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Reidemeister Classes for Coincidences Between Sections of a Fiber Bundle

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ABSTRACT: Let s_0, f_0 be two sections of a fiber bundle $q : E \to B$ and assume the coincidence set $\Gamma(s_0, f_0) \neq \emptyset$. We consider the problem of identifying the algebraic Reidemeister classes for s_0 and f_0 with the geometric classes obtained by the lifting maps on covering spaces. We do this by using the homotopy lifting extension propriety of the fibration q to obtain homotopies over B. When we make this and the basic point is fixed we can use the elements $s_0(\beta), f_0(\beta^{-1})$ where $\beta \in \pi_1(B, b_0)$ and the elements $\gamma \in \pi_1(F_0, e_0)$. So we will introduce the algebraic Reidemeister classes relative to the subgroup $\pi_1(F_0, e_0)$. When the basic points are not fixed we need to consider the classes $[\widetilde{s}]_L$ of lifting of s_0 defined on the universal covering $\widetilde{B}$ to $\widetilde{E}$. The present work relates the lifting classes $[\widetilde{s}]_L$ of s_0 and the algebraic Reidemeister classes $R_A(s_0, f_0; \pi_1(F_0, e_0))$, as given in [2],[3] and [5].

Key Words: Coincidence theory, Reidemeister classes, Fiber bundle.

Contents

1. Introduction

Let $f : (B, b_0) \to (B, b_0)$ be a function and assume that B is compact, locally path connected, semi locally 1-connected and an euclidean neighborhood retract space. Then there is the universal covering $p^{b_0} : \widetilde{B}(b_0) \to B$, constructed from the trivial subgroup $\{[\overline{b}_0]\} \leq \pi_1(B, b_0)$. Let $T : \pi_1(B, b_0) \to Cov(\widetilde{B}(b_0)/B)$ be the isomorphism $\beta \mapsto T_\beta$, the deck transformation associated to β.

Let $\mathcal{L}(f_0)$ be the set of all liftings $\widetilde{f}$ of f_0 with respect to the following commutative diagram. Note that the second diagram corresponds to the case when f is

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the identity map I_B.

$$\begin{tikzcd} \widetilde{B}(b_0) \arrow[r, "\widetilde{f}"] \arrow[d, "p^{b_0}"'] & \widetilde{B}(b_0) \arrow[d, "p^{b_0}"'] \\ B \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} \widetilde{B}(b_0) \arrow[r, "\widetilde{I_B}"] \arrow[d, "p^{b_0}"'] & \widetilde{B}(b_0) \arrow[d, "p^{b_0}"'] \\ B \arrow[r, "I_B"] & B \end{tikzcd} \tag{1.1}$$

Consider the following equivalence relation on the set $\mathcal{L}(f)$: $\widetilde{f}_1 R_L \widetilde{f}_2 \Leftrightarrow \widetilde{f}_1 = T_\beta \circ \widetilde{f}_2 \circ T_\beta^{-1}$, with $\beta \in \pi_1(B, b_0)$. We denote by $R_L(\mathcal{L}(f)$ the quotient set and by $[\widetilde{f}]_L$ the class of $\widetilde{f}$ and $r_L(\mathcal{L}(f)) = |R_L(\mathcal{L}(f))|$.

In [3],[2] and [4] the authors related the relation R_L with the algebraic Reidemeister classes induced by $I_B, f : \pi_1(B, b_0) \to \pi_1(B, b_0)$ whose quotient set is $R_A(f, I_B)$ and $r_A(f, I_B) = |R_A(f, I_B)|$. They proved that:

1. There is an one to one correspondence between $R_L(\mathcal{L}(f)$ and $R_A(f, I_B)$, therefore $r_L(\mathcal{L}(f)) = r_A(f, I_B)$.
2. If $[\widetilde{f}_1]_L = [\widetilde{f}_2]_L$ then $p^{b_0}\left(Fix(\widetilde{f}_1)\right) = p^{b_0}\left(Fix(\widetilde{f}_2)\right)$.
3. If $p^{b_0}\left(Fix(\widetilde{f}_1)\right) \cap p^{b_0}\left(Fix(\widetilde{f}_2)\right) \neq \emptyset$ then $[\widetilde{f}_1]_L = [\widetilde{f}_2]_L$.

In fixed point theory it is usual to put the date $f, I_B : B \to B$ on the context of fiber bundle considering the trivial fiber bundle $q : B \times B \to B$, $q(b_1, b_2) = b_1$ and the sections $s_0, f_0 : B \to B \times B$ of q given by $s_0(b) = (b, b)$ and $f_0(b) = (b, f(b))$.

In this work we consider a general fiber bundle $q : E \to B$ and initially two sections $s_0, f_0 : (B, b_0) \to (E, e_0)$ and F_0 the fiber over b_0 which satisfies good hypotheses on B, E and $F_0 = q^{-1}(b_0)$. The purpose of this work is to prove an analogous result of some results in [2] and [3] in this context of section on fiber bundle.

This work is divided in four sections. In Section 2 we established notations and we listed some results about the construction of covering spaces of a subgroup G of $\pi_1(E, e_0)$ and we explicit the lifting $\widetilde{s}_0, \widetilde{f}_0$ and s_{F_0}, f_{F_0}. We also introduce the equivalence relation R_{f_0} on the set $\mathcal{L}(s_0; f_{F_0})$ and the relation R_{s_0} on $\mathcal{L}(f_0; s_{F_0})$ which the sets are, respectively, specified lifting maps on the universal covering for the sections s_0 and f_0, as in [2],[3] and [5]. The Theorem 2.9 established the first approximations between relations R_{f_0} or R_{s_0} and the Reidemeister relation relative to the subgroup $\pi_1(F_0, e_0)$, as we will see in the next section. In Section 3 we defined the algebraic Reidemeister classes relative of a subgroup $H_0 \leq G_0$ induced by the homomorphisms $\varphi, \psi : G_1 \to G_0$ which the quotient set is $R_A(\psi, \phi; H_0)$. In particular, we will apply this for the homomorphisms on the fundamental groups for two sections $s_0, f_0 : \pi_1(B, b_0) \to \pi_1(E, e_0)$ of a fiber bundle $q : E \to B$ and the subgroup $H_0 = \pi_1(F_0, e_0)$, so we have the set $R_A(s_0, f_0; \pi_1(F_0, e_0))$.

In Section 4 we also defined the set of Nielsen coincidence classes which is indicated by $\widetilde{\Gamma}(s_0, f_0)$ and proved that it is finite under good hypotheses on the spaces

B, E, F_0. We also exhibit an injection map $\widetilde{\Gamma}(s_0, f_0) \to R_A(s_0, f_0; \pi_1(F_0, e_0))$. We finished this section relating the theorems 2.9 and 4.3 and then we proved the main theorem 4.5, which established an one-to-one correspondence between $R_L(\mathcal{L}(f_0; s_{F_0}))$ and $R_A(s_0, f_0; \pi_1(F_0, e_0))$ or similarly for the classes $[\widetilde{s}]_L$ on the set $R_L(\mathcal{L}(s_0; f_{F_0}))$, as in [4] and [5].

