Bol. Soc. Paran. Mat. (3s.) **v. 38** 6 (2020): 99–126.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v38i6.36594

Existence of Solutions for Some Strongly Nonlinear Parabolic Problems Involving Lower Order Terms in Divergence Form in Musielak-Orlicz Spaces

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Abstract: In this paper, we study an existence of solutions for a class of nonlinear parabolic problems with two lower order terms and L^1-data in the context of Musielak-Orlicz spaces.

Key Words: Parabolic problems, Inhomogeneous Musielak-Orlicz-Sobolev space, Lower order term, Truncation.

Contents

1. Introduction:

Let Ω be a bounded open set of $\mathbb{R}^N$, T is a positive real number, and $Q = \Omega \times [0, T]$. We deal with boundary value problem:

$$(\mathcal{P}) \begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\big(a(x,t,u,\nabla u) + \Phi(x,t,u)\big) + g(x,t,u,\nabla u) = f \quad \text{in } Q, \\ u = 0 \quad \text{in } \partial\Omega \times (0,T). \\ u(x,0) = u_0 \quad \text{in} \quad \Omega, \end{cases}$$

where $A(u) = -\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray-Lions Operator defined on $D(A) \subset W_0^{1,x}L_\varphi(Q) \longrightarrow W^{-1,x}L_\psi(Q)$ where φ and ψ are two complementary Musielak-Orlicz functions. The lower order term $\Phi : \Omega \times (0,T) \times \mathbb{R} \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfies the following growth condition for a.e. $(x,t) \in Q$ and for all $s \in \mathbb{R}$,

$$|\Phi(x,t,s)| \leq P(x,t)\, \overline{\gamma_x}^{-1} \gamma_x(|s|).$$

2010 *Mathematics Subject Classification:* 46E35, 35K15, 35K20, 35K60.
Submitted April 06, 2017. Published November 10, 2017

where $P(x,t) \in L^{\infty}(Q)$ and γ is a Musielak-Orlicz function such that $\gamma \prec\prec \varphi$ means that γ grows essentially less rapidly than φ (see Preliminaries). g is a nonlinearity with sign condition and satisfying, for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ and almost all $(x,t) \in Q$ the following natural growth condition:

$$|g(x,t,s,\xi)| \leq b(|s|)(c_2(x,t) + \varphi(x,|\xi|)),$$

where $c_2(x,t) \in L^1(Q)$ and $b : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a continuous and nondecreasing function. The right-hand side f is assumed to belongs to $L^1(Q)$.

On Orlicz-Sobolev spaces, Elmahi had studied in [6] the problem $(\mathcal{P})$ for $\Phi \equiv 0$, without assuming any restriction on the N-function M. In the case where $u \equiv b(x,u)$ and $g \equiv 0$, the existence of solution has been proved in [10] by Hadj Nassar, Moussa and Rhoudaf.

In the framework of variable exponent Sobolev spaces, Azroul, Benboubker, Redwane and Yazough in [2] have proved the existence result of solutions for the problem $(\mathcal{P})$ without sign condition involving nonstandard growth and where $u = b(u)$ and $\Phi \equiv 0$. Fu and Pan have treated in [8] the existence of solutions for the problem $(\mathcal{P})$ where $\Phi \equiv g \equiv 0$ and the second membe f is in $W^{-1,x}L^{p'(x)}(Q)$.

In the setting of Musielak spaces and in variational case, the existence of a weak solution for the problem $(\mathcal{P})$ has been proved by M. L. Ahmed Oubeid, A. Benkirane and M. Sidi El Vally in [1] where $\Phi \equiv 0$, the existence of solutions for the problem $(\mathcal{P})$ has been studied by A. Talha, A. Benkirane, and M.S.B. Elemine vall in [16] when $\Phi \equiv 0$ and the right hand side is a measure data. A large number of papers was devoted to the study the existence solutions of elliptic and parabolic problems under various assumptions and in different contexts for a review on classical results see [1,3,4,11,13,15].

Our main goal in this paper is to study the problem $(\mathcal{P})$ in the context of Musielak-Orlicz spaces without assuming the $\Delta_2$condition, neither on the Musielak function φ nor on its complementary ψ. The main difficulty in our study is due to the fact that the second member is in L^1 and the fact that no hypothesis of coercivity is assumed on Φ. Our result generalizes that of Elmahi and Meskine [7] and that of Ahmed Oubeid, Benkirane, and Sidi El Vally [1].

This research is divided into several parts. In Section 2 we recall some well-know preliminaries, properties and results of Musielak-Orlicz-Sobolev Spaces. Section 3 is devoted to specify the assumptions on a, Φ, g, f and u_0. Section 4 is devoted to some technical lemmas where be used to our results. Final section 5 consecrate to prove the existence of solution of $(\mathcal{P})$.

2. Preliminaries

2.1. Musielak-Orlicz functions

Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions:

(a) $\varphi(x,.)$ is an N-function for all $\in \Omega$ (i.e. convex, strictly increasing, continuous,

$\varphi(x,0)=0$, $\varphi(x,t)>0$, for all $t>0$,

$$\lim_{t\to 0}\sup_{x\in\Omega}\frac{\varphi(x,t)}{t}=0$$

and

$$\lim_{t\to\infty}\inf_{x\in\Omega}\frac{\varphi(x,t)}{t}=\infty).$$

(b) $\varphi(.\,,t)$ is a measurable function.

The function φ is called a Musielak-Orlicz function.

For a Musielak-orlicz function φ we put $\varphi_x(t)=\varphi(x,t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t, that is

$$\varphi_x^{-1}(\varphi(x,t))=\varphi(x,\varphi_x^{-1}(t))=t.$$

The Musielak-orlicz function φ is said to satisfy the Δ_2-condition if for some $k>0$, and a non negative function h, integrable in Ω, we have

$$\varphi(x,2t)\leq k\,\varphi(x,t)+h(x) \text{ for all } x\in\Omega \text{ and } t\geq 0. \tag{2.1}$$

When (2.1) holds only for $t\geq t_0>0$, then φ is said to satisfy the Δ_2-condition near infinity.

Let φ and γ be two Musielak-orlicz functions, we say that φ dominate γ and we write $\gamma\prec\varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x\in\Omega$

$$\gamma(x,t)\leq\varphi(x,ct) \text{ for all } t\geq t_0, \quad (\text{ resp. for all } t\geq 0 \text{ i.e. } t_0=0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma\prec\prec\varphi$ if for every positive constant c we have

$$\lim_{t\longrightarrow 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0, \quad (\text{resp. } \lim_{t\longrightarrow\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0).$$

Remark 2.1. *[4] If $\gamma\prec\prec\varphi$ near infinity, then $\forall\varepsilon>0$ there exist $k(\varepsilon)>0$ such that for almost all $x\in\Omega$ we have*

$$\gamma(x,t)\leq k(\varepsilon)\varphi(x,\varepsilon t), \quad \textit{for all } t\geq 0. \tag{2.2}$$

2.2. Musielak-Orlicz spaces

For a Musielak-Orlicz function φ and a measurable function $u:\Omega\longrightarrow\mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u)=\int_\Omega\varphi(x,|u(x)|)\,dx.$$

The set $K_\varphi(\Omega)=\Big\{u:\Omega\longrightarrow\mathbb{R}$ measurable $/\ \rho_{\varphi,\Omega}(u)<\infty\Big\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \big\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable } / \rho_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) < \infty, \text{ for some } \lambda > 0\big\}.$$

For a Musielak-Orlicz function φ we put: $\psi(x, s) = \sup_{t\geq 0}\{st - \varphi(x, t)\}$, ψ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sens of Young with respect to the variable s.

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\big\{\lambda > 0 / \int_\Omega \varphi\Big(x, \frac{|u(x)|}{\lambda}\Big)\, dx \leq 1\big\}.$$

which is called the Luxemburg norm and the so–called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|\, dx,$$

where ψ is the Musielak Orlicz function complementary to φ. These two norms are equivalent [12].

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$.

A Musielak function φ is called locally integrable on Ω if $\rho_\varphi(t\chi_D) < \infty$ for all $t > 0$ and all measurable $D \subset \Omega$ with meas$(D) < \infty$.
Let φ a Musielak function which is locally integrable. Then $E_\varphi(\Omega)$ is separable ([12] , Theorem 7.10.)

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \big\{u \in L_\varphi(\Omega) : \forall|\alpha| \leq m,\ D^\alpha u \in L_\varphi(\Omega)\big\}.$$

and

$$W^m E_\varphi(\Omega) = \big\{u \in E_\varphi(\Omega) : \forall|\alpha| \leq m,\ D^\alpha u \in E_\varphi(\Omega)\big\}.$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.

Let

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}\Big(D^\alpha u\Big) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\Big\{\lambda > 0 : \overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) \leq 1\Big\}$$

for $u \in W^m L_\varphi(\Omega)$. These functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\left(W^m L_\varphi(\Omega), \|\|^m_{\varphi,\Omega}\right)$ is a Banach space if φ satisfies the following condition [12]:

$$\text{there exist a constant } c_0 > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c_0. \tag{2.3}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.

