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Derivations with Invertible Values in Flexible Algebras

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abstract: Derivations with invertible values of 0 - torsion flexible algebras satisfying $x(yz) = (xz)y$ over an algebraically closed field are described. For this class of algebra with unit element 1 and derivation with invertible value d is either a Cayley-Dickson algebra over its center $Z(A)$ or a factor algebra of polynomial algebra $C[a]/(a^2)$ over a Cayley-Dickson division algebra; also C is 2 - torsion, $d(C) = 0$ and $d(a) = 1 + ua$ for some u in center of C and d is an outer derivation. Moreover, C is a split Cayley-Dickson algebra over its center Z having a derivation with invertible value d if and only if C is obtained by means of Cayley-Dickson process from its associative division subalgebra and can be represented as a direct sum $C = V \oplus aV$.

Key Words: Derivations, Invertible values, Flexible algebras, Cayley-Dickson algebras.

Contents

1. Introduction

Derivation with invertible values as a derivation of a ring with unity that only takes multiplicatively invertible or zero values is defined in 1983 by Bergen, Herstein and Lanski [2] in which they determined the structure of associative rings admitting derivations with invertible values. They proved that such ring must be either a division ring, or the ring of 2×2 matrices over a division ring, or a factor of a polynomial ring over a division ring of characteristic 2. They also characterized those division rings such that a 2×2 matrix ring over them has an inner derivation with invertible values. Later their results were generalized in many cases like generalized derivations, associative superalgebras and alternative algebras. In [8] semiprime associative rings with involution, allowing a derivation with invertible values on the set of symmetric elements, were given an examination. In [3] Bergen and Carini studied associative rings admitting a derivation with invertible values on some non - central Lie ideal. Also in the papers [4] and [9] the structure of associative rings that admit α - derivations with invertible values and their natural generalizations - $(\sigma, ?\tau)$ - derivations with invertible values was described. In [12]

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Komatsu and Nakajima described associative rings that allow generalized derivations with invertible values. The case of associative superalgebras with derivations with invertible values was studied in the paper of Demir, Albas, Argac and Fosner [5]. Nonassociative algebras admitting derivations with invertible values are described in the paper of Kaygorodov, Lopatin and Popov [11], where it was proved that Jordan algebra can be represented as a symmetric bilinear form $J(V, f)$ and as a division algebra of Albert type.

Nowadays, a great interest is shown to the studying of nonassociative algebras and superalgebras with derivations. Nevertheless, the problem of specification of flexible algebras admitting derivations with invertible values remains unconsidered. However, flexible composition algebras and Okubo algebras were studied by Elduque and Myung [6, 7]. Our approach was motivated by the work of Kaygorodov [11] where certain unital composition algebras are effectively constructed from 2, 3 - torsion free Jordan algebras. Some important examples of composition algebras have arisen in the study of real division algebras, flexible algebras and Malcev - admissible algebras and in physical problems related to the $SU(3)$ particle physics. Following technically Elduque and Myung, in the present paper, we come up with an account of derivation with invertible values for 0 - torsion flexible algebra A satisfying the identity $x(yz) = (xz)y$, for all $x, y, z \in A$ over algebraically closed field.

2. Preliminaries

Let A be an algebra with unit element 1 over field F. We denote the set of invertible elements of A by U and consider derivations with invertible values i.e. non - zero derivation d as for every $x \in A$, $d(x) \in U$ or $d(x) = 0$. The nucleus of an algebra A is the set

$$N(A)=\{n \in A \mid (n, A, A) = (A, n, A) = (A, A, n) = (0)\},$$

the commutative center of A is the set

$$K(A) = \{k \in A \mid [k, A] = [A, k] = (0)\},$$

and the center of A is

$$Z(A) = N(A) \cap K(A).$$

Derivation d is called inner if it lies in the smallest subspace of the space of all linear operators an A containing all right and left multiplications by elements of A and closed under commutation. Otherwise d is called outer.

The definition and properties of Cayley-Dickson algebras and the Cayley-Dickson process can be found, for instance, in [14]. Every Cayley-Dickson algebra C over field F is 8 - dimensional, nonassociative simple and has unit element. A nonassociative algebra A over a 2 - torsion free field with a non degenerate symmetric bilinear form (,) permitting composition $(x \cdot y, x \cdot y) = (x, x)(y, y)$ for all $x, y \in A$ [7] where $x \cdot y$ denotes the multiplication in A. If A has a unit element, then it is called unital. An algebra is flexible if the identity $(xy)x = x(yx)$ is satisfied for all $x, y \in A$. It is clear that all unital composition algebras are flexible.