2. Covering projection constructed from a subgroup and relations on the lifting maps

Let G be a subgroup of $\pi_1(E, e_0)$ and let $P(E, e_0)$ be the set of all paths $\alpha : [0,1] \to E$ such that $\alpha(0) = e_0$. We say that $\alpha_1, \alpha_2 \in P(E, e_0)$ are $G-$related if $\alpha_1(1) = \alpha_2(1)$ and the class $[\alpha_1 * \alpha_2^{-1}] \in G \leq \pi_1(E, e_0)$. It is easy to prove that this is an equivalence relation and we denote the class from the path α by $\langle e, \alpha\rangle_G$ with $e = \alpha(1)$. The quotient set of $P(E, e_0)$ by this relation is indicated by $\widetilde{E}(G)$.

In [6] the author defined a basis for a topology on the set $\widetilde{E}(G)$ for which the function $p^G : \widetilde{E}(G) \to E$, $p^G\langle e, \alpha\rangle_G = e$ is continuous. Moreover if E is path connected then p^G is a surjection and we have the following statements:

1. If E is connected, locally path connected and semi-locally 1-connected then $p^G : \left(\widetilde{E}(G), \widetilde{e}_0\right) \to (E, e_0)$ is a covering space with
$$p^G\left(\pi_1\left(\widetilde{E}(G), \widetilde{e}_0\right)\right) = G,$$
where $\widetilde{e}_0 = \langle e_0, \overline{e}_0\rangle_G$ and $\overline{e}_0$ is the constant path on $e_0 \in E$.

2. If $G_1 \leq G_2 \leq \pi_1(E, e_0)$ are subgroups and $p^{G_1} : \widetilde{E}(G_1) \to E$, $p^{G_2} : \widetilde{E}(G_2) \to E$ are covering spaces then there is a covering space $p^{G_1}_{G_2} : \widetilde{E}(G_1) \to \widetilde{E}(G_2)$ so that $p^{G_1} = p^{G_2} \circ p^{G_1}_{G_2}$.

3. If G is a normal subgroup of $\pi_1(E, e_0)$ and $p : (\widetilde{E}, \widetilde{x}_0) \to (E, e_0)$ is a covering space so that $p\left(\pi_1(\widetilde{E}, x_0)\right) = G$ then there is an homeomorphism $\varphi : (\widetilde{E}, \widetilde{x}_0) \to (\widetilde{E}(G), \widetilde{e}_0)$ so that $p = p^G \circ \varphi$.

Now we apply this construction when we have two sections $s_0, f_0 : (B, b_0) \to (E, e_0)$ of a fiber bundle $q : (E, e_0) \to (B, b_0)$. For this we suppose that E, B and the fiber $F_0 = q^{-1}(b_0)$ are compact spaces with B and E satisfying the hypothesis as in (1) above.

More precisely, we construct the universal covering spaces for the trivial subgroups $[\overline{b}_0] \lhd \pi_1(B, b_0)$ and $[\overline{e}_0] \lhd \pi_1(E, e_0)$ which are denoted by $p^{b_0} : \widetilde{B}(b_0) \to B$ and $p^{e_0} : \widetilde{E}(e_0) \to E$. We also consider the regular covering space $p^{F_0} : \widetilde{E}(F_0) \to E$ where $\widetilde{E}(F_0) = \widetilde{E}(\pi_1(F_0, e_0))$. As in (2) above we denote $p^{e_0}_{F_0} : \widetilde{E}(e_0) \to \widetilde{E}(F_0)$ for the covering space so that $p^{e_0} = p^{F_0} \circ p^{e_0}_{F_0}$.

From these constructions it is easy to explicit the covering projections, that is $p^{b_0}\langle b, \beta\rangle_{b_0} = b$, $p^{e_0}\langle e, \alpha\rangle_{e_0} = e$ and $p^{e_0}_{F_0}\langle e, \alpha\rangle_{e_0} = \langle e, \alpha\rangle_{F_0} \in \widetilde{E}(F_0)$. Moreover from the sections $s_0, f_0 : (B, b_0) \to (E, e_0)$ it is possible to explicit two special lifting maps as in the following lemma:

Lemma 2.1. *The maps*

$$\widetilde{s}_0, \widetilde{f}_0 : (\widetilde{B}(b_0), \widetilde{b}_0) \to (\widetilde{E}(e_0), \widetilde{e}_0) \text{ and } s_{F_0}, f_{F_0} : (\widetilde{B}(b_0), \widetilde{b}_0) \to (\widetilde{E}(F_0), \widetilde{e}_0)$$

given by $\widetilde{s}_0 \langle b, \beta \rangle_{b_0} = \langle s_0(b), s_0(\beta) \rangle_{e_0}$, $\widetilde{f}_0 \langle b, \beta \rangle_{b_0} = \langle f_0(b), f_0(\beta) \rangle_{e_0}$, $s_{F_0} \langle b, \beta \rangle_{b_0} = \langle s_0(b), s_0(\beta) \rangle_{F_0}$ *and* $f_{F_0} \langle b, \beta \rangle_{b_0} = \langle f_0(b), f_0(\beta) \rangle_{F_0}$ *are continuous and the following diagram commutes.*

$$\begin{tikzcd}
\left(\widetilde{B}(b_0), \widetilde{b}_0\right) \arrow[r, shift left, "\widetilde{s}_0"] \arrow[r, shift right, "\widetilde{f}_0"'] \arrow[d, "I_{\widetilde{B}}"] & \left(\widetilde{E}(e_0), \widetilde{e}_0\right) \arrow[d, "p^{e_0}_{F_0}"'] \arrow[dd, bend left=60, "p^{e_0}"] \\
\left(\widetilde{B}(b_0), \widetilde{b}_0\right) \arrow[r, shift left, "s_{F_0}"] \arrow[r, shift right, "f_{F_0}"'] \arrow[d, "p^{b_0}"] & \left(\widetilde{E}(F_0), \widetilde{e}_{F_0}\right) \arrow[d, "p^{F_0}"'] \\
(B, b_0) \arrow[r, shift left, "s_0"] \arrow[r, shift right, "f_0"'] & (E, e_0)
\end{tikzcd}$$

Proof: The commutativity is immediate from the constructions. Note that the continuity of the maps is given by the choice of the topology on the sets $\widetilde{B}(b_0)$, $\widetilde{E}(e_0)$ and $\widetilde{E}(F_0)$. In fact, let $\langle b, \beta \rangle_{b_0} \in \widetilde{B}(b_0)$ and $V\left(\langle s_0(b), s_0(\beta) \rangle_{e_0}\right)$ be a basic open set of the topology on $\widetilde{E}(e_0)$ where V is an open neighborhood of $s_0(b)$ in E. From the continuity of s_0 let $U = s_0^{-1}(V) \subseteq B$ an open neighborhood of b on B and note that $\widetilde{s}_0 \left(U\left(\langle b, \beta \rangle_{b_0}\right)\right) \subseteq V\left(\langle s_0(b), s_0(\beta) \rangle_{e_0}\right)$.