The space $W_0^m L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. and the space $W_0^m E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W_0^m L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \big\{ f \in D'(\Omega);\ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \big\}.$$

and

$$W^{-m} E_\psi(\Omega) = \big\{ f \in D'(\Omega);\ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \big\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\Big(\frac{u_n - u}{k}\Big) = 0.$$

For φ and her complementary function ψ, the following inequality is called the Young inequality [12]:

$$ts \leq \varphi(x,t) + \psi(x,s), \quad \forall t, s \geq 0, x \in \Omega. \tag{2.4}$$

This inequality implies that

$$\||u|\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1. \tag{2.5}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} > 1. \tag{2.6}$$

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \text{ if } \|u\|_{\varphi,\Omega} \leq 1. \tag{2.7}$$

For two complementary Musielak Orlicz functions φ and ψ, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, then we have the Hölder inequality [12]:

$$\left| \int_\Omega u(x)v(x)\, dx \right| \leq \|u\|_{\varphi,\Omega} \||v|\|_{\psi,\Omega}. \tag{2.8}$$

2.3. Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let Ω a bounded open subset of $\mathbb{R}^N$ and let $Q = \Omega \times]0,T[$ with some given $T > 0$. Let φ and ψ be two complementary Musielak-Orlicz functions. For each $\alpha \in \mathbb{N}^N$ denote by D_x^α the distributional derivative on Q of order α with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows.

$$W^{1,x}L_\varphi(Q) = \{u \in L_\varphi(Q) : \forall |\alpha| \leq 1 \; D_x^\alpha u \in L_\varphi(Q)\}$$

et

$$W^{1,x}E_\varphi(Q) = \{u \in E_\varphi(Q) : \forall |\alpha| \leq 1 \; D_x^\alpha u \in E_\varphi(Q)\}.$$

This second space is a subspace of the first one, and both are Banach spaces under the norm

$$\|u\| = \sum_{|\alpha| \leq 1} \|D_x^\alpha u\|_{\varphi,Q}.$$

These spaces constitute a complementary system since Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$ which has $(N+1)$ copies.

We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$.

If $u \in W^{1,x}L_\varphi(Q)$ then the function $t \to u(t) = u(\cdot,t)$ is defined on $[0,T]$ with values in $W^1L_\varphi(\Omega)$. If $u \in W^{1,x}E_\varphi(Q)$, then $u \in W^1E_\varphi(\Omega)$ and it is strongly measurable. Furthermore, the imbedding $W^{1,x}E_\varphi(Q) \subset L^1(0,T,W^1E_\varphi(\Omega))$ holds. The space $W^{1,x}L_\varphi(Q)$ is not in general separable, for $u \in W^{1,x}L_\varphi(Q)$ we cannot conclude that the function $u(t)$ is measurable on $[0,T]$.

However, the scalar function $t \to \|u(t)\|_{\varphi,\Omega}$ is in $L^1(0,T)$. The space $W_0^{1,x}E_\varphi(Q)$ is defined as the norm closure of $\mathcal{D}(Q)$ in $W^{1,x}E_\varphi(Q)$. We can easily show as in [9] that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q)$ with respect of the weak* topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is a limit in $W^{1,x}L_\varphi(Q)$ of some subsequence $(v_j) \in \mathcal{D}(Q)$ for the modular convergence, i.e. there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$,

$$\int_Q \varphi(x, (\frac{D_x^\alpha v_j - D_x^\alpha u}{\lambda}))\, dx\, dt \to 0 \text{ as } j \to \infty,$$

this implies that (v_j) converges to u in $W^{1,x}L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_\varphi, \Pi L_\psi)$.

Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi E_\psi)} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi L_\psi)},$$

The space of functions satisfying such a property will be denoted by $W_0^{1,x}L_\psi(Q)$. Furthermore, $W_0^{1,x}E_\varphi(Q) = W_0^{1,x}L_\varphi(Q) \cap \Pi E_\varphi(Q)$. Thus, both sides of the last inequality are equivalent norms on $W_0^{1,x}L_\varphi(Q)$. We then have the following complementary system:

$$\begin{pmatrix} W_0^{1,x}L_\varphi(Q) & F \\ W_0^{1,x}E_\varphi(Q) & F_0 \end{pmatrix},$$

where F states for the dual space of $W_0^{1,x}E_\varphi(Q)$. and can be defined, except for an isomorphism, as the quotient of ΠL_ψ by the polar set $W_0^{1,x}E_\varphi(Q)^\perp$. It will be denoted by $F = W^{-1,x}L_\psi(Q)$, where

$$W^{-1,x}L_\psi(Q) = \big\{f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in L_\psi(Q)\big\}.$$

This space will be equipped with the usual quotient norm

$$\|f\| = \inf \sum_{|\alpha|\le 1} \|f_\alpha\|_{\psi,Q},$$

where the infimum is taken over all possible decompositions

$$f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha, \quad f_\alpha \in L_\psi(Q).$$

The space F_0 is then given by

$$F_0 = \Big\{f = \sum_{|\alpha|\le 1} D_x^\alpha f_\alpha : f_\alpha \in E_\psi(Q)\Big\},$$

and is denoted by $F_0 = W^{-1,x}E_\psi(Q)$, see [1].

3. Essential Assumptions

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and $T > 0$, we denote $Q = \Omega \times [0,T]$, and let φ and γ be two Musielak-Orlicz functions such that φ is locally integrable and $\gamma \prec\prec \varphi$.

Let $A : D(A) \subset W_0^{1,x}L_\varphi(Q) \longrightarrow W^{-1,x}L_\psi(Q)$ be a mapping given by

$$A(u) = -\mathrm{div}(a(x,t,u,\nabla u)),$$

where $a : a(x,t,s,\xi) : \Omega \times [0,t] \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory function satisfying,
for a.e $(x,t) \in Q$ and for all $s \in \mathbb{R}$ and all $\xi, \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$:

$$|a(x,t,s,\xi)| \le \beta\Big(c(x,t) + \psi_x^{-1}\gamma(x,\nu|s|) + \psi_x^{-1}\varphi(x,\nu|s|)\Big), \tag{3.1}$$

$$\Big(a(x,t,s,\xi) - a(x,t,s,\xi')\Big)(\xi - \xi') > 0, \tag{3.2}$$

$$a(x,t,s,\xi).\xi \ge \alpha\varphi(x,|\xi|). \tag{3.3}$$

where $c(x,t)$ a positive function, $c(x,t) \in E_\psi(Q)$ and positive constants ν, β, α.

let $g : \Omega \times [0,t] \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a Caratheodory function satisfying for a.e. $(x,t) \in \Omega \times [0,t]$ and $\forall s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$:

$$|g(x,t,s,\xi)| \le b(|s|)(c_2(x,t) + \varphi(x,|\xi|)), \tag{3.4}$$

$$g(x,t,s,\xi)s \geq 0, \tag{3.5}$$

where $c_2(x,t) \in L^1(Q)$ and $b : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is a continuous and nondecreasing function.

Furthermore the function Φ is a Carathéodory function which satisfies the following growth condition for a.e. $(x,t) \in Q$ and for all $\forall s \in \mathbb{R}$,

$$|\Phi(x,t,s)| \leq P(x,t)\,\overline{\gamma_x}^{-1}\gamma_x(|s|). \tag{3.6}$$

where $P(x,t) \in L^\infty(Q)$.

$$f \text{ is an element of } L^1(Q), \tag{3.7}$$

$$u_0 \text{ is an element of } L^1(\Omega). \tag{3.8}$$

Let us give the following lemma which will be needed later.

4. Some technical Lemmas

Lemma 4.1. *[3]. Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ and let φ and ψ be two complementary Musielak-Orlicz functions which satisfy the following conditions:*
i) *There exist a constant $c > 0$ such that $\inf_{x\in\Omega}\varphi(x,1) \geq c$,*
ii) *There exist a constant $A > 0$ such that for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$ we have*

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{log(\frac{1}{|x-y|})}\right)}, \quad \forall t \geq 1. \tag{4.1}$$

iii)

$$\textit{If } D \subset \Omega \textit{ is a bounded measurable set, then } \int_D \varphi(x,1)dx < \infty. \tag{4.2}$$

iv) *There exist a constant $C > 0$ such that $\psi(x,1) \leq C$ a.e in Ω.*

Under this assumptions, $\mathcal{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ with respect to the modular topology, $\mathcal{D}(\Omega)$ is dense in $W_0^1L_\varphi(\Omega)$ for the modular convergence and $\mathcal{D}(\overline{\Omega})$ is dense in $W^1L_\varphi(\Omega)$ the modular convergence.

Consequently, the action of a distribution S in $W^{-1}L_\psi(\Omega)$ on an element u of $W_0^1L_\varphi(\Omega)$ is well defined. It will be denoted by $< S, u >$.

Lemma 4.2. *[13]. Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak- Orlicz function and let $u \in W_0^1L_\varphi(\Omega)$. Then $F(u) \in W_0^1L_\varphi(\Omega)$. Moreover, if the set D of discontinuity points of F' is finite, we have*

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e \text{ in } \{x \in \Omega : u(x) \in D\}. \\ 0 & a.e \text{ in } \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 4.3 (Poincaré inequality). *[15].Let φ a Musielak Orlicz function which satisfies the assumptions of lemma 4.1, suppose that $\varphi(x,t)$ decreases with respect of one of coordinate of x.*
Then, that exists a constant $c>0$ depends only of Ω such that

$$\int_\Omega \varphi(x,|u(x)|)dx \leq \int_\Omega \varphi(x,c|\nabla u(x)|)dx, \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{4.3}$$

Lemma 4.4. *[3]. Suppose that Ω satisfies the segment property and let $u \in W_0^1 L_\varphi(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1 L_\varphi(\Omega).$$

Furthermore, if $u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $||u_n||_\infty \leq (N+1)||u||_\infty$.

