Bol. Soc. Paran. Mat. (3s.) **v. 38** 6 (2020): 73–83.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v38i6.39291

Unified Integrals Involving Product of Multivariable Polynomials and Generalized Bessel Functions

K.S. Nisar, D.L. Suthar, S.D. Purohit, Hafte Amsalu

ABSTRACT: The intention of the present article is to evaluate two integral formulas associated with a finite product of the generalized Bessel functions of the first kind and multivariable polynomials. The results are formulated in terms of the generalized Lauricella functions. The major outcome conferred here are of general aspect and simply reducible to unique and widely known integral formulae.

Key Words: Multivariable polynomial function, Gamma function, Ggeneralized hypergeometric function ${}_pF_q$, Generalized Lauricella functions, Generalized Bessel function, Oberhettinger's integral formula.

Contents

1 Introduction and Preliminaries **73**

2 Noted Results **76**

3 Particular Cases **80**

4 Concluding Remarks **82**

1. Introduction and Preliminaries

A remarkable authors have developed a huge amount of integral formulas involving a different type of special functions for instance; Suthar *et al.* [19] evaluated unified integrals associated with the hypergeometric function; Choi *et al.* [5], Choi *et al.* [6], Menaria *et al.* [10], Nisar *et al.* [11] and Suthar and Habenom [18] established certain integrals involving Bessel type functions. In this paper, we explore the possibility to obtain certain new integrals involving generalized Bessel function of the first kind and multivariable polynomials.
Recently, the generalized Bessel function of the first kind $w_v(z)$, studied and introduce by Baricz [3], as follow:

$$w_{\upsilon,b,c}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k \left(\frac{z}{2}\right)^{\upsilon+2k}}{k\,!\Gamma\left(\upsilon+k+\frac{b+1}{2}\right)}, \tag{1.1}$$

where $\Gamma(z)$ is a gamma function (see [14], Section 1.1). For additional details of function defined by (1.1), one may refer to [3].

2010 *Mathematics Subject Classification:* 33C05, 33C20, 33C70.
Submitted September 03, 2017. Published December 25, 2017

Here, Bessel function of the first kind $J_v(z)$ and $I_v(z)$ are often accounted in considering solutions of differential type equations and they are connected beside a broad scope of issues in significant fields of mathematical physics, like issues of acoustics, radio physics, hydrodynamics and atomic, nuclear-physics, probability theory and statics. These thoughts have led a variety of hands in the meadow of special functions for analyzing them, likely extensions, fundamental properties and applications for the Bessel functions (see, [2], [4], [7], [8], [13], [20], [17]).
Also, we recall the following multivariable polynomial $S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[z]$ introduced by Srivastava ([16], p, 185, eq.(7)) defined and represented as:

$$S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}[z] = \sum_{l_1=0}^{n_1/m_1} \cdots \sum_{l_r=0}^{n_r/m_r} \prod_{i=1}^{r} \left[\frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,l_i} (z)^{\,l_i} \right]. \tag{1.2}$$

Throughout this paper, we consider $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}^-$ and $\mathbb{N}_0$ as set of complex numbers, set of positive integers, set of negative integers and set of positive integers including zero respectively.
The generalized Lauricella function see [15] is defined as:

$$F_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

$$= F_{C:D^{(1)};\ldots;D^{(n)}}^{A:B^{(1)};\ldots;B^{(n)}} \left(\begin{matrix} \left[(a):\theta^{(1)};\ldots;\theta^{(n)}\right] : \left[(b)^{(1)}:\phi^{(1)}\right];\ldots;\left[(b)^{(n)}:\phi^{(n)}\right] \\ \left[(c):\psi^{(1)};\ldots;\psi^{(n)}\right] : \left[(d)^{(1)}:\delta^{(1)}\right];\ldots;\left[(d)^{(n)}:\delta^{(n)}\right] \end{matrix} \; z_1, \ldots, z_n \right) \tag{1.3}$$

$$= \sum_{k_1,\ldots,k_n=0}^{\infty} \Omega\,(k_1, \ldots, k_n) \frac{z_1^{k_1}}{k_1!}, \ldots, \frac{z_n^{k_n}}{k_n!},$$

where, for convenience

$$\Omega\,(k_1, \ldots, k_n) = \frac{\prod_{j=1}^{A} (a_j)_{k_1\theta_j^1+\ldots+k_n\theta_j^n} \prod_{j=1}^{B^{(1)}} \left(b_j^{(1)}\right)_{k_1\phi_j^1} \cdots \prod_{j=1}^{B^{(n)}} \left(b_j^{(n)}\right)_{k_n\phi_j^n}}{\prod_{j=1}^{C} (c_j)_{k_1\psi_j^1+\ldots+k_n\psi_j^n} \prod_{j=1}^{D^{(1)}} \left(d_j^{(1)}\right)_{k_1\delta_j^1} \cdots \prod_{j=1}^{D^{(n)}} \left(d_j^{(n)}\right)_{k_n\delta_j^n}}, \tag{1.4}$$

