambda_{m_3} = [\lambda_3 m]$ for $m = 1, 2...$ Taking into account the fact that (u_{mn}) is $(\overline{N}, p, *; 1, 0)(C, \alpha, \beta)$ summable to s, by Lemma 3.2 and (1.3), we obtain

$$\begin{aligned} \limsup_{m,n\to\infty} |s - u_{mn}| &\le \limsup_{m,n\to\infty} \left| s - \frac{1}{(P_{\lambda_{m_3}} - P_m)} \sum_{k=m+1}^{\lambda_{m_3}} p_k \sigma_{kn}^{(\alpha\beta)}(u) \right| \\ &+ \limsup_{m,n\to\infty} \left| \frac{1}{(P_{\lambda_{m_3}} - P_m)} \sum_{k=m+1}^{\lambda_{m_3}} p_k(\sigma_{kn}^{(\alpha\beta)}(u) - u_{mn}) \right| \le \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\lim_{m,n\to\infty} u_{mn} = s$. □

4. Tauberian theorems for $(\overline{N}, p, q; 0, 1)(C, \alpha, \beta)$ summability method

The $(\overline{N}, p, q; 0, 1)(C, \alpha, \beta)$ summability method is independent of p. Hence, we write $(\overline{N}, *, q; 0, 1)(C, \alpha, \beta)$ in the place of $(\overline{N}, p, q; 0, 1)(C, \alpha, \beta)$.

Theorem 4.1. *Let*

$$\limsup_{n\to\infty} \frac{Q_n}{Q_{\lambda_n}} < 1, \quad \lambda > 1.$$

Assume that (u_{mn}) is a real sequence and $u_{mn} \to s \left(\overline{N}, *, q; 0, 1\right)(C, \alpha, \beta)$. Then (u_{mn}) is convergent to s if and only if the following two conditions are satisfied:

$$\sup_{\lambda>1} \liminf_{m,n\to\infty} \frac{1}{Q_{\lambda_n} - Q_n} \sum_{l=n+1}^{\lambda_n} q_l(\sigma_{ml}^{(\alpha\beta)}(u) - u_{mn}) \ge 0$$

and

$$\sup_{0<\lambda<1} \liminf_{m,n\to\infty} \frac{1}{Q_n - Q_{\lambda_n}} \sum_{l=\lambda_{n+1}}^{n} q_l(u_{mn} - \sigma_{ml}^{(\alpha\beta)}(u)) \ge 0$$

Lemma 4.2. *Assume that* $\limsup_{m\to\infty} \frac{Q_n}{Q_{\lambda_n}} < 1$ *for* $\lambda > 1$ *is satisfied and let* (u_{mn}) *be a sequence of complex numbers which is* $(\overline{N}, *, q; 0, 1)\,(C, \alpha, \beta)$ *summable to* s*. Then*

$$\lim_{m,n\to\infty} \frac{1}{Q_{\lambda_n} - Q_n} \sum_{l=n+1}^{\lambda_n} q_l \sigma_{ml}^{(\alpha\beta)}(u) = s$$

for $\lambda > 1$ *and*

$$\lim_{m,n\to\infty} \frac{1}{Q_n - Q_{\lambda_n}} \sum_{l=\lambda_{n+1}}^{n} q_l \sigma_{ml}^{(\alpha\beta)}(u) = s$$

for $0 < \lambda < 1$.

The proof of Theorem 4.1 can be done by the similar techniques as in the proof of Theorem 3.1 and by using Lemma 4.2.

Theorem 4.3. *Let*

$$\limsup_{m\to\infty} \frac{Q_n}{Q_{\lambda_n}} < 1, \quad \lambda > 1.$$

Assume that (u_{mn}) *is a sequence of complex numbers which is and* $(\overline{N}, *, q; 0, 1)\,(C, \alpha, \beta)$ *summable to* s*. Then* (u_{mn}) *is convergent to* s *if and only if one the following two conditions is satisfied:*

$$\sup_{\lambda>1} \liminf_{m,n\to\infty} \left| \frac{1}{Q_{\lambda_n} - Q_n} \sum_{l=n+1}^{\lambda_n} q_l(\sigma_{ml}^{(\alpha\beta)}(u) - u_{mn}) \right| = 0$$

and

$$\sup_{0<\lambda<1} \liminf_{m,n\to\infty} \left| \frac{1}{Q_n - Q_{\lambda_n}} \sum_{l=\lambda_{n+1}}^{n} q_l(u_{mn} - \sigma_{ml}^{(\alpha\beta)}(u)) \right| = 0.$$

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