Lemma 4.5 (The Nemytskii Operator). *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak Orlicz functions. Let $f : \Omega \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a Carathodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|f(x,s)| \leq c(x) + k_1 \psi_x^{-1} \varphi(x, k_2|s|). \tag{4.4}$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$.
Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x,u(x))$ is continuous from

$$\mathcal{P}\Big(E_\varphi(\Omega), \frac{1}{k_2}\Big)^p = \prod \Big\{u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2}\Big\}.$$

into $(L_\psi(\Omega))^q$ for the modular convergence.
Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\mathcal{P}\Big(E_\varphi(\Omega), \frac{1}{k_2}\Big)^p$ to $(E_\gamma(\Omega))^q$

Theorem 4.6. *[1] Let φ be an Musielak-Orlicz function satisfies the assumption (4.1). If $u \in W^{1,x} L_\varphi(Q) \cap L^2(Q)$ (respectively $u \in W_0^{1,x} L_\varphi(Q) \cap L^2(Q)$) and $\frac{\partial u}{\partial t} \in W^{-1,x} L_\psi(Q) + L^2(Q)$, then there exists a sequence $(v_j) \in D(\overline{Q})$ (respectively $D(\overline{I}, D(\Omega))$) such that $v_j \longrightarrow u$ in $W^{1,x} L_\varphi(Q) \cap L^2(Q)$ and $\frac{\partial v_j}{\partial t} \longrightarrow \frac{\partial u}{\partial t}$ in $W^{-1,x} L_\psi(Q) + L^2(Q)$ for the modular convergence.*

Lemma 4.7. *[1] Let $a < b \in \mathbb{R}$ and let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then*

$$\big\{u \in W_0^{1,x} L_\varphi(\Omega\times]a,b[) \ : \ \frac{\partial u}{\partial t} \in W^{-1,x} L_\psi(\Omega\times]a,b[) + L^1(\Omega\times]a,b[)\big\}$$

is a subset of $\mathcal{C}(]a,b[, L^1(\Omega))$.

Lemma 4.8. *Let φ be a Musielak function. Let $(u_n)_n$ be a sequence of $W^{1,x} L_\varphi(Q)$ such that*

$$u_n \rightharpoonup u \text{ weakly in } W^{1,x} L_\varphi(Q) \text{ for } \sigma(\Pi L_\varphi, \Pi L_\psi)$$

and

$$\frac{\partial u_n}{\partial t} = h_n + k_n \text{ in } \mathcal{D}'(Q)$$

with $(h_n)_n$ *is bounded in* $W^{-1,x}L_\psi(Q)$ *and* $(k_n)_n$ *bounded in the space* $L^1(Q)$*. Then* $u_n \longrightarrow u$ *strongly in* $L^1_{loc}(Q)$*.*
If further $u_n \in W_0^{1,x}L_\varphi(Q)$ *then* $u_n \longrightarrow u$ *strongly in* $L^1(Q)$*.*

Proof: It is easily adapted from that given in [5] by using Theorem 4.4 and Remark 4.3 instead of Lemma 8 of [14]. □

5. Main results

For $k > 0$ we define the truncation at height k: $T_k : \mathbb{R} \longrightarrow \mathbb{R}$ by:

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k. \\ k\frac{s}{|s|} & \text{if } |s| > k. \end{cases} \tag{5.1}$$

We note also

$$S_k(r) = \int_0^r T_k(\sigma)d\sigma = \begin{cases} \frac{r^2}{2} & \text{if } |r| \leq k, \\ k|r| - \frac{r^2}{2} & \text{if } |r| > k. \end{cases} \tag{5.2}$$

We define

$$T_0^{1,\varphi}(Q) = \Big\{ u : \Omega \longrightarrow \mathbb{R} \ \text{ measurable such that } T_k(u) \in W_0^{1,x}L_\varphi(Q)\ \forall k > 0 \Big\}.$$

We consider the following boundary value problem:

$$(\mathcal{P}) \begin{cases} \frac{\partial u}{\partial t} + \operatorname{div}\big(a(x,t,u,\nabla u) + \Phi(x,t,u)\big) + g(x,t,u,\nabla u) = f & \text{in } Q, \\ u \equiv 0 \quad \text{on } \partial Q = \partial\Omega \times [0,T], \\ u(\cdot,0) = u_0 \text{ in } \Omega. \end{cases}$$

Our goal now is to show the following existence theorem.

Theorem 5.1. *Let* Ω *be a bounded Lipschitz domain in* $\mathbb{R}^N$*,* φ *and* ψ *be two complementary Musielak-Orlicz functions satisfying the assumptions of Lemma 4.1 and* $\varphi(x,t)$ *decreases with respect to one of coordinate of* x*, we assume also that (3.1)–(3.7) are fulfilled, then there exists at least one solution of* $(\mathcal{P})$ *in the following* *sense*

$$\begin{cases} u \in T_0^{1,\varphi}(Q),\ S_k(u) \in L^1(Q),\ g(.,.u,\nabla u) \in L^1(Q) \\ \displaystyle\int_\Omega S_k(u(T)-v(T))dx + \Big\langle \frac{\partial v}{\partial t}, T_k(u-v)\Big\rangle + \int_Q a(x,t,u,\nabla u)\cdot \nabla T_k(u-v)\,dx\,dt \\ \displaystyle +\int_Q \Phi(x,t,u)\cdot\nabla T_k(u-v)\,dx\,dt + \int_Q g(x,t,u,\nabla u)T_k(u-v)\,dx\,dt \\ \displaystyle \le \int_Q fT_k(u-v)\,dx\,dt + \int_\Omega S_k(u_0-v(0))dx \\ \text{and} \\ u(x,0)=u_0(x) \text{ for a.e. } \quad x\in\Omega, \\ \forall v \in W_0^{1,x}L_\varphi(Q)\cap L^\infty(Q) \text{ such that } \frac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q)+L^1(Q). \end{cases} \tag{5.3}$$

The following remarks are concerned with a few comments on Theorem 5.1.

Remark 5.2. *Equation (5.3) is formally obtained through pointwise multiplication of the problem ($\mathcal{P}$) by $T_k(u-v)$. Note that each term in (5.3) has a meaning since $T_k(u-v) \in W_0^{1,x}L_\varphi(Q)\cap L^\infty(Q)$. In addition by Lemma 4.7, we have $v \in C([0,T];L^1(\Omega))$ and then the first and last terms of Eq. (5.3) are well defined.*

Proof: The proof of Theorem 5.1 is done in 6 steps.

Step 1: Approximate problem.

Let us introduce the following regularization of the data:

$$a(x,t,r,\xi) = a(x,t,T_n(r),\xi) \text{ a.e } (x,t)\in Q, \quad \forall r\in\mathbb{R}, \quad \forall\xi\in\mathbb{R}^N, \tag{5.4}$$

$$g_n(x,t,r,\xi) = g(x,t,T_n(r),\xi) \text{ a.e } (x,t)\in Q, \quad \forall r\in\mathbb{R}, \quad \forall\xi\in\mathbb{R}^N, \tag{5.5}$$

$$\Phi_n(x,t,r) = \Phi(x,t,T_n(r)) \quad \text{ a.e } (x,t)\in Q, \quad \forall r\in\mathbb{R}, \tag{5.6}$$

$$f_n \in C_0^\infty(Q) \quad : \|f_n\|_{L^1} \le \|f\|_{L^1} \text{ and } f_n \longrightarrow f \text{ in } L^1(Q) \text{ as n tends to } +\infty, \tag{5.7}$$

$$u_{0n} \in C_0^\infty(\Omega) \quad : \|u_{0n}\|_{L^1} \le \|u\|_{L^1} \text{ and } u_{0n} \longrightarrow u_0 \text{ in } L^1(\Omega) \text{ as n tends to } +\infty. \tag{5.8}$$

Let us now consider the following regularized problem:

$$(\mathcal{P}_n)\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}\Big(a(x,t,u_n,\nabla u_n) + \Phi_n(x,t,u_n)\Big) + g_n(x,t,u_n,\nabla u_n) = f_n \quad \text{in } Q, \\ u_n = 0 \quad \text{on} \quad \partial\Omega\times(0,T), \\ u_n(x,t=0) = u_{0n} \quad \text{in } \Omega. \end{cases}$$

Since g_n is bounded for any fixed n, as a consequence, proving of a weak solution $u_n \in W_0^{1,x}L_\varphi(Q)$ of ($\mathcal{P}_n$) is an easy task (see e.g. [1,11])

Step 2: A priori estimates.

The estimates derived in this step rely on usual techniques for problems of the type $(\mathcal{P}_n)$.

We take $T_k(u_n)\chi_{(0,\tau)}$ as test function in $(\mathcal{P}_n)$, we get for every $\tau \in (0,T)$

$$\begin{aligned} &\langle \frac{\partial u_n}{\partial t}, T_k(u_n)\chi_{(0,\tau)}\rangle + \int_{Q_\tau} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot \nabla T_k(u_n)\,dx\,dt \\ &+ \int_{Q_\tau} \Phi_n(x,t,u_n)\cdot \nabla T_k(u_n)\,dx\,dt + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_k(u_n)\,dx\,dt \\ &= \int_{Q_\tau} f_n T_k(u_n)\,dx\,dt \end{aligned} \tag{5.9}$$

which implies that

$$\begin{aligned} &\int_\Omega S_k(u_n)(\tau)dx + \int_{Q_\tau} a(x,t,T_k(u_n),\nabla T_k(u_n))\cdot \nabla T_k(u_n)\,dx\,dt \\ &\quad + \int_{Q_\tau} \Phi_n(x,t,u_n)\cdot \nabla T_k(u_n)\,dx\,dt + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_k(u_n)\,dx\,dt \\ &\quad = \int_{Q_\tau} f_n T_k(u_n)\,dx\,dt + \int_\Omega S_k(u_{0n})dx \end{aligned} \tag{5.10}$$

While $\gamma \prec\prec \varphi$, we have, for all $\varepsilon > 0$ there exists a constant $d_\varepsilon > 0$ depending on $\varepsilon > 0$ such that for almost all $x \in \Omega$

$$\gamma(x,t) \le \varphi(x,\varepsilon t) + d_\varepsilon, \qquad \text{for all } t \ge 0. \tag{5.11}$$

Without loss of generality, we can assume that $\varepsilon = \frac{\alpha}{(\alpha+C_p)(\lambda+1)}$, (with α is the constant of (3.3)).