The coefficients defined as follows

$$\begin{cases} \theta_j^{(m)} (j = 1, \ldots, A);\ \phi_j^{(m)} (j = 1, \ldots, B^{(m)}); \\ \psi_j^{(m)} (j = 1, \ldots, C);\ \delta_j^{(m)} (j = 1, \ldots, D^{(m)}); \end{cases} \forall\, m \in \{1, \ldots, n\}, \tag{1.5}$$

are real, positive and (a) shortens the array of A parameters $a_1, a_2, \ldots, a_A$, $\left(b^{(m)}\right)$ shortens the array of $B^{(m)}$ parameters $b_j^{(m)} \left(j = 1, \ldots, B^{(m)}\right)$;$\forall\, m \in \{1, \ldots, n\}$, with identical explanations for (c) and $\left(d^{(m)}\right)$ $(m = 1, \ldots, n)$.

Further, the multiple series (1.2), converges absolutely either

1. $\Delta_i > 0$ $(i = 1, 2, \ldots, n), \forall z_1, \ldots, z_n \in \mathbb{C}$,

2. $\Delta_i = 0$ $(i = 1, 2, \ldots, n)$, $\forall z_1, \ldots, z_n \in \mathbb{C}$, $|\mathrm{z_i}| < \rho_i$ $(i = 1, 2, \ldots, n)$.

The multiple series defined in (1.2) is divergent when $\Delta_i < 0\,(i = 1, 2, \ldots, n)$ omitting for the worthless case $z_1 = 0, \ldots, z_n = 0$. Here

$$\Delta_i \equiv 1 + \sum_{j=1}^{C} \psi_j^{(i)} + \sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{A} \theta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)} \quad (i = 1, ..., n), \tag{1.6}$$

$$\rho_i = \min_{\mu_1, \ldots, \mu_n \geq 0} \{E_i\} \qquad (i = 1, 2, ..., n), \tag{1.7}$$

with

$$E_i = (\mu_i)^{1+\sum_{j=1}^{D^{(i)}} \delta_j^{(i)} - \sum_{j=1}^{B^{(i)}} \phi_j^{(i)}} \frac{\left\{\prod_{j=1}^{C} \left(\sum_{i=1}^{n} \mu_i \psi_j^{(i)}\right)^{\psi_j^{(i)}}\right\} \left\{\prod_{j=1}^{D^{(i)}} \left(\delta_j^{(i)}\right)^{\delta_j^{(i)}}\right\}}{\left\{\prod_{j=1}^{A} \left(\sum_{i=1}^{n} \mu_i \theta_j^{(i)}\right)^{\theta_j^{(i)}}\right\} \left\{\prod_{j=1}^{B^{(i)}} \left(\phi_j^{(i)}\right)^{\phi_j^{(i)}}\right\}}. \tag{1.8}$$

We also required (see [14], Section 1.5)

$$\begin{aligned} {}_pF_q &= \left[\begin{array}{cc} \alpha_1, \cdots, \alpha_p & ; \\ \beta_1, \cdots, \beta_q & ; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n \, z^n}{(\beta_1)_n \cdots (\beta_q)_n \, n!} \\ &= {}_pF_q \left(\alpha_1, \ldots, \alpha_p; \beta_1 \ldots, \beta_q; z\right). \end{aligned} \tag{1.9}$$

where $(\lambda)_n$ is the pochhammer symbol defined as:

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)},$$

where $\lambda \in \mathbb{C}$ (see [14], p. 2 and pp. 4-6).

The Oberhettinger's integral formula [12] is defined as:

$$\int_0^\infty x^{\sigma-1} \left(x + a + \sqrt{x^2 + 2ax}\right)^{-\eta} dx = 2\,\eta\, a^{-\eta} \left(\frac{a}{2}\right)^{\sigma} \frac{\Gamma(2\sigma)\,\Gamma(\lambda - \sigma)}{\Gamma(1 + \eta + \sigma)}, \tag{1.10}$$

accommodated with $0 < \Re(\sigma) < \Re(\eta)$.
The intention of this note is to evaluate the Oberhettinger type integrals associated with a finite product of the generalized Bessel functions (1.1) and multivariable polynomials (1.2).