The continuity of the $\widetilde{f}_0$, $\widetilde{s}_{F_0}$ and $\widetilde{f}_{F_0}$ is shown by similar argument. □

For $\beta \in \pi_1(B, b_0)$ the correspondent deck transformation we denote by $T_\beta \in Cov(\widetilde{B}(b_0)/B)$ and similarly, T_α, T_γ for $\alpha \in \pi_1(E, e_0)$ and $\gamma \in \pi_1(F_0, e_0) \lhd \pi_1(E, e_0)$.

From the covering map constructions we can to explicit the following fibers, where we use the same symbol to express the loop path and its class on fundamental groups:

$$\left(p^{b_0}\right)^{-1}(b_0) = \{\langle b_0, \beta \rangle_{b_0}, \beta \in \pi_1(B, b_0)\};$$

$$\left(p^{e_0}_{F_0}\right)^{-1}\left(\langle e_0, \overline{e}_0 \rangle_{F_0}\right) = \left(p^{e_0}_{F_0}\right)^{-1}\left(\widetilde{e}_{F_0}\right)\{\langle e_0, \gamma \rangle_{e_0}; \gamma \in \pi_1(F_0, e_0)\};$$

$$\left(p^{F_0}\right)^{-1}(e_0) = \{\langle e_0, s_0(\beta) \rangle_{F_0} = \langle e_0, f_0(\beta) \rangle_{F_0}; \beta \in \pi_1(B, b_0)\};$$

We know that on theses fibers we have a right transitive action of the fundamental group and a left action of the deck transformation group. For example, if $\beta \in \pi_1(B, b_0)$ then

$$T_\beta\left(\langle b_0, \beta_1 \rangle_{b_0}\right) = \langle b_0, \beta_1 \rangle_{b_0} \star \beta^{-1} = \langle b_0, \beta_1 * \beta^{-1} \rangle_{b_0}, \tag{2.1}$$

where $\star$ denotes the right action $\left(p^{b_0}\right)^{-1}(b_0) \times \pi_1(B, b_0) \to \left(p^{b_0}\right)^{-1}(b_0)$.

In order to relate the set of coincidences between the sections with the liftings maps involved we define the following sets.

Definition 2.2.

1. $\mathfrak{L}(s_0) = \{\widetilde{s} : \widetilde{B}(b_0) \to \widetilde{E}(e_0), p^{e_0} \circ \widetilde{s} = s_0 \circ p^{b_0}\}$
2. $\mathfrak{L}(f_0) = \{\widetilde{f} : \widetilde{B}(b_0) \to \widetilde{E}(e_0), p^{e_0} \circ \widetilde{f} = f_0 \circ p^{b_0}\}$
3. $\mathfrak{L}(s_0, f_0) = \{(\widetilde{s}, \widetilde{f}); p^{e_0} \circ \widetilde{s} = s_0 \circ p^{b_0},\ p^{e_0} \circ \widetilde{f} = f_0 \circ p^{b_0}\}$
4. $\mathfrak{L}(s_0; f_{F_0}) = \{(\widetilde{s}, \widetilde{f}_0),\ s_{F_0} = p^{e_0}_{F_0} \circ \widetilde{s},\ \widetilde{s} \in \mathfrak{L}(s_0)\}$
5. $\mathfrak{L}(f_0; s_{F_0}) = \{(\widetilde{s}_0, \widetilde{f}),\ f_{F_0} = p^{e_0}_{F_0} \circ \widetilde{f},\ \widetilde{f} \in \mathfrak{L}(f_0)\}$

Lemma 2.3.

1. $\mathfrak{L}(s_0; f_{F_0}) = \left\{(\widetilde{s}, \widetilde{f}_0);\quad \widetilde{s} = T_{\gamma_1} \circ \widetilde{s}_0;\ \gamma_1 \in \pi_1(F_0, e_0)\right\}$
2. $\mathfrak{L}(f_0; s_{F_0}) = \left\{(\widetilde{s}_0, \widetilde{f});\ \widetilde{f} = T_{\gamma_2} \circ \widetilde{f}_0;\ \gamma_2 \in \pi_1(F_0, e_0)\right\}$
3. $\mathfrak{L}(s_0, f_0) = \left\{\left(T_{\alpha_1} \circ \widetilde{s}_0, T_{\alpha_2} \circ \widetilde{f}_0\right); \alpha_1, \alpha_2 \in \pi_1(E, e_0)\right\} = \mathfrak{L}(s_0) \times \mathfrak{L}(f_0)$.

Proof: Standard results in covering space theory when we identified $Cov(\widetilde{E}(e_0)/\widetilde{E}(F_0)) \simeq \pi_1(F_0, e_0)$ and $Cov(\widetilde{E}(F_0)/E) \simeq \pi_1(B, b_0) \simeq Cov(\widetilde{B}(b_0)/B)$. □

Lemma 2.4.

1. *If $\widetilde{s} \in \mathfrak{L}(s_0)$ and $\widetilde{s} = \widetilde{s}_0 \circ T_\beta$ then there is only one $\alpha(\widetilde{s}) \in \pi_1(E_0, e_0)$ so that $T_{\alpha(\widetilde{s})} \circ \widetilde{s}_0 = \widetilde{s}_0 \circ T_\beta$, moreover $\alpha(\widetilde{s}) = s_0(\beta)$.*
2. *If $\widetilde{f} \in \mathfrak{L}(f_0)$ and $\widetilde{f} = \widetilde{f}_0 \circ T_\beta$ then there is only one $\alpha(\widetilde{f}) \in \pi_1(E, e_0)$ so that $T_{\alpha(\widetilde{f})} \circ \widetilde{f}_0 = \widetilde{f}_0 \circ T_\beta$ and moreover $\alpha(\widetilde{f}) = f_0(\beta)$.*

Proof: Just use the uniqueness of each lifting and apply on the basic point $\widetilde{b}_0$ using the equality (2.1). □

Remark 2.5. *From lemmas 2.4 and 2.3 part* (3)*, for each pair $(\widetilde{s}, \widetilde{f}) \in \mathfrak{L}(s_0, f_0)$ we can write in the form*

$$\begin{aligned}
(\widetilde{s}, \widetilde{f}) &= (T_{\alpha_1} \circ \widetilde{s}_0, T_{\alpha_2} \circ \widetilde{f}_0) \\
&= (T_{\alpha_1 * s_0(q(\alpha_1^{-1})) * s_0(q(\alpha_1))} \circ \widetilde{s}_0, T_{\alpha_2 * f_0(q(\alpha_2^{-1})) * f_0(q(\alpha_2))} \circ \widetilde{f}_0) \\
&= \left(T_{\alpha_1 * s_0(q(\alpha_1^{-1}))} \circ \widetilde{s}_0 \circ T_{q(\alpha_1)},\ T_{\alpha_2 * f_0(q(\alpha_2^{-1}))} \circ \widetilde{f}_0 \circ T_{q(\alpha_2)}\right) \\
&= \left(T_{\gamma_1} \circ \widetilde{s}_0 \circ T_{q(\alpha_1)}, T_{\gamma_2} \circ \widetilde{f}_0 \circ T_{q(\alpha_2)}\right) \\
&= \left(T_{\gamma_2^{-1} * \gamma_1} \circ \widetilde{s}_0,\ \widetilde{f}_0 \circ T_{q(\alpha_2) * q(\alpha_1^{-1})}\right) \text{ or} \\
&= \left(\widetilde{s}_0 \circ T_{q(\alpha_1) * q(\alpha_2^{-1})}, T_{\gamma_1^{-1} * \gamma_2} \circ \widetilde{f}_0\right),
\end{aligned}$$