Using (3.6) we get

$$\int_{Q_\tau} \Phi_n(x,t,u_n)\nabla T_k(u_n)dxdt \le \int_{Q_\tau} P(x,t)\,\overline{\gamma_x}^{-1}\gamma_x(|T_k(u_n)|)\,\nabla T_k(u_n)dxdt. \tag{5.12}$$

Recall that $\gamma \prec\prec \varphi \Longleftrightarrow \overline{\varphi} = \psi \prec\prec \overline{\gamma}$ then, with Young inequality and bearing in mind that $P \in L^\infty(Q_\tau)$, we obtain

$$\begin{aligned} \int_{Q_\tau} \Phi_n(x,t,u_n)\nabla T_k(u_n)\,dx\,dt \le C_p \int_{Q_\tau} \varphi\Big(x, \frac{\varepsilon\lambda|T_k(u_n)|}{\lambda}\Big) + 2d_\varepsilon \text{meas}(Q_\tau) \\ +\ \varepsilon\, C_p \int_{Q_\tau} \varphi(x,|\nabla T_k(u_n)|)\,dx\,dt, \end{aligned} \tag{5.13}$$

by Lemma 4.3 and the convexity of φ with $\lambda\varepsilon \le 1$, we get

$$\begin{aligned} \int_{Q_\tau} \Phi_n(x,t,u_n)\nabla T_k(u_n)\,dx\,dt \le (\varepsilon\, C_p + \varepsilon\lambda\, C_p)\int_{Q_\tau} \varphi(x,|\nabla T_k(u_n)|)\,dx\,dt \\ + 2d_\varepsilon \text{meas}(Q_\tau). \end{aligned} \tag{5.14}$$

By using (3.5), (5.7), (5.8), (5.14), and the fact that $S_k(u_n)(\tau) \geq 0$, $S_k(u_{0n}) \leq k|u_{0n}|$, permit to deduce from (5.10) that

$$\begin{aligned}\int_{Q_\tau} a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)\,dx\,dt \leq (\varepsilon\, C_p + \varepsilon\lambda\, C_p)\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx\,dt \\ + 2d_\varepsilon \text{meas}(Q_\tau) \\ + k\Big(||f||_{L^1(Q_\tau)} + ||u_0||_{L^1(Q_\tau)}\Big),\end{aligned} \tag{5.15}$$

by (3.3) and since $\Big(\alpha - \varepsilon\, C_p(1+\lambda)\Big) > 0$, then

$$\int_{Q_\tau} \varphi(x,|\nabla T_k(u_n)|)\,dx\,dt \leq \frac{1}{\alpha}\int_\Omega a(x,u_n,\nabla u_n)\nabla T_k(u_n)\,dx\,dt \leq kC_1. \tag{5.16}$$

where C_1 is a constant independently of n,

Using Lemma 4.3, one has

$$\int_{Q_\tau} \varphi(x,\frac{|T_k(u_n)|}{\lambda})\,dx\,dt \leq kC_1. \tag{5.17}$$

Then we deduce by using (5.17), that

$$\begin{aligned} meas\{|u_n| > k\} &\leq \frac{1}{\inf\limits_{x\in\Omega}\varphi(x,\frac{k}{\lambda})}\int_{\{|u_n|>k\}} \varphi(x,\frac{k}{\lambda})\,dx\,dt \\ &\leq \frac{1}{\inf\limits_{x\in\Omega}\varphi(x,\frac{k}{\lambda})}\int_{Q_\tau} \varphi(x,\frac{1}{\lambda}|T_k(u_n)|)\,dx\,dt \\ &\leq \frac{C_1 k}{\inf\limits_{x\in\Omega}\varphi(x,\frac{k}{\lambda})} \quad \forall n, \quad \forall k \geq 0. \end{aligned} \tag{5.18}$$

For every $\lambda > 0$ we have

$$\begin{aligned} meas\{|u_n - u_m| > \lambda\} \;&\leq\; meas\{|u_n| > k\} \\ &+\; meas\{|u_m| > k\} \\ &+\; meas\{|T_k(u_n) - T_k(u_m)| > \lambda\}. \end{aligned} \tag{5.19}$$

Consequently, by (5.17) we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in Q.

Let $\varepsilon > 0$, then by (5.19) there exists some $k = k(\varepsilon) > 0$ such that

$$meas\{|u_n - u_m| > \lambda\} < \varepsilon, \qquad \text{for all } n,m \geq h_0(k(\varepsilon),\lambda).$$

Which means that $(u_n)_n$ is a Cauchy sequence in measure in Q, thus converge almost every where to some measurable functions u.

We have from (5.17) that $T_k(u_n)$ is bounded in $W_0^{1,x}L_\varphi(Q)$ for every $k>0$. Consider now a $C^2(\mathbb{R})$ nondecreasing function $\zeta_k(s)=s$ for $|s|\le\frac{k}{2}$ and $\zeta_k(s)=k$ sign (s).

Multiplying the approximating equation by $\zeta_k'(u_n)$, we obtain

$$\begin{aligned}\frac{\partial(\zeta_k(u_n))}{\partial t} &= \text{div}\left(a(x,t,u_n,\nabla u_n)\zeta_k'(u_n)\right)-\zeta_k''(u_n)a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\\ &+\text{div}\left(\Phi_n(x,t,u_n)\zeta_k'(u_n)\right)-\zeta_k''(u_n)\Phi_n(x,t,u_n)\cdot\nabla u_n-g_n(x,t,u_n,\nabla u_n)\zeta_k'(u_n)\\ &+f_n\zeta_k'(u_n),\end{aligned}\tag{5.20}$$

Due to (3.1), (3.4), (5.4), (5.5) and the fact that

$$T_k(u_n)\text{ is bounded in}W_0^{1,x}L_\varphi(Q),$$

and

$$\text{div}\left(a(x,t,u_n,\nabla u_n)\zeta_k'(u_n)\right)-\zeta_k''(u_n)a(x,t,u_n,\nabla u_n)\cdot\nabla u_n-g_n(x,t,u_n,\nabla u_n)\zeta_k'(u_n)+f_n\zeta_k'(u_n),$$

is bounded in $L^1(Q)+W_0^{-1,x}L_\psi(Q)$, so $\zeta_n(u_n)$ is bounded in $L^1(Q)+W_0^{1,x}L_\varphi(Q)$.

Moreover since $supp(\zeta_k')$ and $supp(\zeta_k'')$ are both included in $[-k,k]$ by (3.6) and (5.6) if follows that,

$|\int_Q\zeta_k'(u_n)\Phi_n(x,t,u_n)\,dx\,dt|\le||\zeta_k'||_{L^\infty}\int_Q P(x,t)\,\overline{\gamma_x}^{-1}\gamma_x(|T_k(u_n)|)\,dx\,dt.$

Furthermore, We have $P\in L^\infty(Q)$ and $\overline{\gamma_x}^{-1}\gamma_x$ is increasing function, hence $|\int_Q\zeta_k'(u_n)\Phi_n(x,t,u_n)\,dx\,dt|\le C_2$, where C_2 is a positive constant independent of n.

In the same way, we get $|\int_Q\zeta_k''(u_n)\Phi_n(x,t,u_n)\,dx\,dt|\le C_3$, where C_3 is a positive constant independent of n.

Then all above implies that

$$\frac{\partial(\zeta_k(u_n))}{\partial t}\text{ is bounded in }L^1(Q)+W_0^{-1,x}L_\psi(Q).\tag{5.21}$$

Hence by Lemma 4.8 and using the same technics in [13], we can see that there exists a measurable function $u\in L^\infty(0,T;L^1(\Omega))$ such that for every $k>0$ and a subsequence, not relabeled,

$$u_n\to u\text{ a. e. in }Q,\tag{5.22}$$

and

$$\begin{aligned}T_k(u_n)\rightharpoonup T_k(u)\text{ weakly in }W_0^{1,x}L_\varphi(Q)\text{ for }\sigma(\Pi L_\varphi,\Pi E_\psi),\\ \text{strongly in }L^1(Q)\text{ and a. e. in }Q.\end{aligned}\tag{5.23}$$

Step 3: Boundedness of $a(x,t,T_k(u_n),\nabla T_k(u_n))$ ***in*** $(L_\psi(Q))^N$***.***

Now we shall to prove the boundness of $(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n$ in $(L_\psi(Q))^N$. Let $\phi \in (E_\varphi(Q))^N$ with $||\phi||_{\varphi,Q} = 1$. In view of the monotonicity of a one easily has,

$$\Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\frac{w}{\nu})\Big)(\nabla T_k(u_n) - \frac{w}{\nu}) > 0,$$

hence

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\frac{w}{\nu}\,dx\,dt \le \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\,dt - \int_Q a(x,t,T_k(u_n),\frac{w}{\nu})(\nabla T_k(u_n) - \frac{w}{\nu})\,dx\,dt. \tag{5.24}$$

Thanks to (5.16), we have

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\,dt \le C_4.$$

where C_4 is a positive constant which is independent of n.