2. Noted Results

We state the subsequent results:

Theorem 2.1. *If $x > 0, \eta, \sigma, \upsilon_j, b, c \in \mathbb{C}$ with $\Re(\upsilon_j) > -1$, $(n_r, l_r) \geq 0$, $0 < \Re(\sigma) < \Re(\eta + \upsilon_j)$ $(j = 1, 2, \ldots, n)$. Then the following integral formula holds:*

$$\int_0^\infty x^{\sigma-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\eta} S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[\frac{y}{x+a+\sqrt{x^2+2ax}}\right]$$
$$\times\prod_{j=1}^{h} w_{\upsilon_j}\left[\frac{y_j}{x+a+\sqrt{x^2+2ax}}\right]dx$$

$$=\Gamma(2\sigma)\,a^{\sigma-\eta}2^{1-\sigma}\sum_{l_1=0}^{n_1/m_1}\cdots\sum_{l_r=0}^{n_r/m_r}\prod_{i=1}^{r}\left[\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\left(\frac{y}{a}\right)^{l_i}\right]\prod_{j=1}^{h}\left[\left(\frac{y_j}{4}\right)^{\upsilon_j}\right]$$

$$\times\left[\frac{1}{\Gamma\left(\upsilon_j+\frac{b+1}{2}\right)}\right]\frac{\Gamma\left(\eta-\sigma+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i\right)\Gamma\left(1+\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i\right)}{\Gamma\left(1+\eta+\sigma+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i\right)\Gamma\left(\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i\right)}$$

$$\times F_{2:1;\ldots;1}^{2:0;\ldots;0}\left[\begin{array}{l}\left(1+\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i;\ 2,\ldots,2\right),\\ \left(\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i;\ 2,\ldots,2\right),\end{array}\right.$$
$$\begin{array}{l}\left(\eta-\sigma+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i;\ 2,\ldots,2\right):\\ \left(1+\eta+\sigma+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i;\ 2,\ldots,2\right):\end{array}$$
$$\left.\begin{array}{l}\text{--------};\ \ldots;\ \text{--------};\\ \left(\upsilon_1+\frac{b+1}{2},\ 1\right);\ldots;\left(\upsilon_n+\frac{b+1}{2},\ 1\right);\end{array}\ \frac{-c}{4a^2}y_1^2,\ \cdots,\ \frac{-c}{4a^2}y_n^2\right]. \tag{2.1}$$

Proof: By making use of product of (1.1) and (1.2) in the integrand of (2.1) and interchanging the order of integral and summation, which is confirmed by uniform convergence of the series, we obtain

$$=\sum_{l_1=0}^{n_1/m_1}\cdots\sum_{l_r=0}^{n_r/m_r}\prod_{i=1}^{r}\left[\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}(y)^{l_i}\right]$$

$$\times\sum_{k_1,\ldots,k_h=0}^{\infty}(-1)^{k_1}\frac{c^{k_1}\left(\frac{y_1}{2}\right)^{\upsilon_1+2k_1}}{k_1!\Gamma\left(\upsilon_1+k_1+\frac{b+1}{2}\right)}\cdots(-1)^{k_h}\frac{c^{k_n}\left(\frac{y_n}{2}\right)^{\upsilon_h+2k_h}}{k_n!\Gamma\left(\upsilon_h+k_h+\frac{b+1}{2}\right)}$$

$$\times\int_0^\infty x^{\sigma-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\left(\eta+\sum_{j=1}^{h}(\upsilon_j+2k_j)+\sum_{i=1}^{r}l_i\right)}dx, \tag{2.2}$$

In sight of the circumstances disposed in Theorem (2.1), since

$$0 < \Re(\sigma) < \Re(\eta + \upsilon_j) \leq \Re\left(\eta + \upsilon_j + l_i + 2k_j\right), \quad \Re(\upsilon_j) > -1,$$

we apply the integral formula (1.10) to the above integral (2.2) and get subsequent expression:

$$= \Gamma\left(2\sigma\right) a^{\sigma-\eta} 2^{1-\sigma} \sum_{l_1=0}^{n_1/m_1} \cdots \sum_{l_r=0}^{n_r/m_r} \prod_{i=1}^{r} \left[\frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i, l_i} \left(y\right)^{l_i} \right]$$

$$\times \sum_{k_1, \ldots, k_h=0}^{\infty} (-1)^{k_1} \frac{c^{k_1} \left(\frac{y_1}{2}\right)^{\upsilon_1+2k_1}}{k_1! \Gamma\left(\upsilon_1 + k_1 + \frac{b+1}{2}\right)} \cdots (-1)^{k_h} \frac{c^{k_h} \left(\frac{y_n}{2}\right)^{\upsilon_h+2k_h}}{k_h! \Gamma\left(\upsilon_h + k_h + \frac{b+1}{2}\right)}$$