where $\gamma_1 = \alpha_1 * s_0(q(\alpha_1^{-1})) \in \pi_1(F_0, e_0)$ *and* $\gamma_2 = \alpha_2 * f_0(q(\alpha_2^{-1})) \in \pi_1(F_0, e_0)$. *Because we are considering the constant homotopy on the basic space* $\overline{I}_B : B \times [0,1] \to B$ *to deform the initial sections* s_0, f_0 *over* B, *so we assume that* $q(\alpha_1) = q(\alpha_2)$. *From this we consider only the pairs of liftings* $\left(T_{\gamma_2^{-1}*\gamma_1} \circ \widetilde{s}_0,\ \widetilde{f}_0\right) \in \mathcal{L}(s_0; f_{F_0})$ *or* $\left(\widetilde{s}_0,\ T_{\gamma_1^{-1}*\gamma_2} \circ \widetilde{f}_0\right) \in \mathcal{L}(f_0; s_{F_0})$.

Definition 2.6.

1. *Given* $(\widetilde{s}_1, \widetilde{f}_0)$ *and* $(\widetilde{s}_2, \widetilde{f}_0) \in \mathcal{L}(s_0; f_{F_0})$ *we say that* $(\widetilde{s}_1, \widetilde{f}_0)$ *is lifting related with* $(\widetilde{s}_2, \widetilde{f}_0)$ *for the* f_0, *in symbols* $\widetilde{s}_1 R_{f_0} \widetilde{s}_2$, *or* $(\widetilde{s}_1, \widetilde{f}_0) R_{f_0} (\widetilde{s}_2, \widetilde{f}_0)$, *if and only if* $T_{f_0(\beta)} \circ \widetilde{s}_1 = \widetilde{s}_2 \circ T_\beta$ *for some* $\beta \in \pi_1(B, b_0)$.

2. *Similarly for the elements* $(\widetilde{s}_0, \widetilde{f}_1)$ $(\widetilde{s}_0, \widetilde{f}_2) \in \mathcal{L}(f_0; s_{F_0})$ *we define the relation* R_{s_0} *by* $\widetilde{f}_1 R_{s_0} \widetilde{f}_2 \Leftrightarrow T_{s_0(\beta)} \circ \widetilde{f}_1 = \widetilde{f}_2 \circ T_\beta$ *for some* $\beta \in \pi_1(B, b_0)$.

Proposition 2.7.

1. *The relation* R_{f_0} *is an equivalence relation on the set* $\mathcal{L}(s_0; f_{F_0})$.

2. *The relation* R_{s_0} *is an equivalence relation on the set* $\mathcal{L}(f_0; s_{F_0})$.

Proof: If $\beta = [\overline{b}_0] \in \pi_1(B, b_0)$ then $\widetilde{s}_1 R_{f_0} \widetilde{s}_1$. If $\widetilde{s}_1 R_{f_0} \widetilde{s}_2$ with $T_{f_0(\beta)} \circ \widetilde{s}_1 = \widetilde{s}_2 \circ T_\beta$ thus $T_{f_0(\beta^{-1})} \circ \widetilde{s}_2 = \widetilde{s}_1 \circ T_{\beta^{-1}}$. If $\widetilde{s}_1 R_{f_0} \widetilde{s}_2$ and $\widetilde{s}_2 R_{f_0} \widetilde{s}_3$ which implies that there are $\beta_1, \beta_2 \in \pi_1(B, b_0)$ so that $T_{f_0(\beta_1)} \circ \widetilde{s}_1 = \widetilde{s}_2 \circ T_{\beta_1}$ and $T_{f_0(\beta_2)} \circ \widetilde{s}_2 = \widetilde{s}_3 \circ T_{\beta 2}$. Therefore,

$$\begin{aligned} T_{f_0(\beta_2 * \beta_1)} \circ \widetilde{s}_1 &= T_{f_0(\beta_2)} \circ \left(T_{f_0(\beta_1)} \circ \widetilde{s}_1\right) \\ &= T_{f_0(\beta_1)} \circ \widetilde{s}_2 \circ T_{\beta_1} \\ &= \left(T_{f_0(\beta_2)} \circ \widetilde{s}_2\right) \circ T_{\beta_1} \\ &= \widetilde{s}_3 \circ T_{\beta_2} \circ T_{\beta_1} \\ &= \widetilde{s}_3 \circ T_{\beta_2 * \beta_1} \end{aligned}$$

The proof of (2) is analogous. □

Let $R_{f_0}(\mathcal{L}(s_0; f_{F_0}))$ and $R_{s_0}(\mathcal{L}(f_0; s_{F_0}))$ be the quotient spaces by the relations R_{f_0} and R_{s_0} on the spaces $\mathcal{L}(s_0; f_{F_0})$ and $\mathcal{L}(f_0, s_{F_0})$ respectively. Denote by $r_{f_0}(\mathcal{L}(s_0; f_{F_0}))$ and $r_{s_0}(\mathcal{L}(f_0; s_{F_0}))$ the respective cardinals of the quotient spaces.

The following definition is approximation between the relation R_{f_0}, or R_{s_0}, and the Reidemeister relation relative to the subgroup $\pi_1(F_0, e_0)$ as we will view in the the next section.

Definition 2.8. *Let* $\widetilde{s}_1 = T_{\gamma_1} \circ \widetilde{s}_0, \widetilde{s}_2 = T_{\gamma_2} \circ \widetilde{s}_0$ *be in* $\mathcal{L}(s_0; f_{F_0})$ *where* $\gamma_1, \gamma_2 \in \pi_1(F_0, e_0)$. *We say that* $\widetilde{s}_1$ *is lifting related with* $\widetilde{s}_2$, *in symbol* $\widetilde{s}_1 R_L \widetilde{s}_2$, *if there is* $\beta \in \pi_1(B, b_0)$ *so that* $f_0(\beta) * \gamma_1 = \gamma_2 * s_0(\beta)$. *Similarly we define for* $\widetilde{f}_1 = T_{\gamma_1} \circ \widetilde{f}_0, \widetilde{f}_2 = T_{\gamma_2} \circ \widetilde{f}_0$ *be in* $\mathcal{L}(f_0; s_{F_0})$. *That is* $\widetilde{f}_1 R_L \widetilde{f}_2$ *if and only if there is* $\beta \in \pi_1(B, b_0)$ *such that* $f_0(\beta) * \gamma_1 = \gamma_2 * s_0(\beta)$.

Theorem 2.9.