A mapping n: $A \to F$ is called a quadratic form if

(i) $n(\lambda x) = \lambda^2 n(x)$, where $x \in A, \lambda \in F$

(ii) the function $f(x, y) = n(x + y) - n(x) - n(y)$ is a bilinear form on A.

A quadratic form $n(x)$ is called strictly nondegenerate if the symmetric bilinear form $f(x, y)$ which corresponds to it is nondegenerate, and it is called nondegenerate if from $n(a) = f(a, x) = 0$ for all $x \in A$ it follows that $a = 0$.

It is well known that Cayley-Dickson algebra C is an example of flexible algebra. C is quadratic over F, that is, for every $x \in C$ the following relation holds:

$$x^2 - t(x)x + n(x) = 0, \tag{2.1}$$

where $t(x), n(x) \in F, t(x)$ is a F - linear mapping and $n(x)$ is a strictly nondegenerate quadratic form satisfying $n(xy) = n(x)n(y)$ for all $x, y \in C$, where xy denotes the multiplication in A.

C is also equipped with a symmetric bilinear nondegenerate form $f(x, y) = n(x + y) - n(x) - n(y)$. For a subset $M \subseteq C$, by $M^{\perp}$, we mean the orthogonal complement to M with respect to f.

A Cayley-Dickson algebra is called split if it contains zero divisors. Element x of a split Cayley-Dickson algebra is invertible if and only if $n(x) \neq 0$ [14].

Let A be a simple flexible algebra. Then the center of the algebra A is a field and let us suppose that A is a Cayley-Dickson algebra over its center. Flexible algebra in associative form is $(x, y, x) = 0$ for all $x, y \in A$. This on linearization gives

$$(x, y, z) + (z, y, x) = 0 \tag{2.2}$$

for all $x, y, z \in A$.
Throughout this paper, let A satisfy the identity

$$x(yz) = (xz)y \tag{2.3}$$

for all $x, y, z \in A$.
It is easy to see that A with (3) satisfies the following identity

$$(x, y, z)y = (x, y, yz). \tag{2.4}$$

3. Derivations with Invertible values

We begin this section with the following Lemmas.

Lemma 3.1.

If $d(a) = 0$ then either $a = 0$ or a is invertible.

Proof: In every flexible algebra satisfying (3), the following identity holds.

$$(u^{-1}, u, a) = 0. \tag{3.1}$$

We now show that product of two invertible elements is invertible. If u and v are invertible then

$$(v^{-1}u^{-1})uv = u^{-1}((uv)v^{-1}) - (u^{-1}(uv))v^{-1} + (v^{-1}u^{-1})uv$$

$$= -(u^{-1}, uv, v^{-1}) + (v^{-1}u^{-1})uv = (v^{-1}, uv, u^{-1}) + (v^{-1}u^{-1})uv$$

$$= ((v^{-1}v)u)u^{-1} = 1.$$

Assume that $a \neq 0$. Since $d \neq 0$, there exists $b \in A$ such that $d(b) \in U$. Hence $d(ba) = d(b)a \in U$ and $d(b)^{-1}d(ba) = a$.
Since $d(b)$ and $d(ba)$ are invertible, a is also invertible. □

Lemma 3.2.

Let I, J be the ideals of a flexible algebra A , then the product IJ is also an ideal of the algebra A.

Proof: Let $i \in I$ and $j \in J$. Then for any $u \in A, (ij)u = i(uj) \in IJ$. And $u(ij) = (ui)j + (iu)j - i(uj) \in IJ$. Hence the product IJ is also an ideal of A. □

Lemma 3.3.

(i) If $L \neq 0$ is a one - sided ideal in A then $d(L) \neq 0$.
(ii) If I is a proper one - sided ideal of A, then I is both minimal and maximal.
(iii) If I is a proper ideal of A then $I^2 = (0)$.
(iv) If A is 2-torsion free, then A is simple.