$$\times a^{-\left(\sum_{j=1}^{h}(\upsilon_j+2k_j)+\sum_{i=1}^{r} l_i\right)} \frac{\Gamma\left(1 + \eta + \sum_{j=1}^{h}\left(\upsilon_j + 2k_j\right) + \sum_{i=1}^{r} l_i\right)}{\Gamma\left(\eta + \sum_{j=1}^{h}\left(\upsilon_j + 2k_j\right) + \sum_{i=1}^{r} l_i\right)}$$

$$\times \frac{\Gamma\left(\eta + \sum_{j=1}^{h}\left(\upsilon_j\right) + \sum_{i=1}^{r} l_i - \mu\right) \left(\eta - \sigma + \sum_{j=1}^{h}\left(\upsilon_j\right) + \sum_{i=1}^{r} l_i\right)_{\sum_{j=1}^{h} 2k_j}}{\Gamma\left(1 + \sigma + \eta + \sum_{j=1}^{h}\left(\upsilon_j\right) + \sum_{i=1}^{r} l_i\right) \left(1 + \sigma + \eta + \sum_{j=1}^{h}\left(\upsilon_j\right) + \sum_{i=1}^{r} l_i\right)_{\sum_{j=1}^{h} 2k_j}},$$

and, we acquire

$$= \Gamma\left(2\sigma\right) a^{\sigma-\eta} 2^{1-\sigma} \sum_{l_1=0}^{n_1/m_1} \cdots \sum_{l_r=0}^{n_r/m_r} \prod_{i=1}^{r} \left[\frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i, l_i} \left(\frac{y}{a}\right)^{l_i} \right] \prod_{j=1}^{h} \left[\frac{y_j}{2a}\right]^{\upsilon_j}$$

$$\times \frac{\Gamma\left(\eta - \sigma + \sum_{j=1}^{h}\left(\upsilon_j\right) + \sum_{i=1}^{r} l_i\right)}{\Gamma\left(1 + \sigma + \eta + \sum_{j=1}^{h}\left(\upsilon_j\right) + \sum_{i=1}^{r} l_i\right)} \frac{\Gamma\left(1 + \eta + \sum_{j=1}^{h}\left(\upsilon_j\right) + \sum_{i=1}^{r} l_i\right)}{\Gamma\left(\eta + \sum_{j=1}^{h}\left(\upsilon_j\right) + \sum_{i=1}^{r} l_i\right)}$$

$$\times \sum_{k_1, \cdots, k_h=0}^{\infty} \frac{\Gamma\left(1 + \eta + \sum_{j=1}^{h}\left(\upsilon_j\right) + \sum_{i=1}^{r} l_i\right)_{\sum_{j=1}^{h} 2k_j}}{\Gamma\left(\eta + \sum_{j=1}^{h}\left(\upsilon_j\right) + \sum_{i=1}^{r} l_i\right)_{\sum_{j=1}^{h} 2k_j}}$$

$$\times \frac{\left(\eta - \sigma + \sum_{j=1}^{h}\left(\upsilon_j\right) + \sum_{i=1}^{r} l_i\right)_{\sum_{j=1}^{h} 2k_j}}{\left(1 + \sigma + \eta + \sum_{j=1}^{h}\left(\upsilon_j\right) + \sum_{i=1}^{r} l_i\right)_{\sum_{j=1}^{h} 2k_j}}$$

$$\times \left(\frac{1}{\Gamma\left(\upsilon_1 + \frac{b+1}{2}\right) \left(\upsilon_1 + \frac{b+1}{2}\right)_{k_1}} \cdots \frac{1}{\Gamma\left(\upsilon_h + \frac{b+1}{2}\right) \left(\upsilon_h + \frac{b+1}{2}\right)_{k_h}} \right)$$

$$\times \left(\frac{\left(-c\frac{y_1^2}{4a^2}\right)^{k_1}}{k_1!} \cdots \frac{\left(-c\frac{y_h^2}{4a^2}\right)^{k_h}}{k_h!} \right), \tag{2.3}$$

Straightaway, we employ (1.3) to pick up the aimed formula (2.1). □

If we set $r = 1$, then the multivariable polynomials reduces to Srivastava's polynomials, i.e. $S_n^m[x]$, and we get the following result:

Corollary 2.2. *If $x > 0, \eta, \sigma, \upsilon_j, b, c \in \mathbb{C}$ with $\Re(\upsilon_j) > -1$, $(n, l) \geq 0$, $0 < \Re(\sigma) < \Re(\eta+\upsilon_j)$ $(j = 1, 2, \ldots, n)$. Then there holds the following integral formula*

$$\int_0^\infty x^{\sigma-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\eta} S_n^m\left[\frac{y}{x+a+\sqrt{x^2+2ax}}\right]$$

$$\times\prod_{j=1}^{h} w_{\upsilon_j}\left[\frac{y_j}{x+a+\sqrt{x^2+2ax}}\right]dx$$