1. *The relation R_L defined on $\mathcal{L}(s_0; f_{F_0})$is an equivalence relation.*

2. *The relation R_L defined on $\mathcal{L}(f_0; s_{F_0})$ is an equivalence relation.*

3. $[\widetilde{s}]_{f_0} = [\widetilde{s}]_L$ *and* $[\widetilde{f}]_{s_0} = [\widetilde{f}]_L$. *Therefore if* $R_L(\mathcal{L}(s_0; f_{F_0}))$ *and* $R_L(\mathcal{L}(f_0; s_{F_0}))$ *are the quotient set by the relation R_L then there is an one to one correspondence between the followings sets:*

$$R_{f_0}(\mathcal{L}(s_0; f_{F_0})) \leftrightarrow R_L(\mathcal{L}(s_0; f_{F_0})) \leftrightarrow R_{s_0}(\mathcal{L}(f_0; s_{F_0})) \leftrightarrow R_L(\mathcal{L}(f_0; s_{F_0})).$$

Proof: We will prove the item (1). Obviously the relation R_L is reflexive and symmetric. If $\widetilde{s}_i = T_{\gamma_i} \circ \widetilde{s}_0$, for $i = 1, 2, 3$ and $\widetilde{s}_1 R_L \widetilde{s}_2$ and $\widetilde{s}_2 R_L \widetilde{s}_3$ then there exists β_1 and β_2 in $\pi_1(B, b_0)$ such that $f_0(\beta_1) * \gamma_1 = \gamma_2 * s_0(\beta_1)$ and $f_0(\beta_2) * \gamma_2 = \gamma_3 * s_0(\beta_2)$. So we have

$$\begin{aligned} f_0(\beta_2 * \beta_1) * \gamma_1 &= f_0(\beta_2) * (f_0(\beta_1) * \gamma_1) \\ &= f_0(\beta_2) * \gamma_2 * s_0(\beta_1) \\ &= \gamma_3 * s_0(\beta_2) * s_0(\beta_1). \end{aligned}$$

Therefore $\widetilde{s}_1 R_L \widetilde{s}_3$. The proof of (2) is analogous.

In fact $[\widetilde{s}_1]_{f_0} = [\widetilde{s}_1]_L$. If $\widetilde{s}_1 R_{f_0} \widetilde{s}_2$, then there is $\beta \in \pi_1(B, b_0)$ such that $T_{f_0(\beta)} \circ \widetilde{s}_1 = \widetilde{s}_2 \circ T_\beta$. But $\widetilde{s}_1, \widetilde{s}_2 \in \mathcal{L}(s_0; f_{F_0})$ so there are $\gamma_1, \gamma_2 \in \pi_1(F_0, e_0)$ such that $\widetilde{s}_1 = T_{\gamma_1} \circ \widetilde{s}_0$ and $\widetilde{s}_2 = T_{\gamma_2} \circ \widetilde{s}_0$. Since $\widetilde{s}_1 R_{f_0} \widetilde{s}_2$ we have:

$$\begin{aligned} T_{f_0(\beta)} \circ \widetilde{s}_1 &= \widetilde{s}_2 \circ T_\beta \\ T_{f_0(\beta)} \circ T_{\gamma_1} \circ \widetilde{s}_0 &= T_{\gamma_2} \circ \widetilde{s}_0 \circ T_\beta \\ T_{f_0(\beta) * \gamma_1} \circ \widetilde{s}_0 &= T_{\gamma_2 * s_0(\beta)} \circ \widetilde{s}_0 \end{aligned}$$

The last equation means that $\widetilde{s}_1 R_L \widetilde{s}_2$. Therefore there is an one to one correspondence betwen the sets. The second part is analogous. □

3. Algebraic Reidemeister classes relative of a subgroup

Definition 3.1. *Let $\psi, \varphi : G_1 \to G_0$ be group homomorphisms and H_0 a subgroup of G_0. We say that two elements $h_1, h_2 \in H_0$ are $(\psi, \varphi;\ H)-$algebraic Reidemeister related, in symbols $h_1 R_{(\psi,\varphi;H_0)} h_2 = h_1 R_{H_0} h_2$ or $h_1 R_A h_2$, if there is $g \in G_0$ such that $\varphi(g) h_1 = h_2 \psi(g)$.*

It is easy to prove that $R_{(\psi,\varphi;H_0)}$ is an equivalence relation on H_0, called the algebraic Reidemeister relation of φ and ψ relative to the subgroup H_0. We denoted by $[h]_{(\psi,\varphi;H_0)} = [h]_{H_0}$ or $[h]_A$ the algebraic Reidemeister class determined by $h \in H_0$ and by $R_A(\varphi, \psi; H_0)$ to the quotient set. The cardinal of $R_A(\varphi, \psi; H_0)$ which is indicated by $r(\varphi, \psi; H_0)$ is called $(\varphi, \psi;\ H_0)-$Reidemeister number. When $H_0 = G_0$ we denoted $R(\varphi, \psi;\ G_0) = R(\varphi, \psi)$ and $r(\varphi, \psi;\ G_0) = r(\varphi, \psi)$.

Proposition 3.2. *Let $\varphi, \psi : G_1 \to G_0$ be homomorphisms and H_0, K_0 subgroups of G_0. If $H_0 \leq K_0$ then $r(\varphi, \psi; H_0) \leq r(\varphi, \psi; K_0)$.*

Proof: Just set the injection $R_A(\varphi,\psi;H_0) \hookrightarrow R_A(\varphi,\psi;K_0)$, $[a]_{H_0} \mapsto [a]_{K_0}$. $\square$

Remark 3.3. *If $\{e_{G_0}\}$ is the trivial subgroup of G_0 then for any subgroup H_0 of G_0 we have $1 = r(\varphi,\psi;\{e_{G_0}\}) \leq r(\varphi,\psi;H_0) \leq r(\varphi,\psi)$.*

Proposition 3.4. *Let $\varphi,\psi : G_2 \to G_1$ be homomorphisms, $K_1 \leq G_1$ and $\Phi : G_1 \to G_0$ a homomorphism with $H_0 = \Phi(K_1)$. The following map $\Phi_A : R_A(\varphi,\psi;\,K_1) \to R_A(\Phi\circ\varphi,\Phi\circ\psi;H_0))$ given by $\Phi_A([k]_{K_1}) = [\Phi(k)]_{H_0}$ is surjective. Therefore $r(\varphi,\psi;K_1) \geq r(\Phi\circ\varphi,\Phi\circ\psi;H_0)$.*

If Φ has the left inverse homomorphism $\Psi : G_0 \to G_1$ then Φ_A is an one to one correspondence and $\Phi_A^{-1}[\Phi(k)]_{H_0} = [k]_{K_1}$ so $r(\varphi,\psi;K_1) = r(\Phi\circ\varphi,\Phi\circ\psi;H_0)$.

Proof: If $[k_1]_{K_1} = [k_2]_{K_1}$ there is g_2 such that $\varphi(g_2)k_1 = k_2\psi(g_2)$, then $\Phi(\varphi(g_2))\Phi(k_1) = \Phi(k_2)\Phi(\psi(g_2))$. Therefore we have a well defined map $\Phi_{K_1} : R_A(\varphi,\psi;K_1) \to R_A(\Phi\circ\varphi,\Phi\circ\psi;H_0)$ given by $[k_1]_{K_1} \mapsto [\Phi(k_1)]_{H_0}$. As $H_0 = \Phi(K_1)$, it is easy to prove that the map Φ_A is surjective.