On the other hand, for λ large enough ($\lambda > \beta$), we have by using (3.1).

$$\begin{aligned}
&\int_Q \psi_x\Big(\frac{a(x,t,T_k(u_n),\frac{w}{\nu})}{3\lambda}\Big)\,dx\,dt \\
&\le \int_Q \psi_x\Big(\frac{\beta\Big(c(x,t) + \psi_x^{-1}(\gamma(x,|T_k(u_n)|)) + \psi_x^{-1}(\varphi(x,|w|))\Big)}{3\lambda}\Big)\,dx\,dt \\
&\le \frac{\beta}{\lambda}\int_Q \psi_x\Big(\frac{c(x,t) + \psi_x^{-1}(\gamma(x,|T_k(u_n)|)) + \psi_x^{-1}(\varphi(x,|w|))}{3}\Big)\,dx\,dt \\
&\le \frac{\beta}{3\lambda}\Big(\int_Q \psi_x(c(x,t))\,dx\,dt + \int_Q \gamma(x,|T_k(u_n)|)\,dx\,dt + \int_Q \varphi(x,|w|)\,dx\,dt\Big) \\
&\le \frac{\beta}{3\lambda}\Big(\int_Q \psi_x(c(x,t))\,dx\,dt + \int_Q \gamma(x,|T_k(u_n)|)\,dx\,dt + \int_Q \varphi(x,|w|)\,dx\,dt\Big)
\end{aligned}$$

Now, since γ grows essentially less rapidly than φ near infinity and by using the Remark 2.1, there exists $r(\varepsilon) > 0$ such that $\gamma(x,|T_k(u_n)|) \le r(\varepsilon)\varphi(x,\varepsilon|T_k(u_n)|)$ and so we have

$$\int_Q \psi_x\Big(\frac{a(x,t,T_k(u_n),\frac{w}{\nu})}{3\lambda}\Big)\,dx\,dt \le \frac{\beta}{3\lambda}\Big(\int_Q \psi_x(c(x,t))\,dx\,dt + r(k)\int_Q \varphi(x,\varepsilon|T_k(u_n)|)\,dx\,dt + \int_Q \varphi(x,|w|)\,dx\,dt\Big).$$

hence $a(x,t,T_k(u_n),\frac{w}{\nu})$ is bounded in $(L_\psi(Q))^N$. Which implies that second term of the right hand side of (5.24) is bounded, consequently we obtain

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))w\,dx\,dt \le C_5, \quad \text{for all } w \in (E_\varphi(Q))^N \text{ with } \|w\|_{\varphi,Q} \le 1,$$

where C_5 is a positive constant which is independent of n.

Hence, thanks the Banach-Steinhaus Theorem, the sequence

$$(a(x,t,T_k(u_n),\nabla T_k(u_n)))_n$$

is a bounded sequence in $(L_\psi(Q))^N$, thus up to a subsequence

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \phi_k \text{ weakly star in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi, \Pi E_\varphi) \tag{5.25}$$

for some $\phi_k \in (L_\psi(Q))^N$.

Step 4: Almost everywhere convergence of the gradients.

Fix $k > 0$ and let $\phi(s) = s\exp(\delta s^2), \delta > 0$. It is well known that when $\delta \geq (\frac{b(k)}{2\alpha})^2$ one has

$$\phi'(s) - \frac{b(k)}{\alpha}|\phi(s)| \geq \frac{1}{2} \text{ for all } s \in \mathbb{R}. \tag{5.26}$$

Let $v_j \in \mathcal{D}(Q)$ be a sequence such that

$$v_j \to u \text{ for the modular convergence in } W_0^{1,x}L_\varphi(Q). \tag{5.27}$$

and let $\omega_i \in \mathcal{D}(Q)$ be a sequence which converges strongly to u_0 in $L^2(\Omega)$.

Set $\omega_{i,j}^\mu = T_k(v_j)_\mu + \exp(-\mu t)T_k(w_i)$ where $T_k(v_j)_\mu$ is the mollification with respect to time of $T_k(v_j)$, see [4].

Note that $\omega_{i,j}^\mu$ is a smooth function having the following properties

$$\frac{\partial}{\partial t}(\omega_{i,j}^\mu) = \mu(T_k(v_j) - \omega_{i,j}^\mu), \omega_{i,j}^\mu(0) = T_k(\omega_i), |\omega_{i,j}^\mu| \leq k, \tag{5.28}$$

$$\omega_{i,j}^\mu \to T_k(u)_\mu + \exp(-\mu t)T_k(w_i) \text{ in } W_0^{1,x}L_\varphi(Q) \tag{5.29}$$

for the modular convergence as $j \to \infty$,

$$T_k(u)_\mu + \exp(-\mu t)T_k(w_i) \to T_k(u) \text{ in } W_0^{1,x}L_\varphi(Q) \tag{5.30}$$

for the modular convergence as $\mu \to \infty$.

Let now the function ρ_m defined on $\mathbb{R}$ with $m \geq k$ by:

$$\rho_m(s) = \begin{cases} 1 & \text{if } |s| \leq m, \\ m+1-|s| & \text{if } m \leq |s| \leq m+1, \\ 0 & \text{if } |s| \geq m+1. \end{cases} \tag{5.31}$$

we set

$$R_m(s) = \int_0^s \rho_m(r)dr, \quad \theta_{i,j}^{\mu,n} = T_k(u_n) - \omega_{i,j}^\mu.$$

Using the admissible test function $Z^{\mu,m}_{i,j,n} = \phi(\theta^{\mu,n}_{i,j})\rho_m(u_n)$ as test function in $(\mathcal{P}_n)$ leads to

$$\begin{aligned}
&\langle\frac{\partial u_n}{\partial t}, Z^{\mu,m}_{i,j,n}\rangle + \int_Q a(x,t,u_n,\nabla u_n)\cdot\Big(\nabla T_k(u_n) - \nabla\omega^{\mu}_{i,j}\Big)\phi'(\theta^{\mu,n}_{i,j})\rho_m(u_n)\,dx\,dt\\
&+\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\phi(\theta^{\mu,n}_{i,j})\rho'_m(u_n)\,dx\,dt\\
&+\int_{\{m\le|u_n|\le m+1\}} \Phi_n(x,t,u_n)\cdot\nabla u_n\phi(\theta^{\mu,n}_{i,j})\rho'_m(u_n)\,dx\,dt\\
&+\int_Q \Phi_n(x,t,u_n)\cdot(\nabla T_k(u_n) - \nabla\omega^{\mu}_{i,j})\phi'(\theta^{\mu,n}_{i,j})\rho_m(u_n)\,dx\,dt\\
&+\int_Q g_n(x,t,u_n,\nabla u_n)Z^{\mu,m}_{i,j,n}\,dx\,dt\\
&=\int_Q f_n Z^{\mu,m}_{i,j,n}\,dx\,dt.
\end{aligned}\tag{5.32}$$

Since $g_n(x,t,u_n,\nabla u_n)\phi(\theta^{\mu,n}_{i,j})\rho_m(u_n)\ge 0$ on $\{|u_n|>k\}$, yields

$$\begin{aligned}
&\langle\frac{\partial u_n}{\partial t}, Z^{\mu,m}_{i,j,n}\rangle + \int_Q a(x,t,u_n,\nabla u_n)\cdot(\nabla T_k(u_n) - \nabla\omega^{\mu}_{i,j})\phi'(\theta^{\mu,n}_{i,j})\rho_m(u_n)\,dx\,dt\\
&+\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\phi(\theta^{\mu,n}_{i,j})\rho'_m(u_n)\,dx\,dt\\
&+\int_{\{m\le|u_n|\le m+1\}} \Phi_n(x,t,u_n)\cdot\nabla u_n\phi(\theta^{\mu,n}_{i,j})\rho'_m(u_n)\,dx\,dt\\
&+\int_Q \Phi_n(x,t,u_n)\cdot(\nabla T_k(u_n) - \nabla\omega^{\mu}_{i,j})\phi'(\theta^{\mu,n}_{i,j})\rho_m(u_n)\,dx\,dt\\
&+\int_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\phi(\theta^{\mu,n}_{i,j})\rho_m(u_n)\,dx\,dt\\
&\le\int_Q f_n Z^{\mu,m}_{i,j,n}\,dx\,dt.
\end{aligned}\tag{5.33}$$

Denoting by $\epsilon(n,j,\mu,i)$ any quantity such that

$$\lim_{i\to\infty}\lim_{\mu\to\infty}\lim_{j\to\infty}\lim_{n\to\infty}\epsilon(n,j,\mu,i) = 0.$$

Now, we prove below the following results for any fixed $k\ge 0$.