Proof: (i) When $L = A$, the statement is obvious. So, let $L \neq A$. Let $u \neq 0$ and $u \in L$. Then by Lemma 3.1, $d(u) \neq 0$ since u is not invertible.
(ii) It suffices to show that every proper one - sided ideal in A is maximal. Let $I \subset J$ be a proper one - sided ideal of A. Then it is easy to check that $d(I) \cap I = (0)$ and $I \oplus d(I)$ is also one - sided ideal in A. By Lemma 3.3 (i), $d(I) \neq 0$ and $d(I)$ contains invertible elements. Hence $I \oplus d(I) = A$. For any $j \in J$ and for $u, v \in I$, we have $j = u + d(v)$. So $d(v) = j - u \in J \cap d(I) = (0)$ and hence $j = u \in I$.
(iii) Let $I \neq A$ be an ideal of A. Then

$$d(I^2) \subset d(I) + Id(I) \subset I.$$

Since product of two ideals in a flexible algebra satisfying (3) is an ideal and I does not contain any invertible elements, by Lemma 3.3(i), $I^2 = (0)$.
(iv) Let $2A \neq 0$ and $I \neq 0$. Then by Lemma 3.3(i), $d(I) \neq 0$. So, there exists $v \in I$ such that $d(v) \in U$. As $v^2 = 0$,

$$0 = d^2(v^2) = d^2(v)v + 2d(v)^2 + vd^2(v).$$

Hence $2d(v)^{2} \in I$. As $d(v)$ is invertible,$d(v)^{2}$ is also invertible and $2d(v)^2 = 0$ which implies that $2 = 0$, and it is a contradiction . So A does not contain any non - trivial ideals and hence A is simple. □

The set of all derivations of algebra A is denoted by $D(A)$. Let us fix some subset $D \subseteq Der(A)$. The ideal I is called D - ideal, if for all $\partial \in D, a \in I$ we have $\partial(a) \in I$. Algebra A is called D - simple if $A^2 \neq 0$ and A contains no proper D - ideals.

Lemma 3.4.

If flexible algebra A admits a derivation with invertible values d, then A is d - simple.

Proof: This is an immediate consequence of Lemma 3.3(iii). □

Lemma 3.5.

If A is not simple and not associative, then $A = C[a]/(a^2)$, where C is a Cayley-Dickson algebra over its center $Z(C)$, C is a division algebra, C is 2 - torsion, $d(C) = 0$, $d(a) = 1 + ua$ for some $u \in Z(C)$ and d is an outer derivation.

Proof: By Lemma 3.3(ii) and (iv), we can see that A is 2 - torsion. If I is any proper ideal in A then $I^2 = (0)$ and all proper one - sided ideals in A are both minimal and maximal. So, we can easily deduce that A contains a unique ideal M and $M^2 = 0$. As in the proof of Lemma 3.3(ii), we have $A = M \oplus d(M)$. For any $u \in A$, there exists $p, q \in M$ such that $d(u) = p + d(q)$. Therefore $p = d(u - q) \in M \cap d(A) = (0)$. By denoting $C = ker(d)$, We have $A = C + M$. By Lemma 3.1, C is a division algebra and hence $A = C \oplus M$. Let $\eta : M \longrightarrow C$ and $\theta : M \longrightarrow M$ be two linear mappings defined by $d(m) = \eta(m) + \theta(m)$ for $m \in M$. For any $u \in C$ and $v \in M$,

$$u\theta(v) + u\eta(v) = ud(v) = d(uv) = \eta(uv) + \theta(uv),$$

where $u\theta(v)$, $\theta(uv) \in M$. Hence $u\eta(v) = \eta(uv) \in \eta(M)$. Similarly $\eta(v)u = \eta(vu) \in \eta(M)$. Hence $\eta(M)$ is an ideal in C. Therefore C is isomorphic to M as a left C - module as C is simple and $\eta(M) \neq 0$. By replacing $a = \eta^{-1}(1)$, we obtain $A = C \oplus Ca$. Since η is a module isomorphism, it is easy to check that $[a, C] = 0$. By the identity

$$3(k, a, b) = 3(b, k, a) = 3(a, b, k) = [ab, k] - a[b, k] - [a, k]b = 0,$$

satisfied for any $k \in K(V)$, $a, b \in V$ in any flexible algebra V, and we can deduce that $a \in Z(A)$. Thus we obtain $A \cong C[a]/(a^2)$. Hence C is a Cayley-Dickson algebra over its center $Z(C)$. We can write $\theta(a) = ua$ for some $u \in C$. As $a \in Z(A)$ and A is 2 - torsion, for any $w \in C$, we obtain

$$0 = d(wa + aw) = w(1 + ua) + (1 + ua)w = wua + uaw = (wu + uw)a.$$

As C is a division algebra, we have $wu + uw = 0$, hence $u \in Z(C)$. Finally, since every ideal of A is invariant under the action of any inner derivation, $a \in M$ and $d(a) \notin M$, it is clear that d is an outer derivation. □

Theorem 3.6.