Otherwise if $\Psi : G_0 \to G_1$ is a left inverse of Φ then when we apply Ψ in the equation $\Phi(\varphi(g_2))\Phi(k_1) = \Phi(k_2)\Phi(\psi(g_2))$ we have a well defined map $\Psi_{\Phi(K_0)} : R_A(\Phi\circ\varphi,\Phi\circ\psi;\Phi(K_1)) \to R_A(\varphi,\psi;K_1)$ such that $\Phi_{K_1}\circ\Psi_{\Phi(K_1)}$ is identity of $R_A(\varphi,\psi;K_1)$

Therefore Φ_{K_1} is an one to one correspondence and we have the equivalence on the Reidemeister numbers $r(\varphi,\psi;\,K_1) = r(\Phi\circ\varphi,\Phi\circ\psi;H_0)$ with $H_0 = \Phi(K_1)$ $\square$

Example 3.5 (Case trivial fiber bundle)**.** *Let $f,g : (B,b_0) \to (F,y_0)$ be continuous maps. So we have $f,g : \pi_1(B,b_0) \to \pi_1(F,y_0)$ and the set of algebraic Reidemeister classes $R_A(f,g)$. Now we consider the trivial fiber bundle $q : (B\times F,(b_0,y_0)) \to (B,b_0)$ so the maps f,g induces two sections $s_f,s_g : (B,b_0) \to (B\times F,(b_0,y_0))$ given by $s_f(b) = (b,f(b))$ and $s_g(b) = (b,g(b))$. Let $F_0 = \{b_0\}\times F = q^{-1}(b_0)$ be the fiber over b_0 with base point $e_0 = (b_0,y_0)$ so $\pi_1(F_0)$. Then we can consider the algebraic classes of Reidemeister $R_A(s_g,s_f;\pi_1(F_0,e_0))$ and $\Phi(\pi_1(F_0,b_0,y_0)) = \pi_1(F,y_0)$. Then we conclude that:*

$$\begin{aligned} R_A(s_g,s_f;\pi_1(F_0,e_0)) &\leftrightarrow R_A(\Phi\circ s_g,\Phi\circ s_f;\Phi(\pi_1(F_0,e_0))) \\ &\leftrightarrow R_A(g,f;\pi_1(F,y_0)) = R_A(g,f) \end{aligned} \tag{3.1}$$

Example 3.6 (Case not trivial fiber bundle)**.** *We consider two sections $s_0,f_0 : (B,b_0) \to (E,e_0)$ of the fiber bundle $q : (E,e_0) \to (B,b_0)$. We used s_0 to describe the structure of the group $\pi_1(E,e_0)$ as the semi direct product $\pi_1(F_0,e_0)\rtimes\pi_1(B,b_0)$. Formally, let $\Phi : \pi_1(E,e_0) \to \pi_1(F_0,e_0)\rtimes\pi_1(B,b_0)$ be the isomorphism given by*

$$\Phi(\alpha) = \Big(\alpha * s_0\left(q\left(\alpha^{-1}\right)\right), q(\alpha)\Big) \in \pi_1\left(F_0,e_0\right)\rtimes\pi_1\left(B,b_0\right), \tag{3.2}$$

The operation on $\pi_1(F_0,e_0)\rtimes\pi_1(B,b_0)$ is expressed by

$$(\gamma_1,\beta_1)\bullet(\gamma_2,\beta_2) := \Big(\gamma_1 * s_0(\beta_1) * \gamma_2 * s_0(\beta_1^{-1}),\ \beta_1 * \beta_2\Big). \tag{3.3}$$

Let $\Psi := \Phi^{-1} : \pi_1(F_0, e_0) \rtimes \pi_1(B, b_0) \to \pi_1(E, e_0)$ be the inverse isomorphism of Φ given by $\Psi(\gamma, \beta) = \gamma * s_0(\beta_1^{-1})$ and let $H_0 = \pi_1(F_0, e_0) \times \{[\overline{b}_0]\} = \Phi(\pi_1(F_0, e_0))$ be the subgroup of $\pi_1(F_0, e_0) \rtimes \pi_1(B, b_0)$. By the proposition 3.4 we have $R_A(s_0, f_0; \pi_1(F_0, e_0)) \leftrightarrow R_A(\Phi \circ s_0, \Phi \circ f_0; H_0)$.

For the operation $\bullet$ in $\pi_1(F_0, e_0) \rtimes \pi_1(E, e_0)$ expressed in (3.3) when we describe the classes of $R_A(\Phi \circ s_0, \Phi \circ f_0; H_0)$ we have the same classes on

$$R_A(s_0, f_0; \pi_1(F_0, e_0))$$

$$\begin{array}{rcl} \Phi \circ s_0(\beta) \bullet (\gamma_1, [\overline{b}_0]) & = & (\gamma_2, [\overline{b}_0]) \bullet (\Phi \circ f_0(\beta)) \\ ([\overline{e}_0], \beta) \bullet (\gamma_1, [\overline{b}_0]) & = & (\gamma_2, [\overline{b}_0]) \bullet (f_0(\beta) * s_0(\beta^{-1}), \beta) \\ (s_0(\beta) * \gamma_1 * s_0(\beta^{-1}), \beta) & = & (\gamma_2 * f_0(\beta) * s_0(\beta^{-1}), \beta) \\ s_0(\beta) * \gamma_1 & = & \gamma_2 * f_0(\beta). \end{array} \tag{3.4}$$

4. The coincidence set and the Nielsen classes for sections on the fiber bundle

Let $s_0, f_0 : (B, b_0) \to (E, e_0)$ be the sections of a fiber bundle $q : (E, e_0) \to (B, b_0)$ and $\Gamma_E^B(s_0, f_0) = \{b \in B, s_0(b) = f_0(b)\} \neq \emptyset$ be the coincidence topological space induced from B. Note that $\Gamma_E^B(s_0, f_0) = s_0^{-1}(f_0(B)) = f_0^{-1}(s_0(B))$.

In $\Gamma_E^B(s_0, f_0)$ we defined the Nielsen classes for $b_1, b_2 \in \Gamma_E^B(s_0, f_0)$ saying that b_1 is Nielsen related to b_2, in symbols $b_1 N b_2$, if and only if there is a path $\beta_{b_2}^{b_1}$ on B connecting b_1 to b_2 such that $s_0(\beta_{b_2}^{b_1})$ is homotopic to $f_0(\beta_{b_2}^{b_1})$ relative to $\{0, 1\}$. It easy is to verify that N is an equivalent relation and we denote by $[b_1]_N$ the class determined by b_1. If $\widetilde{\Gamma}_E^B(s_0, f_0)$ is the quotient set of $\Gamma_E^B(s_0, f_0)$ by the Nielsen relation, we denote by $p_N : \Gamma_E^B(s_0, f_0) \to \widetilde{\Gamma}_E^B(s_0, f_0)$ the canonical projection map.