$$=\Gamma(2\sigma)\,a^{\sigma-\eta}2^{1-\sigma}\sum_{l=0}^{n/m}\frac{(-n)_{ml}}{l!}A_{n,l}\left(\frac{y}{a}\right)\prod_{j=1}^{h}\left[\left(\frac{y_j}{2a}\right)^{\upsilon_j}\frac{1}{\Gamma\left(\upsilon_j+\frac{b+1}{2}\right)}\right]$$

$$\times\frac{\Gamma\left(\eta-\sigma+l+\sum_{j=1}^{h}\upsilon_j\right)}{\Gamma\left(1+\eta+\sigma+l+\sum_{j=1}^{h}\upsilon_j\right)}\frac{\Gamma\left(1+\eta+l+\sum_{j=1}^{h}\upsilon_j\right)}{\Gamma\left(\eta+l+\sum_{j=1}^{h}\upsilon_j\right)}$$

$$\times F_{2:1;\ldots;1}^{2:0;\ldots;0}\left[\begin{array}{l}\left(1+\eta+l+\sum_{j=1}^{h}\upsilon_j;2,\ldots,2\right),\quad\left(\eta-\sigma+l+\sum_{j=1}^{h}\upsilon_j;2,\ldots,2\right):\\ \left(\eta+l+\sum_{j=1}^{h}\upsilon_j;2,\ldots,2\right),\quad\left(1+\eta+\sigma+l+\sum_{j=1}^{h}\upsilon_j;2,\ldots,2\right):\end{array}\right.$$

$$\left.\begin{array}{l}\text{———};\;\cdots\;;\text{———};\\ \left(\upsilon_1+\frac{b+1}{2},1\right);\ldots;\left(\upsilon_h+\frac{b+1}{2},1\right);\end{array}\;\frac{-c}{4a^2}y_1^2,\cdots,\frac{-c}{4a^2}y_h^2\right]. \tag{2.4}$$

Theorem 2.3. *If $x > 0, \sigma, \eta, \upsilon_j, b, c \in \mathbb{C}$ with $\Re(\upsilon_j) > -1$, $(n_r, l_r) \geq 0$, $0 < \Re(\sigma) < \Re(\eta+\upsilon_j)$ $(j = 1, \ldots, n)$. Then the following integral formula holds true:*

$$\int_0^\infty x^{\sigma-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\eta} S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[\frac{x\,y}{x+a+\sqrt{x^2+2ax}}\right]$$

$$\times\prod_{j=1}^{n} w_{\upsilon_j}\left[\frac{x\,y_j}{x+a+\sqrt{x^2+2ax}}\right]dx$$

$$=a^{\sigma-\eta}2^{1-\sigma}\,\Gamma(\eta-\sigma)\sum_{l_1=0}^{n_1/m_1}\cdots\sum_{l_r=0}^{n_r/m_r}\prod_{i=1}^{r}\left[\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,l_i}\left(\frac{y}{2}\right)^{l_i}\right]\prod_{j=1}^{h}\left[\left(\frac{y_j}{4}\right)^{\upsilon_j}\right]$$

$$\times\left[\frac{1}{\Gamma\left(\upsilon_j+\frac{b+1}{2}\right)}\right]\frac{\Gamma\left(1+\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i\right)}{\Gamma\left(\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i\right)}$$

$$\times \frac{\Gamma\left(2\sigma + 2\sum_{j=1}^{h} \upsilon_j + 2\sum_{i=1}^{r} l_i\right)}{\Gamma\left(1 + \sigma + \eta + 2\sum_{j=1}^{h} \upsilon_j + 2\sum_{i=1}^{r} l_i\right)}$$

$$\times F^{2:0;\ldots;0}_{2:1;\ldots;1}\left[\begin{array}{l} \left(1+\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r} l_i \;:\; 2,\ldots,2\right), \\ \left(\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r} l_i :\; 2,\ldots,2\right), \\ \left(2\sigma+2\sum_{j=1}^{h}\upsilon_j+2\sum_{i=1}^{r} l_i \;: 4,\ldots,4\right) \;: \\ \left(1+\sigma+\eta+2\sum_{j=1}^{h}\upsilon_j+2\sum_{i=1}^{r} l_i : 4,\ldots,4\right) \;: \\ \text{-------};\ \ldots\ ;\ \text{-------}; \\ \left(\upsilon_1+\frac{b+1}{2},\ 1\right);\ldots;\left(\upsilon_h+\frac{b+1}{2},\ 1\right); \end{array} \frac{-c}{16}y_1^2,\cdots,\frac{-c}{16}y_h^2\right]. \tag{2.5}$$

Proof: By analogous way as in demonstration of the Theorem (2.1), we can fix the integral formula (2.5). □