Let A be a flexible algebra with unit element 1, admitting a derivation with invertible values d. Then one of the following conditions holds:
(i) A is a Cayley-Dickson algebra over its center $Z(A)$;
(ii)A is a factor algebra of polynomial algebra $C[a]/(a^2)$ over a Cayley-Dickson division algebra; also, C is 2 - torsion, $d(C) = 0$ and $d(a) = 1 + ua$ for some u in the center of C and d is an outer derivation.

Proof: The proof follows from Lemmas 3.3 and 3.5. □

Theorem 3.7.

An algebra C, which is a split Cayley-Dickson algebra over its center Z, admits a derivation with invertible values d if and only if one of the following conditions holds:
(i) C is obtained by means of the Cayley-Dickson process from its associative division subalgebra V: $C = V + sV$, $s^2 = \beta \in Z$, $\beta \neq 0$ where $V = ker(d)$ and $dim_Z V = 4$. Also, an arbitrary derivation with invertible values d is of the form $d(u + sv) = s(vr)$, where $u, v \in V$ and $r \in V$ is a fixed element with $t(r) = 0$.
(ii) C can be represented as a direct sum: $C = V + aV$, where $t(a) = 0$, $V = ker(d)$, V is a subfield of C, $V = V^{\perp}$ and $dim_Z V = 4$. Also, an arbitrary derivation with invertible values d is of the form $d(u + av) = v$, where $u, v \in V$.

Proof: Every derivation of C is inner. It is easy to check that $Z \subseteq ker(d)$ and d is a Z - linear mapping. So C is considered as a Z - algebra. Suppose that C allows a derivation with invertible values d. Take a subspace $W \subset C$ such that $dim_Z W = 4$ and W does not contain invertible elements. From Lemma 3.1, we have $dim_Z d(W) = 4$ and $W \cap d(W) = (0)$, hence $C = W \oplus d(W)$. In particular, for any $a \in C$ there exists $r, s \in W$ such that $d(a) = r + d(s)$. Hence, $r = d(a - s) \in W \cap d(A) = (0)$. By denoting $V = ker(d)$, we have $C = V + W$. By Lemma 3.1, V is a division algebra and so $C = V \oplus W$ and $dim_Z V = 4$. Using the facts that V is simple and $Z(C) \subseteq Z(V)$, we have that V is an associative subalgebra in C. The following relation is valid in C [14]:

$$u \circ v - t(u)v - t(v)u - f(u, v) = 0. \tag{3.2}$$

Replacing $v = d(u)$, we obtain

$$u \circ d(u) - t(u)d(u) - t(d(u))u - f(u, d(u)) = 0. \tag{3.3}$$

By applying d on (1), we obtain

$$u \circ d(u) - t(u)d(u) = 0. \tag{3.4}$$

By subtracting (7) from (8), we get $t(d(u))u + f(u, d(u)) = 0$. If u and 1 are linearly independent over Z, we have

$$f(u, d(u)) = 0. \tag{3.5}$$

If $u \in Z$ then $u \in ker(d)$ and the relation (9) is obvious. By linearizing (9), we obtain $f(u, d(v)) + f(d(u), v) = 0$. Since $V = ker(d)$, for arbitrary $u \in C$ we have $f(d(u), V) = -f(u, d(V)) = 0$ and hence $d(C) \subseteq V^{\perp}$. We will now study two cases:

Case(i). If the restriction of the form f on V is nondegenerate, then C can be obtained from V by means of the Cayley-Dickson process [14], that is, $C = V + sV$, $s^2 = \beta \neq 0$, $V^{\perp} = sV$. Then $d(s) = sr$ for some $r \in V$ and therefore for arbitrary $u, v \in V$ we have $d(u + sv) = d(s)v = (sr)v = s(vr)$. For any $a, b, c \in C$ we have $n(a)f(b, c) = f(ab, ac)$. For $a = s, b = 1, c = r$, we obtain $f(s, sr) = n(s)t(r) = 0$ by using (9). Since $s^2 = \beta \in Z, \beta \neq 0$, we have $n(s) \neq 0$ and $t(r) = 0$.

Case(ii). If the restriction of the form f on V is degenerate then there exists $0 \neq v \in V$ such that $f(v, V) = 0$. Hence $f(v, v) = 2n(v) = 0$. As v is invertible, $n(v) \neq 0$ and so C is of 2 - torsion. In C, the following relation holds [14]:

$$f(a, e)f(b, c) = f(ab, ec) + f(ac, be). \tag{3.6}$$

By replacing $a = v$, $e = u$, $b = v^{-1}w, c = 1$, where $u, w \in V$, we obtain $f(v, u)f(v^{-1}w, 1) = f(w, u) + f(v, uv^{-1}w)$, and so by the arbitrariness of u, w we obtain $f(V, V) = 0$, that is, $V \subseteq V^{\perp}$.