Considering $\widetilde{\Gamma}_E^B(s_0, f_0)$ with the topology co-induced by p_N we have the following statements.

Theorem 4.1.

1. *If $[b_1]_{cc}$ is the connected component by path of $b_1 \in \Gamma_E^B(s_0, f_0)$ then $[b_1]_{cc} \subset [b_1]_N$.*

2. *If E is a Hausdorff topological space then $\Gamma_E^B(s_0, f_0)$ is closed in B.*

3. *If B is locally path connected and E is Hausdorff and semilocally 1-connected topological space then $\widetilde{\Gamma}_E^B(s_0, f_0)$ is discrete topological space.*

4. *If B and E satisfies the before conditions and $\Gamma_E^B(s_0, f_0)$ is compact then $\widetilde{\Gamma}_E^B(s_0, f_0)$ is finite.*

Proof: The $(1), (2)$ and (4) items are easy to prove. We will prove only the item (3). Let $b_2 \in [b_1]_N$ and consider an open set V_{e_2} such that $i : \pi_1(V_{e_2}, e_2) \to \pi_1(E, e_2)$ is trivial homomorphism. Now $W_{b_2} = s_0^{-1}(V_{e_2}) \cap f_0(V_{e_2}) \cap U_{b_2}$ where U_{b_2} is connected path neighborhood of b_2. It is immediate to verify that $W_{b_2} \subset [b_1]_N$ so

$p_N^{-1}([b_1]_N) = \bigcup\limits_{b_2 \in [b_1]} W_{b_2}$. In others words $[b_1]_N$ is open and closed set in $\widetilde{\Gamma}_E^B(s_0, f_0)$.

$\square$

In this work we assume that B, E and F_0 are compact, ENR (Euclidean neighborhood Retracts), path connected, locally path connected and semilocally 1-connected space so the set $\widetilde{\Gamma}_E^B(s_0, f_0)$ is finite.

Let $\mathcal{B}$ be the collection of all the pairs $\left(b_i; \beta_{b_i}^{b_0}\right)$, where $\beta_{b_i}^{b_0}$ is a path on B connecting b_0 to $b_i \in \Gamma_E^B(s_0, f_0)$. Let $\gamma(b_i) \in \pi_1(F_0, e_0)$ be the homotopy class given by the loop $s(\beta_{b_i}^{b_0}) * f(\beta_{b_i}^{b_0})^{-1}$. Now we define $P_R : \mathcal{B} \to R_A(s_0, f_0; \pi_1(F_0, e_0))$ given by $P_R(b_i; \beta_{b_i}^{b_0}) = \left[[s_0(\beta_{b_i}^{b_0}) * f_0(\beta_{b_i}^{b_0})^{-1}]\right]_A = [\gamma(b_i)]_A$ and $P_N : \mathcal{B} \to \widetilde{\Gamma}_E^B(s_0, f_0)$ by $P_N(b_i, \beta_{b_i}^{b_0}) = [b_i]_N$

Theorem 4.2. *The map P_R does not depend of the path $\beta_{b_1}^{b_0}$, it is that, if $(b_1, \beta_{b_1}^{b_0}(1)), (b_1, \beta_{b_1}^{b_0}(2)) \in \mathcal{B}$ then $P_R(b_1, \beta_{b_1}^{b_0}(1)) = P_R(b_1, \beta_{b_1}^{b_0}(2))$. The map P_R splits by P_N through the injective map $\overline{P}_R$ on the bellow diagram, so that we have $|\widetilde{\Gamma}_E^B(s_0, f_0)| \leq r_A(s_0, f_0; \pi_1(F_0, e_0))$.*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "P_R"] \arrow[d, "P_N"] & R_A(s_0, f_0; \pi_1(F_0, e_0)) \\
\widetilde{\Gamma}_E^B(s_0, f_0) \arrow[ur, "\overline{P}_R"] &
\end{tikzcd}$$

Proof: For the first part we consider $\beta = \beta_{b_1}^{b_0}(2) * (\beta_{b_1}^{b_0}(1))^{-1}$. Now

$$\begin{aligned}
P_R(b_1, \beta_{b_1}^{b_0}(1)) &= [[s_0(\beta_{b_1}^{b_0}(1) * f_0(\beta_{b_1}^{b_0}(1))^{-1}]]_A \\
&= [[s_0(\beta) * (s_0(\beta_{b_1}^{b_0}(1) * f_0(\beta_{b_1}^{b_0}(1))^{-1}) * f_0(\beta)^{-1}]]_A \\
&= P_R(b_1, \beta_{b_1}^{b_0}(2))
\end{aligned}$$

For the second part, if $(b_1, \beta_{b_1}^{b_0}), (b_2, \beta_{b_2}^{b_0}) \in \mathcal{B}$ and $[b_1]_N = [b_2]_N$ on $\widetilde{\Gamma}_E^B(s_0, f_0)$ then there is a path $\beta_{b_2}^{b_1}(N)$ between b_1 and b_2 such that $f_0(\beta_{b_2}^{b_1}(N))$ is homotopic to $s_0(\beta_{b_2}^{b_1}(N))$ relative to $\{0, 1\}$. So we have:

$$\begin{aligned}
P_R(b_1, \beta_{b_1}^{b_0}) &= [[s_0(\beta_{b_1}^{b_0}) * f(\beta_{b_1}^{b_0})^{-1}]]_A \\
&= [[s_0(\beta_{b_1}^{b_0}) * s_0(\beta_{b_2}^{b_1}(N)) * f(\beta_{b_2}^{b_1}(N))^{-1} * f(\beta_{b_1}^{b_0})^{-1}]]_A \\
&= [[s_0(\beta_{b_2}^{b_0}) * f_0(\beta_{b_2}^{b_0})^{-1}]]_A = P_R(b_2, \beta_{b_2}^{b_0})
\end{aligned}$$

So $\overline{P}_R([b_1]) = P_R(b_1, \beta_{b_1}^{b_0})$ is a well defined map as on the commutative diagram and it is easy to see that $\overline{P}_R$ is an injection. $\square$

Theorem 4.3. *Let $\Gamma_{\widetilde{E}(e_0)}^{\widetilde{B}(b_0)}(\widetilde{s}_0, \widetilde{f}_1)$ and $\Gamma_{\widetilde{E}(e_0)}^{\widetilde{B}(b_0)}(\widetilde{s}_0, \widetilde{f}_2)$ be the coincidence set for $\widetilde{f}_1, \widetilde{f}_2 \in \mathcal{L}(f_0; s_{F_0})$.*

1. *If $[\widetilde{f}_1]_{s_0} = [\widetilde{f}_2]_{s_0}$ then $p^{b_0}\left(\Gamma_{\widetilde{E}(e_0)}^{\widetilde{B}(b_0)}(\widetilde{s}_0, \widetilde{f}_1)\right) = p^{b_0}\left(\Gamma_{\widetilde{E}(e_0)}^{\widetilde{B}(b_0)}(\widetilde{s}_0, \widetilde{f}_2)\right)$.*

2. If $p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_1)\right) \cap p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_2)\right) \neq \emptyset$ *then* $[\widetilde{f}_1]_{s_0} = [\widetilde{f}_2]_{s_0}$.