$$\int_Q f_n Z^{\mu,m}_{i,j,n}\,dx\,dt = \epsilon(n,j,\mu).\tag{5.34}$$

$$\int_Q \Phi_n(x,t,u_n)\cdot(\nabla T_k(u_n) - \nabla\omega^{\mu}_{i,j})\phi'(\theta^{\mu,n}_{i,j})\rho_m(u_n)\,dx\,dt = \epsilon(n,j,\mu).\tag{5.35}$$

$$\int_{\{m\le|u_n|\le m+1\}} \Phi_n(x,t,u_n)\cdot\nabla u_n\phi(\theta_{i,j}^{\mu,n})\rho'_m(u_n)\,dx\,dt=\epsilon(n,j,\mu). \tag{5.36}$$

$$\langle\frac{\partial u_n}{\partial t},Z_{i,j,n}^{\mu,m}\rangle\ge\epsilon(n,j,\mu,i). \tag{5.37}$$

$$\int_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\phi(\theta_{i,j}^{\mu,n})\rho'_m(u_n)\,dx\,dt\le\epsilon(n,j,\mu,m). \tag{5.38}$$

$$\begin{aligned}&\int_Q\Big[a\Big(x,t,T_k(u_n)),\nabla T_k(u_n)\Big)-a\Big(x,t,T_k(u_n)\Big),\nabla T_k(u)\chi_s)\Big]\\&\quad\times\Big[\nabla T_k(u_n)-\nabla T_k(u)\chi_s\Big]\,dx\,dt\le\epsilon(n,j,\mu,i).\end{aligned} \tag{5.39}$$

Proof of (5.34) :
By the almost every where convergence of u_n, we have $\phi(T_k(u_n)-\omega_{i,j}^{\mu})\rho_m(u_n)\rightharpoonup\phi(T_k(u)-\omega_{i,j}^{\mu})\rho_m(u)$ weakly-* in $L^\infty(Q)$ as $n\to\infty$, and then,

$$\int_Q f_n\phi(T_k(u_n)-\omega_{i,j}^{\mu})\rho_m(u_n)\,dx\,dt\to\int_Q f\phi(T_k(u)-\omega_{i,j}^{\mu})\rho_m(n)\,dx\,dt,$$

so that, $\phi(T_k(u)-\omega_{i,j}^{\mu})\rho_m(u)\rightharpoonup\phi(T_k(u)-T_k(u)_\mu-\exp(-\mu t)T_k(w_i))\rho_m(u)$ weakly star in $L^\infty(Q)$ as $j\to\infty$, and finally,

$$\phi(T_k(u)-T_k(u)_\mu-\exp(-\mu t)T_k(w_i))\rho_m(u)\rightharpoonup 0 \text{ weakly star as } \mu\to\infty.$$

Then, we deduce that,

$$\langle f_n,\phi(T_k(u_n)-\omega_{i,j}^{\mu})\rho_m(u_n)\rangle=\epsilon(n,j,\mu). \tag{5.40}$$

Proof of (5.35) and (5.36), Similarly, Lebesgue's convergence theorem shows that,

$$\Phi_n(x,t,u_n)\rho_m(u_n)\to\Phi(x,t,u)\rho_m(u) \text{ strongly in } (E_\psi(Q)^N) \text{ as } n\to\infty,$$

and

$$\Phi_n(x,t,u_n)\chi_{\{m\le|u_n|\le m+1\}}\phi'(T_k(u_n)-\omega_{i,j}^{\mu})\to\Phi(x,t,u)\chi_{\{m\le u\le m+1\}}\phi'(T_k(u)-\omega_{i,j}^{\mu})$$

strongly in $(E_\psi(Q)^N)$. Then by virtue of

$$\nabla T_k(u_n)\rightharpoonup\nabla T_k(u) \text{ weak star in } (L_\varphi(Q)^N),$$

and $\nabla u_n\chi_{\{m\le|u_n|\le m+1\}}=\nabla T_{m+1}(u_n)\chi_{\{m\le|u_n|\le m+1\}}$ a. e. in Q, one has,

$$\int_Q\Phi_n(x,t,u_n)\cdot(\nabla T_k(u_n)-\nabla\omega_{i,j}^{\mu})\phi'(T_k(u_n)-\omega_{i,j}^{\mu})\rho_m(u_n)\,dx\,dt$$

$$\to\int_Q\Phi(x,t,u)\nabla(\nabla T_k(u)-\nabla\omega_{i,j}^{\mu})\phi'(T_k(u)-\omega_{i,j}^{\mu})\rho_m(u)\,dx\,dt$$

as $n \to \infty$, and

$$\int_{\{m\leq|u_n|\leq m+1\}} \Phi_n(x,t,u_n)\phi(T_k(u_n)-\omega^{\mu}_{i,j})\nabla u_n \rho'_m(u_n)\,dx\,dt$$

$$\to \int_{\{m\leq|u_n|\leq m+1\}} \Phi(x,t,u)\phi(T_k(u_n)-\omega^{\mu}_{i,j})\nabla u \rho'_m(u)\,dx\,dt$$

,
as $n \to +\infty$.

Thus, by using the modular convergence of $\omega^{\mu}_{i,j}$ as $j \to +\infty$ and letting μ tend to infinity, we get (5.35) and (5.36).

Proof of (5.37) : Since $u_n \in W^{1,x}_0 L_\varphi(Q)$, there exists a smooth function $u_{n\sigma}$ (see [1]) such that:

$$u_{n\sigma} \to u_n \text{ for the modular convergence in } W^{1,x}_0 L_\varphi(Q) \cap L^2(Q),$$

$$\frac{\partial u_{n\sigma}}{\partial t} \to \frac{\partial u_n}{\partial t} \text{ for the modular convergence in } W^{-1,x} L\psi(Q) + L^2(Q).$$

Then,

$$\begin{aligned}
\left\langle \frac{\partial u_n}{\partial t}, Z^{\mu,m}_{i,j,n} \right\rangle &= \lim_{\sigma\to 0^+} \int_Q (u_{n\sigma})'\phi(T_k(u_{n\sigma})-\omega^{i}_{\mu,j})\rho_m(u_n)\,dx\,dt \\
&= \lim_{\sigma\to 0^+} \int_Q (R_m(u_{n\sigma}))'\phi((T_k(u_{n\sigma})-\omega^{i}_{\mu,j})\,dx\,dt \\
&= \lim_{\sigma\to 0^+} \Big[\int_Q (R_m(u_{n\sigma})-T_k(u_{n\sigma}))'\phi(T_k(u_{n\sigma})-\omega^{i}_{\mu,j})\,dx\,dt \\
&+ \int_Q (T_k(u_{n\sigma}))'\phi(T_k(u_{n\sigma})-\omega^{i}_{\mu,j})\,dx\,dt\Big] \\
&= \lim_{\sigma\to 0^+} \int_\Omega \Big[(R_m(u_{n\sigma})-T_k(u_{n\sigma}))\phi(T_k(u_{n\sigma})-\omega^{i}_{\mu,j})dx\Big]_0^T \\
&- \int_Q (R_m(u_{n\sigma})-T_k(u_{n\sigma}))\phi'(T_k(u_{n\sigma})-\omega^{i}_{\mu,j})'\,dx\,dt \\
&+ \int_Q (T_k(u_{n\sigma}))'\phi(T_k(u_{n\sigma})-\omega^{i}_{\mu,j})\,dx\,dt \\
&= \lim_{\sigma\to 0^+} \Big[I_1(\sigma)+I_2(\sigma)+I_3(\sigma)\Big].
\end{aligned}$$

Observe that for $|s| \leq k$, we have $R_m(s) = T_k(s) = s$ and for $|s| > k$ we have $|R_m(s)| \geq |T_k(s)|$ and, since both $R_m(s)$ and $T_k(s)$ have the same sign of s, we deduce that sign $(s)(R_m(s) - T_k(s)) \geq 0$. Consequently

$$I_1(\sigma) = \left[\int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\phi(T_k(u_{n\sigma}) - \omega^{\mu}_{i,j})dx\right]_0^T$$
$$\geq -\int_{\{|u_{n\sigma}(0)|>k\}} (R_m(u_{n\sigma}(0)) - T_k(u_{n\sigma}(0)))\phi(T_k(u_{n\sigma}(0)) - \omega^{\mu}_{i,j}(0))dx$$

and so, by letting $\sigma \to 0^+$ in the last integral, we get

$$\limsup_{\sigma\to 0^+} I_1(\sigma) \geq -\int_{\{|u_{0n}|>k\}} (R_m(u_{0n}) - T_k(u_{0n}))\phi(T_k(u_{0n}) - T_k(w_i))dx.$$

Letting $n \to \infty$, the right hand side of the above inequality clearly tends to

$$-\int_{\{|u_0|>k\}} (R_m(u_0) - T_k(u_0))\phi(T_k(u_0) - T_k(w_i))dx$$

which obviously goes to 0 as $i \to \infty$.