If we intent $r = 1$, then the multivariable polynomials reduces to Srivastava's polynomials, i.e. $S_n^m[x]$, and we get the following result:

Corollary 2.4. *If $x > 0, \sigma, \eta, \upsilon_j, b, c \in \mathbb{C}$ with $\Re(\upsilon_j) > -1$, $(n, l) \geq 0$, $0 < \Re(\sigma) < \Re(\eta + \upsilon_j)$ $(j = 1, \ldots, n)$. Then there holds the following integral formula*

$$\int_0^\infty x^{\sigma-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\eta} S_n^m\left[\frac{x\,y}{x+a+\sqrt{x^2+2ax}}\right]$$

$$\times \prod_{j=1}^{h} w_{\upsilon_j}\left[\frac{x\,y_j}{x+a+\sqrt{x^2+2ax}}\right] dx$$

$$= a^{\sigma-\eta}2^{1-\sigma}\,\Gamma(\eta-\sigma)\sum_{l=0}^{n/m}\frac{(-n)_{m\,l}}{l\,!}A_{n,l}\left(\frac{y}{2}\right)^l\prod_{j=1}^{h}\left[\left(\frac{y_j}{4}\right)^{\upsilon_j}\frac{1}{\Gamma\left(\upsilon_j+\frac{b+1}{2}\right)}\right]$$

$$\times \frac{\Gamma\left(1+\eta+l+\sum_{j=1}^{h}\upsilon_j\right)}{\Gamma\left(l+\eta+\sum_{j=1}^{h}\upsilon_j\right)}\frac{\Gamma\left(l+2\sigma+2\sum_{j=1}^{h}\upsilon_j\right)}{\Gamma\left(1+\sigma+\eta+l+2\sum_{j=1}^{h}\upsilon_j\right)}$$

$$\times F^{2:0;\ldots;0}_{2:1;\ldots;1}\left[\begin{array}{l} \left(1+\eta+l+\sum_{j=1}^{h}\upsilon_j : 2,\ldots,2\right), \left(l+2\sigma+2\sum_{j=1}^{h}\upsilon_j : 4,\ldots,4\right) : \\ \left(l+\eta+\sum_{j=1}^{h}\upsilon_j : 2,\ldots,2\right), \left(1+\sigma+\eta+l+2\sum_{j=1}^{h}\upsilon_j : 4,\ldots,4\right) : \\ \text{-------};\ \ldots\ ;\ \text{-------}; \\ \left(\upsilon_1+\frac{b+1}{2},\ 1\right);\ldots;\left(\upsilon_h+\frac{b+1}{2},\ 1\right); \end{array} \frac{-c}{16}y_1^2,\cdots,\frac{-c}{16}y_h^2\right]. \tag{2.6}$$

Remark 2.5. *If we plan $n = 0$ in corollary (2.2) and (2.4) , then in observation of Srivastava's polynomials $S_n^m[x]$ yields to unity, i.e. $S_0^m[x] \to 1$, and we arrive at the subsequent noted results due to Agarwal et al. [1].*

Remark 2.6. *If we set $h = 1$ in corollary (2.2) and (2.4), then we get the subsequent established results due to Menaria et al. [9].*

3. Particular Cases

In this part, we deal with some particular cases of above outcome presented in term of generalized Lauricella type functions.

(i) For $b-1 = 1 = c$ equation (1.1) reduces to Spherical Bessel function $K_\upsilon(z)$ and is express as:

$$K_\upsilon(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\upsilon + k + 3/2)\, k!} \left(\frac{z}{2}\right)^{\upsilon+2k}, \qquad (z \in \mathbb{C}), \tag{3.1}$$

(ii) For $b = c = 1$ in equation (1.1) reduces to Bessel function $J_\upsilon(z)$ and is express as:

$$J_\upsilon(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\upsilon + k + 1)\, k!} \left(\frac{z}{2}\right)^{\upsilon+2k}, \qquad (z \in \mathbb{C}). \tag{3.2}$$

Now, by setting $c = 1$ and $b = 2$ in equation (2.1) and (2.5), the new unified integrals presents in terms of the spherical Bessel function $K_\upsilon(z)$. we get two next couples of integral results, as under:

Corollary 3.1. *Let the observation of Theorem (2.1) be fulfilled. Then the subsequent integral formula holds true:*

$$\int_0^\infty x^{\sigma-1}\left(x + a + \sqrt{x^2+2ax}\right)^{-\eta} S_{n_1,\ldots,n_r}^{m_1,\ldots,m_r}\left[\frac{y}{x+a+\sqrt{x^2+2ax}}\right]$$

$$\times \prod_{j=1}^{h} K_{\upsilon_j}\left[\frac{y_j}{x+a+\sqrt{x^2+2ax}}\right] dx$$

$$= \Gamma(2\sigma)\, a^{\sigma-\eta} 2^{1-\sigma} \sum_{l_1=0}^{n_1/m_1} \cdots\cdots \sum_{l_r=0}^{n_r/m_r} \prod_{i=1}^{r}\left[\frac{(-n_i)_{m_i l_i}}{l_i!} A_{n_i,\, l_i} \left(\frac{y}{a}\right)^{l_i}\right]$$

$$\times \prod_{j=1}^{h}\left[\left(\frac{y_j}{2a}\right)^{\upsilon_j} \frac{1}{\Gamma(\upsilon_j + 3/2)}\right]$$

$$\times \frac{\Gamma\left(\eta - \sigma + \sum_{j=1}^{h}\upsilon_j + \sum_{i=1}^{r} l_i\right)}{\Gamma\left(1 + \eta + \sigma + \sum_{j=1}^{h}\upsilon_j + \sum_{i=1}^{r} l_i\right)} \frac{\Gamma\left(1 + \eta + \sum_{j=1}^{h}\upsilon_j + \sum_{i=1}^{r} l_i\right)}{\Gamma\left(\eta + \sum_{j=1}^{h}\upsilon_j + \sum_{i=1}^{r} l_i\right)}$$

$$\times F^{2:0;\ldots;0}_{2:1;\ldots;1}\left[\begin{array}{c}\left(1+\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i;2,...,2\right),\\ \left(\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i;2,...,2\right),\end{array}\right.$$

$$\begin{array}{c}\left(\eta-\sigma+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i;2,...,2\right):\\ \left(1+\eta+\sigma+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i;2,...,2\right):\end{array}$$

$$\left.\begin{array}{c}\text{--------};\ \cdots;\ \text{--------};\\ (\upsilon_1+3/2,\ 1)\ ;\ldots;\ (h_n+3/2,\ 1)\ ;\end{array}\ \frac{-c}{4a^2}y_1^2,\ \cdots\ ,\frac{-c}{4a^2}y_h^2\right]. \tag{3.3}$$

Corollary 3.2. *Let the observation of Theorem (2.3) be fulfilled. Then the subsequent formula exists:*

$$\int_0^\infty x^{\sigma-1}\left(x+a+\sqrt{x^2+2ax}\right)^{-\eta}S^{m_1,\ldots,m_r}_{n_1,\ldots,n_r}\left[\frac{x\,y}{x+a+\sqrt{x^2+2ax}}\right]$$

$$\times\prod_{j=1}^{h}K_{\upsilon_j}\left[\frac{x\,y_j}{x+a+\sqrt{x^2+2ax}}\right]dx$$

$$=a^{\sigma-\eta}2^{1-\sigma}\,\Gamma(\eta-\sigma)\sum_{l_1=0}^{n_1/m_1}\cdots\cdots\sum_{l_r=0}^{n_r/m_r}\prod_{i=1}^{r}\left[\frac{(-n_i)_{m_i l_i}}{l_i!}A_{n_i,\,l_i}\left(\frac{y}{2}\right)^{l_i}\right]$$

$$\times\prod_{j=1}^{h}\left[\left(\frac{y_j}{4}\right)^{\upsilon_j}\frac{1}{\Gamma(\upsilon_j+3/2)}\right]$$

$$\times\frac{\Gamma\left(1+\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i\right)}{\Gamma\left(\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i\right)}\frac{\Gamma\left(2\sigma+2\sum_{j=1}^{h}\upsilon_j+2\sum_{i=1}^{r}l_i\right)}{\Gamma\left(1+\sigma+\eta+2\sum_{j=1}^{h}\upsilon_j+2\sum_{i=1}^{r}l_i\right)}$$

$$\times F^{2:0;\ldots;0}_{2:1;\ldots;1}\left[\begin{array}{c}\left(1+\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i:2,\ldots,2\right),\\ \left(\eta+\sum_{j=1}^{h}\upsilon_j+\sum_{i=1}^{r}l_i:2,\ldots,2\right),\end{array}\right.$$

$$\begin{array}{c}\left(2\sigma+2\sum_{j=1}^{h}\upsilon_j+2\sum_{i=1}^{r}l_i:4,\ldots,4\right):\\ \left(1+\sigma+\eta+2\sum_{j=1}^{h}\upsilon_j+2\sum_{i=1}^{r}l_i:4,\ldots,4\right):\end{array}$$

$$\left.\begin{array}{c}\text{--------};\ \cdots;\ \text{--------};\\ (\upsilon_1+3/2,\ 1)\ ;\ldots;\ (\upsilon_h+3/2,\ 1)\ ;\end{array}\ \frac{-c}{16}y_1^2,\ \cdots\ ,\frac{-c}{16}y_h^2\right]. \tag{3.4}$$

4. Concluding Remarks

We put the lid on the comment that, by accounting our finest developments, one can find many other impressive integrals associated with a variety of Bessel functions, hyperbolic functions and trigonometric functions, after appropriate parametric replacements. Additionally, on putting convenient particular characters to the coefficient $A_{n,\,l}$ the Srivastava's polynomials provide various acknowledged classical orthogonal polynomials as its special cases as comprise Hermite, Laguerre, Jacobi, the Konhauser polynomials along with others.