Proof: (1). Since $[\widetilde{f}_1]_{s_0} = [\widetilde{f}_2]_{s_0}$ there is $\beta \in \pi_1(B, b_0)$ which satisfies $T_{s_0(\beta)} \circ \widetilde{f}_1 = \widetilde{f}_2 \circ T_\beta$. If $\widetilde{b} \in \left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}\left(\widetilde{s}_0, \widetilde{f}_1\right)\right)$ then $\widetilde{s}_0(\widetilde{b}) = \widetilde{f}_1(\widetilde{b})$, so we have

$$\begin{array}{rcl} \widetilde{s}_0 \circ T_\beta(\widetilde{b}) & = & T_{s_0(\beta)} \circ \widetilde{s}_0(\widetilde{b}) \\ & = & T_{s_0(\beta)} \circ \widetilde{f}_1(\widetilde{b}) \\ & = & \widetilde{f}_2 \circ T_\beta(\widetilde{b}). \end{array}$$

Therefore $T_\beta(\widetilde{b}) \in \Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}_{e_0}}\left(\widetilde{s}_0, \widetilde{f}_2\right)$. The verification of the inverse inclusion is analogous. Since T_β established an one to one correspondence between $\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}\left(\widetilde{s}_0, \widetilde{f}_1\right)$ and $\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}_{e_0}}\left(\widetilde{s}_0, \widetilde{f}_2\right)$ then when we apply p^{b_0} we have

$$p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_1)\right) = p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_2)\right).$$

Since (3) is equivalent to (2) it is sufficient to prove the item (2). If

$$p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_1)\right) \cap p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_2)\right) \neq \emptyset,$$

then $\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_1) \neq \emptyset$ and $\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_2) \neq \emptyset$. Then there are $\widetilde{b}_1 \in \Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_1)$ and $\widetilde{b}_2 \in \Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_2)$ such that $p^{b_0}(\widetilde{b}_1) = p^{b_0}(\widetilde{b}_2) := b$. Since the action of the fundamental group $\pi_1(B, b_0)$ on the fibers is transitive, there is $\beta \in \pi_1(B, b_0)$ such that $T_\beta(\widetilde{b}_1) = \widetilde{b}_2$ and

$$\begin{array}{rcl} \widetilde{e} := \widetilde{f}_2(\widetilde{b}_2) & = & \widetilde{s}_0(\widetilde{b}_2) \\ \widetilde{f}_2 \circ T_\beta(\widetilde{b}_1) & = & \widetilde{s}_0 \circ T_\beta(\widetilde{b}_1) \\ & = & T_{s_0(\beta)} \circ \widetilde{s}_0(\widetilde{b}_1) \\ & = & T_{s_0(\beta)} \circ \widetilde{f}_1(\widetilde{b}_1) \end{array}$$

Since $\widetilde{f}_1, \widetilde{f}_2 \in \mathcal{L}(f_0; s_{F_0})$ and the coincidence occurs in the $\widetilde{b}_1$ it follows that $\widetilde{f}_2 \circ T_\beta = T_{s_0(\beta)} \circ \widetilde{f}_1$ as the bellow diagram. Therefore we have $[\widetilde{f}_2]_{s_0} = [\widetilde{f}_1]_{s_0}$.

$$\begin{array}{ccccc} (\widetilde{B}(b_0), \widetilde{b}_2) & \xrightarrow{\widetilde{f}_2} & (\widetilde{E}(e_0), \widetilde{e}) & \xleftarrow{T_{s_0(\beta)}} & (\widetilde{E}(e_0), \widetilde{f}_1(\widetilde{b}_1)) \\ {\scriptstyle T_\beta}\uparrow & & \downarrow{\scriptstyle p^{e_0}_{F_0}} & & \uparrow{\scriptstyle \widetilde{f}_1} \\ (\widetilde{B}(b_0), \widetilde{b}_1) & \xrightarrow{s_{F_0}} & (\widetilde{E}(F_0), s_{F_0}(\widetilde{b}_1)) & \xleftarrow{s_{F_0}} & (\widetilde{B}(b_0), \widetilde{b}_1) \end{array}$$

$\square$

Theorem 4.4. *Let* $\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{f}_0, \widetilde{s}_1)$ *and* $\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{f}_0, \widetilde{s}_2)$ *be the coincidence set for* $\widetilde{s}_1, \widetilde{s}_2 \in \mathcal{L}(s_0; f_{F_0})$.

1. *If* $[\widetilde{s}_1]_{f_0} = [\widetilde{s}_2]_{f_0}$ *then* $p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{f}_0, \widetilde{s}_1)\right) = p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{f}_0, \widetilde{s}_2)\right)$.

2. *If* $p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{f}_0, \widetilde{s}_1)\right) \cap p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{f}_0, \widetilde{s}_2)\right) \neq \emptyset$ *then* $[\widetilde{s}_1]_{f_0} = [\widetilde{s}_2]_{f_0}$.

Now, $[\widetilde{s}_1]_{f_0} = [\widetilde{s}_2]_L \in R_L(\mathcal{L}(\widetilde{s}_0; f_{F_0})$ by theorem 2.9. If $\widetilde{s}_1 = T_{\gamma_1} \circ \widetilde{s}_0$ and $\widetilde{s}_2 = T_{\gamma_2} \circ \widetilde{s}_0$ with $\gamma_1, \gamma_2 \in \pi_1(F_0, e_0)$ then, by definition 2.8, we have $[T_{\gamma_1} \circ \widetilde{s}_0]_L = [T_{\gamma_2} \circ \widetilde{s}_0]_L$ if and only if $[\gamma_1]_A = [\gamma_2]_A \in R_A(s_0, f_0; \pi_1(F_0, e_0))$. From this, it follows the main theorem:

Theorem 4.5. *Let* $\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_1)$ *and* $\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_2)$ *be the coincidence set for* $\widetilde{f}_1, \widetilde{f}_2 \in \mathcal{L}(f_0; s_{F_0})$.

1. *There is an one to one correspondence*

$$\Psi : R_L(\mathcal{L}(f_0; s_{F_0})) \to R_A(f_0, s_0; \pi_1(F_0, e_0)).$$

2. *If* $[\widetilde{f}_1]_L = [\widetilde{f}_2]_L$ *then* $p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_1)\right) = p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_2)\right)$.

3. *If* $p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_1)\right) \cap p^{b_0}\left(\Gamma^{\widetilde{B}(b_0)}_{\widetilde{E}(e_0)}(\widetilde{s}_0, \widetilde{f}_2)\right) \neq \emptyset$ *then* $[\widetilde{f}_1]_L = [\widetilde{f}_2]_L$.

Remark 4.6. *Note that the theorem follows from the theorems 2.1 and 4.3, and is true if we replace* $f_0, \widetilde{f}_1, \widetilde{f}_2$ *by* $s_0, \widetilde{s}_1, \widetilde{s}_2$ *and* s_{F_0} *by* f_{F_0}.

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