Which yields that

$$\limsup_{\sigma\to 0^+} I_1(\sigma) \geq \epsilon(n, i).$$

About $I_2(\sigma)$, since $(R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma})' = 0$, one has

$$\begin{aligned} I_2(\sigma) &= \int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\phi'(T_k(u_{n\sigma}) - \omega^{\mu}_{i,j})(\omega^{\mu}_{i,j})'\,dx\,dt \\ &= \mu\int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\phi'(T_k(u_{n\sigma}) - \omega^{\mu}_{i,j})(T_k(v_j) - \omega^{\mu}_{i,j})\,dx\,dt \\ &\leq \mu\int_{\{|u_{n\sigma}|>k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\phi'(T_k(u_{n\sigma}) - \omega^{\mu}_{i,j})(T_k(v_j) - T_k(u_{n\sigma}))\,dx\,dt, \end{aligned}$$

by using the fact $\phi' \geq 0$ and that $(R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega^{\mu}_{i,j})\chi_{\{|u_{n\sigma}|>k\}} \geq 0$ and so by letting $\sigma \to 0^+$ in the last integral, we get

$$\limsup_{\sigma\to 0^+} I_2(\sigma) \geq \mu\int_{\{|u_n|\geq k\}} (R_m(u_n) - T_k(u_n))\phi'(T_k(u_n) - \omega^{\mu}_{i,j})(T_k(v_j) - T_k(u_n))\,dx\,dt,$$

and since, as it can be easily seen, the last integral is of the form $\epsilon(n, j)$, we deduce that

$$\limsup_{\sigma\to 0^+} I_2(\sigma) \geq \epsilon(n, j).$$

For what concerns $I_3(\sigma)$, one

$$I_3(\sigma) = \int_Q (R_m(u_{n\sigma}) - \omega^{\mu}_{i,j})'\phi(T_k(u_{n\sigma}) - \omega^{\mu}_{i,j})\,dx\,dt$$
$$+\int_Q (\omega^{\mu}_{i,j})'\phi(T_k(u_{n\sigma}) - \omega^{\mu}_{i,j})\,dx\,dt$$

and then, by setting $\xi(s) = \int_0^s \phi(\eta)d\eta$ and integrating by parts

$$I_3(\sigma) = \left[\int_\Omega \xi(T_k(u_{n\sigma}) - \omega^\mu_{i,j})(t)dx\right]_0^T + \mu\int_Q (T_k(v_j) - \omega^\mu_{i,j})\phi(T_k(u_{n\sigma}) - \omega^\mu_{i,j})\,dx\,dt,$$

Since $\xi \geq 0$ and $(T_k(v_j) - \omega^\mu_{i,j})\phi(T_k(u_{n\sigma}) - \omega^\mu_{i,j}) \geq 0$, yields

$$\begin{aligned} I_3(\sigma) &\geq -\int_\Omega \xi(T_k(u_{n\sigma}(0)) - T_k(w_i))dx \\ &+ \mu\int_Q (T_k(v_j) - T_k(u_{n\sigma})\phi(T_k(u_{n\sigma}) - \omega^\mu_{i,j})\,dx\,dt, \end{aligned}$$

so that,

$$\limsup_{\sigma\to 0^+} I_3(\sigma) \geq -\int_\Omega \xi(T_k(u_{0n}) - T_k(w_i))dx$$
$$+\mu\int_Q (T_k(v_j) - T_k(u_n)\phi(T_k(u_n) - \omega^\mu_{i,j})\,dx\,dt.$$

Hence, by letting $n \to \infty$ in the last side, we obtain

$$\begin{aligned} \limsup_{\sigma\to 0^+} I_3(\sigma) &\geq -\int_\Omega \xi(T_k(u_0) - T_k(w_i))dx \\ &+ \mu\int_Q (T_k(v_j) - T_k(u)\phi(T_k(u) - \omega^\mu_{i,j})\,dx\,dt + \epsilon(n). \end{aligned}$$

since the first integral of the last side is of the from $\epsilon(i)$ while the second one is of the form $\epsilon(j)$, we deduce that

$$\limsup_{\sigma\to 0^+} I_3(\sigma) \geq \epsilon(n, j, i).$$

where we have used the fact that (recall that $|\omega^\mu_{i,j}| \leq k$)

$$\begin{aligned} \int_Q G_k(u)\phi'(T_k(u) - \omega^\mu_{i,j})(T_k(u) - \omega^\mu_{i,j})\,dx\,dt &= \int_{\{u>k\}} (u-k)\phi'(k - \omega^\mu_{i,j})(k - \omega^\mu_{i,j})\,dx\,dt \\ &+ \int_{\{u<-k\}} (u+k)\phi'(-k - \omega^\mu_{i,j})(-k - \omega^\mu_{i,j})\,dx\,dt \\ &\geq 0. \end{aligned}$$

Combining these estimates, we conclude that

$$\langle u'_n, \phi(T_k(u_n) - \omega^\mu_{i,j})\rho_m(u_n)\rangle \geq \epsilon(n, j, i). \tag{5.41}$$

Proof of (5.38) : Concerning the third term of the right hand side of (5.33) we obtain that

$$\begin{aligned} &\left|\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n \phi(\theta^{\mu,i}_{n,j})\rho'_m(u_n)\,dx\,dt\right| \\ &\leq \phi(2k)\int_{\{m\leq|u_n|\leq m+1\}} a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\,dx\,dt. \end{aligned}$$

Then by (5.16) we deduce that,

$$|\int_Q a(x,t,u_n,\nabla u_n)\cdot\nabla u_n\phi(\theta^{\mu,i}_{n,j})\rho'_m(u_n)\,dx\,dt| \le \epsilon(n,\mu,m). \tag{5.42}$$

Proof of (5.39) : Now, concerning the sixth term of the right hand side of (5.33), We can write

$$\begin{aligned}|\int_{\{|u_n|\le k\}} & g_n(x,t,u_n,\nabla u_n)\phi(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)|\,dx\,dt\\ &\le b(k)\int_Q c_2(x,t)|\phi(T_k(u_n)-\omega^i_{\mu,j})|\,dx\,dt\\ +\frac{b(k)}{\alpha}&\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n)|\phi(T_k(u_n)-\omega^i_{\mu,j})|\,dx\,dt. \end{aligned}\tag{5.43}$$

Since $c_2(x,t)$ belongs to $L^1(Q)$ it is easy to see that

$$b(k)\int_Q c_2(x,t)|\phi(T_k(u_n)-\omega^i_{\mu,j})|\,dx\,dt = \varepsilon(n,j,\mu).$$

On the other hand, the second term of the right hand side of (5.43) reads as

$$\begin{aligned}&\frac{b(k)}{\alpha}\int_Q a(T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\phi(T_k(u_n)-\omega^i_{\mu,j})|\,dx\,dt\\ &=\frac{b(k)}{\alpha}\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(v_j)\chi^s_j)\Big)\\ &\times\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j\Big)|\phi(T_k(u_n)-\omega^i_{\mu,j})|\,dx\,dt\\ &+\frac{b(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(v_j)\chi^s_j)\\ &\times\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j\Big)|\phi(T_k(u_n)-\omega^i_{\mu,j})|\,dx\,dt\\ &+\frac{b(k)}{\alpha}\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi^s_j)|\phi(T_k(u_n)-\omega^i_{\mu,j})|\,dx\,dt\end{aligned}$$

and, as above, by letting successively first n, then j,μ and finally s go to infinity, we can easily see that each one of last two integrals of the right-hand side of the last equality is of the form $\varepsilon(n,j,\mu)$.

This implies that

$$\begin{aligned}&|\int_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\phi(T_k(u_n)-\omega^i_{\mu,j})\rho_m(u_n)\,dx\,dt\,|\\ &\le\frac{b(k)}{\alpha}\int_Q\Big(a(x,t,T_k(u_n),\nabla T_k(u_n))-a(x,t,T_k(u_n),\nabla T_k(v_j)\chi^s_j)\Big)\\ &\times\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi^s_j\Big)|\phi(T_k(u_n)-\omega^i_{\mu,j})|\,dx\,dt+\varepsilon(n,j,\mu).\end{aligned}$$

Combining (5.33),(5.38), (5.37) (5.39),(5.43) and (5.44), we get

$$\int_Q \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big)\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)$$
$$\times\Big(\phi'(T_k(u_n) - \omega_{\mu,j}^i) - \frac{b(k)}{\alpha}|\phi(T_k(u_n) - \omega_{\mu,j}^i)|\Big)\,dx\,dt$$
$$\leq \varepsilon(n,j,\mu,i,s,m).$$

and so, thanks to (5.26),

$$\begin{aligned}&\int_Q \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big)\\&\times\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)\,dx\,dt\\&\leq 2\varepsilon(n,j,\mu,i,s,m).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\int_Q \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)\Big)\\&\times\Big(\nabla T_k(u_n) - \nabla T_k(u)\chi^s\Big)\,dx\,dt\\&-\int_Q \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big)\\&\times\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)\,dx\,dt\\&=\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\Big(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\Big)\,dx\,dt\\&-\int_Q a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)\Big(\nabla T_k(u_n) - \nabla T_k(u)\chi^s\Big)\,dx\,dt\\&+\int_Q a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)\,dx\,dt\end{aligned}$$

and, as it can be easily seen, each integral of the right-hand side is of the form $\varepsilon(n,j,s)$, implying that

$$\begin{aligned}&\int_Q \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi^s)\Big)\\&\times\Big(\nabla T_k(u_n) - \nabla T_k(u)\chi^s\Big)\,dx\,dt\\&=\int_Q \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\Big)\\&\times\Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\Big)\,dx\,dt + \varepsilon(n,j,s).\end{aligned}$$

For $r \leq s$, we have

$$\begin{aligned}
0 &\leq \int_{Q^r} \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)) \Big) \\
&\quad \times \Big(\nabla T_k(u_n) - \nabla T_k(u) \Big)\, dx\, dt \\
&\leq \int_{Q^s} \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)) \Big) \\
&\quad \times \Big(\nabla T_k(u_n) - \nabla T_k(u) \Big)\, dx\, dt \\
&= \int_{Q^s} \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi^s) \Big) \\
&\quad \times \Big(\nabla T_k(u_n) - \nabla T_k(u)\chi^s \Big)\, dx\, dt \\
&\leq \int_{Q} \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)\chi^s) \Big) \\
&\quad \times \Big(\nabla T_k(u_n) - \nabla T_k(u)\chi^s \Big)\, dx\, dt \\
&= \int_{Q} \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(v_j)\chi_j^s) \Big) \\
&\quad \times \Big(\nabla T_k(u_n) - \nabla T_k(u)\chi_j^s \Big)\, dx\, dt + \varepsilon(n,j,s). \\
&\leq \varepsilon(n,j,\mu,i,s,m).
\end{aligned}$$