References

1. P. Agarwal, D. Ritelli, A. Kilicman and S. Jain, *Certain new unified integrals associated with the product of generalized Bessel functions*, Commun. Numer. Anal., **2016** (2016), 50-56.

2. W.N. Bailey, *Some infinite integrals involving Bessel functions*, Proc. London Math. Soc., **40**(2) (1936), 37-48.

3. A. Baricz, *Generalized Bessel Functions of the First Kind*, Springer-Verlag Berlin, Heidelberg, 2010.

4. Y.A. Brychkov, *Handbook of Special Functions, Derivatives, Integrals, Series and Other Formulas*, CRC Press, Boca Raton, FL, 2008.

5. J. Choi, P. Agarwal, S. Mathur and S.D. Purohit, *Certain new integral formulas involving the generalized Bessel functions*, Bull. Korean Math. Soc., **51**(4) (2014), 995-1003.

6. J. Choi, D. Kumar and S.D. Purohit, *Integral formulas involving a product of generalized Bessel functions of the first kind*, Kyungpook Math. J., **56** (2016), 131-136.

7. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Mc Graw Hill Book Company, New York, Toronto, and London, II, 1953.

8. A. Gray and G.B. Mathews, *A treatise on Bessel functions and their applications to physics*. 2nd ed, prepared by A. Gray and T.M. Mac Robert. London: Macmillan, 1922.

9. N. Menaria, D. Baleanu and S.D. Purohit, *Integral formulas involving product of general class of polynomials and generalized Bessel function*, Sohag J. Math., **3**(2) (2016), 77-81.

10. N. Menaria, R.K. Parmar, S.D. Purohit and K.S. Nisar, *Certain unified integrals involving product of generalized k-Bessel function and gneral calss of polynomials*, Honam Math. J., **39**(3) (2017), 349-361.

11. K.S. Nisa, D.L. Suthar, S.D. Purohit and M. Aldhaifallah, *Some unified integral associated with the generalized Struve function*, Proc. Jangjeon Math. Soc., **20**(2) (2017), 261-267.

12. F. Oberhettinger, *Tables of Mellin Transforms*, Springer-Verlag, New York-Heidelberg, 1974.

13. S.D. Purohit, D.L. Suthar and S.L. Kalla, *Marichev-Saigo-Maeda fractional integration operators of the Bessel function*, Matematiche (Catania), **67**(1) (2012), 21-32.

14. H.M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, Inc., Amsterdam, 2012.

15. H.M. Srivastava and M.C. Daoust, *A note on the convergence of Kampé de Fériet's double hypergeometric series*, Math. Nachr., **53** (1985), 151-159.

16. H.M. Srivastava and P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis HorwoodLimited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

17. D.L. Suthar and H. Amsalu, *Certain integrals associated with the generalized Bessel-Maitland function*, Appl. Appl. Math., **12**(2) (2017), 1002-1016.

18. D.L. Suthar and H. Habenom, *Integrals involving generalized Bessel-Maitland Function*, J. Sci. Arts, **37**(4) (2016), 357-362.

19. D.L. Suthar, S.D. Purohit and S. Agarwal, *Class of Integrals Involving Generalized Hypergeometric Function and Srivastava's Polynomials*, Int. J. Appl. Comput. Math., **3**(1) (2017), 1197-1203.

20. G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Reprint of the second (1944) edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1995.

K.S. Nisar,
Department of Mathematics,
College of Arts and Science-Wadi Aldawaser,
11991, Prince Sattam bin Abdulaziz University, Alkharj,
Kingdom of Saudi Arabia.
E-mail address: `n.sooppy@psau.edu.sa`

and

D.L. Suthar,
Department of Mathematics,
Wollo University, P.O.Box: 1145, Dessie, South Wollo, Amhara Region,
Ethiopia.
E-mail address: `dlsuthar@gmail.com`

and

S.D. Purohit,
Department of HEAS (Mathematics),
Rajasthan Technical University, Kota,
India.
E-mail address: `sunil_a_purohit@yahoo.com`

and

Hafte Amsalu
Department of Mathematics, College of Natural Sciences,
Wollo University, P.O. Box: 1145, Dessie,
South Wollo, Amhara Region,
Ethiopia.
E-mail address: `hafamsalu@gmail.com`