Hence, by passing to the limit sup over n and the limit successively on $j \to \infty, \mu \to, i \to \infty, s \to \infty$, and $m \to \infty$, we get

$$\limsup_{n\to\infty} \int_{Q^r} \Big[(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u)) \Big] \times \Big[\nabla T_k(u_n) - \nabla T_k(u) \Big]\, dx\, dt = 0.$$

Using a similar tools as in [16], we get

$$T_k(u_n) \to T_k(u) \text{ for the modular convergence in } W_0^{1,x}L(Q). \tag{5.44}$$

Which implies that exists a subsequence still denote by u_n such that

$$\nabla u_n \to \nabla u \text{ a.e. in } Q. \tag{5.45}$$

We deduce then that,for all $k > 0$, one has

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u)) \text{ weak star in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi, \Pi E_\varphi). \tag{5.46}$$

Step 5: Equi-integrability of $g_n(x, u_n, \nabla u_n)$***.***

We shall now prove that $g_n(x,t,u_n,\nabla u_n) \to g(x,t,u,\nabla u)$ strongly in $L^1(Q)$ by using Vitli's theorem.

Since $g_n(x,t,u_n,\nabla u_n) \to g(x,t,u,\nabla u)$ a.e. in Q, thanks to (5.22) and (5.44) and Vitali's theorem, it suffices to prove that $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q.

Let $E \subset Q$ be a measurable subset of Q. Then for any $m > 0$, one has

$$\int_E |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt = \int_{E\cap\{u_n\le m\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt + \int_{E\cap\{u_n>m\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt.$$

On the one hand,

$$\int_{E\cap\{u_n>m\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt \le \frac{1}{m}\int_Q g_n(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt \le \frac{C}{m}$$

where C is the constant in (3.4). Therefore, there exists $m = m(\varepsilon)$ large enough such that

$$\int_{E\cap\{u_n>m\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt \le \frac{\varepsilon}{2} \quad \forall n.$$

On the other hand

$$\begin{aligned}\int_{E\cap\{u_n\le m\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt &\le \int_E |g_n(x,t,T_m(u_n),\nabla T_m(u_n))|\,dx\,dt\\ &\le b(m)\int_E \Big(c_2(x,t) + \varphi(x,\nabla|T_m(u_n))|\Big)\,dx\,dt\\ &\le b(m)\int_E \Big(c_2(x,t) + \frac{1}{\alpha}d(x,t)\Big)\,dx\,dt\\ &+ \frac{b(m)}{\alpha}\int_E a(x,t,T_m(u_n),\nabla T_m(u_n))\cdot\nabla T_m(u_n)\,dx\,dt\end{aligned}$$

where we have used (3.4). Therefore, it is easy to see that there exists $\nu > 0$ such that

$$|E| < \nu \Longrightarrow \int_{E\cap\{u_n\le m\}} |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt \le \frac{\varepsilon}{2} \quad \forall n.$$

Consequently,

$$|E| < \nu \Longrightarrow \int_E |g_n(x,t,u_n,\nabla u_n)|\,dx\,dt \le \varepsilon \quad \forall n.$$

Which shows that $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q as required.

Moreover, we get

$$g_n(x,t,u_n,\nabla u_n) \longrightarrow g(x,t,u,\nabla u) \quad \text{strongly in } L^1(Q). \tag{5.47}$$

Step 6: Passage to the limit.

Let $v \in W_0^{1,x}L_\varphi(Q)$ such that $\frac{\partial v}{\partial t} \in W^{-1,x}L_\psi(Q) + L^1(Q)$. There exists a prolongation $\overline{v}$ of v such that (see the proof of Lemma 4.7 and Theorem 4.6. in [1])

$$\begin{cases} \overline{v} = v \quad \text{on } Q, \\ \overline{v} \in W_0^{1,x}L_\varphi(\Omega\times\mathbb{R}) \cap L^1(\Omega\times\mathbb{R}) \cap L^\infty(\Omega\times\mathbb{R}), \\ \text{and} \quad \frac{\partial \overline{v}}{\partial t} \in W^{-1,x}L_\psi(\Omega\times\mathbb{R}) + L^1(\Omega\times\mathbb{R}). \end{cases}$$

By Lemma 4.7, there exists a sequence $(w_j)_j$ in $D(\Omega\times\mathbb{R})$ such that $w_j \longrightarrow \overline{v}$ in $W_0^{1,x}L_\varphi(\Omega\times\mathbb{R})$ and $\frac{\partial w_j}{\partial t} \longrightarrow \frac{\partial \overline{v}}{\partial t}$ in $W^{-1,x}L_\psi(\Omega\times\mathbb{R}) + L^1(\Omega\times\mathbb{R})$ for the modular convergence and
$\|w_j\|_{\infty,Q} \le (N+2)\|v\|_{\infty,Q}$.

Go back to approximate equations $(\mathcal{P}_n)$ and use $T_k(u_n - w_j)\chi_{[0,\tau]}$ for every $\tau \in [0,T]$, as a test function one has

$$\begin{aligned} &\int_{Q_\tau} \frac{\partial u_n}{\partial t} T_k(u_n - w_j)\,dx\,dt \\ &\quad + \int_{Q_\tau} a(x,t,u_n,\nabla u_n)\cdot\nabla T_k(u_n - w_j)\,dx\,dt \\ &\qquad + \int_{Q_\tau} \Phi_n(x,t,u_n)\cdot\nabla T_k(u_n - w_j)\,dx\,dt \\ &\qquad\quad \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n) T_k(u_n - w_j)\,dx\,dt \\ &\qquad\qquad \le \int_{Q_\tau} f_n T_k(u_n - w_j)\,dx\,dt. \end{aligned} \tag{5.48}$$

For the first term of (5.48), we get

$$\begin{aligned} \int_{Q_\tau} \frac{\partial u_n}{\partial t} T_k(u_n - w_j)\,dx\,dt &= \left[\int_\Omega T_k(u_n - w_j)dx\right]_0^\tau + \int_{Q_\tau} \frac{\partial w_j}{\partial t} T_k(u_n - w_j)\,dx\,dt \\ &= \left[\int_\Omega T_k(u - w_j)dx\right]_0^\tau + \int_{Q_\tau} \frac{\partial w_j}{\partial t} T_k(u - w_j)\,dx\,dt \\ &+ \varepsilon(n) \\ &= \int_{Q_\tau} \frac{\partial u}{\partial t} T_k(u - w_j)\,dx\,dt. \end{aligned}$$

For the second term of (5.48), we have if $|u_n| > \lambda$ then $|u_n - w_j| \geq |u_n| - \|w_j\|_\infty > k$, therefore $\{|u_n - w_j| \leq k\} \subseteq \{|u_n| \leq k + (N+2)\|v\|_\infty\}$, which implies that, we get

$$\begin{aligned}
&\liminf_{n\to+\infty} \int_Q a(x,t,u_n,\nabla u_n)\nabla T_k(u_n - w_j)\ dx\,dt \\
&\geq \int_Q a(x,t,T_{k+(N+2)\|v\|_\infty}(u), \nabla T_{k+(N+2)\|v\|_\infty}(u)) \\
&\Big(\nabla T_{k+(N+2)\|v\|_\infty}(u) - \nabla w_j\Big)\chi_{\{|u-v|\leq k\}}\ dx\,dt, \qquad (5.49)\\
&= \int_Q a(x,t,u,\nabla u)(\nabla u - \nabla w_j)\chi_{\{|u-w_j|\leq k\}}\ dx\,dt \\
&= \int_Q a(x,t,u,\nabla u)\nabla T_k(u-w_j)\ dx\,dt.
\end{aligned}$$

Since $\nabla T_k(u_n - w_j) \rightharpoonup \nabla T_k(u-w_j)$ in $L_\varphi(Q)$ as $n \to +\infty$, we have (as $n \to +\infty$)

$$\begin{aligned}
&\int_{Q_\tau} \Phi_n(x,t,u_n)\cdot\nabla T_k(u_n - w_j)\,dx\,dt \\
&\to \int_{Q_\tau} \Phi(x,t,u)\cdot\nabla T_k(u-w_j)\,dx\,dt.
\end{aligned}$$

Consequently, by using the strong convergence of $(g_n(x,t,u_n,\nabla u_n))_n$ and $((f_n))_n$, one has

$$\begin{aligned}
&\int_{Q_\tau} \frac{\partial u}{\partial t} T_k(u-w_j)\,dx\,dt \\
&\quad + \int_{Q_\tau} a(x,t,u,\nabla u)\cdot\nabla T_k(u-w_j)\,dx\,dt \\
&\qquad + \int_{Q_\tau} \Phi(x,t,u)\cdot\nabla T_k(u-w_j) d\,dx\,dt \qquad (5.50)\\
&\qquad\quad + \int_{Q_\tau} g(x,t,u,\nabla u) T_k(u-w_j)\,dx\,dt \\
&\qquad\qquad \leq \int_{Q_\tau} f T_k(u-w_j)\,dx\,dt.
\end{aligned}$$

Thus , by using the modular convergence of j, we achieve this step.

As a conclusion of Step 1 to Step 6, the proof of Theorem 5.1 is complete.

□

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