Now let us suppose that there exists $a \in V^{\perp}, a \notin V$. Then by the skew symmetry of the associator and (5), we have $dim_Z aV = 4$ and $A = V \oplus aV$. By (10), we obtain

$$f(u, aw) = f(u \cdot 1, aw) = -f(uw, a) + f(u, a)f(1, w) = 0$$

for any $u, w \in V$. Hence $aV \subset V^{\perp}$ and $C = V^{\perp}$. This is a contradiction to the nondegeneracy of the form f. Let $a = d^{-1}(1)$. Then $a \notin V$ and $C = V \oplus aV$. Equation (9) implies that $f(a, 1) = t(a) = 0$. By the definition of f and from $f(V, V) = 0$, we obtain $f(u, w) = n(u + w) - n(u) - n(w) = 0$, for any u, w. Hence n is a ring homomorphism from V to Z. Since V is simple, $n(1) = 1$ and $ker(n) = 0$, V is a subfield of Z.

Conversely let us suppose that condition (i) is true, which means that C is obtained from V by the Cayley-Dickson process. Let $0 \neq r \in V$ such that $t(r) = 0$. Let $d : u+sv \longmapsto s(vr), u, v \in V$ be a mapping. We now show that d is a derivation. Let $u_1, v_1, u_2, v_2 \in V$. Then

$$d(u_1 + sv_1)(u_2 + sv_2) + (u_1 + sv_1)d(u_2 + sv_2)$$

$$= \beta(v_2(r + \bar{r})\bar{v}_1) + s((u_2v_1 + \bar{u}_1v_2)r) = \beta(v_2t(r)\bar{v}_1) + s((u_2v_1 + \bar{u}_1v_2)r)$$

$$= s((u_2v_1 + \bar{u}_1v_2)r) = d((u_1u_2 + \beta v_2\bar{v}_1) + s(\bar{u}_1v_2 + u_2v_1)) = d((u_1 + sv_1)(u_2 + sv_2)).$$

Since V is a division algebra, $n(d(u+sv)) = n(s(vr)) = n(s)n(v)n(r) = -\beta n(v)n(r)$ $\neq 0$, if $v \neq 0$. So $d(u+sv)$ is invertible for any $u \in V, 0 \neq v \in V$. Hence d takes invertible values.

Now let us suppose that condition (ii) is true. Let $d : u+av \longmapsto v$ be a mapping. We now show that d is a derivation with invertible values. As $V = V^{\perp}, t(u) = 0$ for any $u \in V$. Since C is 2 - torsion, from equation (6) we have

$$[a, u] = a \circ u = t(u)a + t(a)u + f(u, a) = f(u, a) \in Z. \tag{3.7}$$

Hence $d([a, u]) = 0$. By substituting a in (1), we have $a^2 \in Z$. By(11), it is easy to verify that for $u, w \in V$ the following identity holds:

$$(u, w, a) = uf(w, a) + f(u, a)w + f(a, uw). \tag{3.8}$$

Hence $d((u, w, a)) = 0$. For any $u, v, w, h \in V$ we have

$$d((ua + v)(wa + h)) = d((ua)(wa) + (ua)h + v(wa) + vh)$$

$$= d((ua)(wa)) + d((ua)h) + d(v(wa)).$$

Now $d((ua)h) = d((au)h) = d(a(uh)) = uh$ and $d(v(wa)) = d((vw)a) = vw$.
Also

$$d((ua + v)(wa + h)) + (ua + v)d(wa + h)$$

$$= u(wa + h) + (ua + v)w = u(wa) + uh + (ua)w + vw.$$

Thus we have to show that $d((ua)(wa)) = u(wa) + (ua)w$.
But

$$u(wa) + (ua)w = (uw)a - (u, w, a) + u(aw) + (u, a, w)$$

$$= (uw)a + u(wa + f(w, a)) = (u, w, a) + uf(w, a).$$

Now, since $a^2 \in Z$ and $d(a^2(uw)) = n(a)d(uw) = 0$, we have

$$d((ua)(wa)) = d((au + f(u, a))wa) = d((au)(wa)) + f(u, a)d(wa)$$

$$= d(a(ua)a) + f(u, a)w = d(a(a(uw) + f(a, uw))) + f(u, a)w$$

$$= d(a^2(uw)) + f(a, uw) + f(u, a)w = f(a, uw) + f(u, a)w.$$

By equating the expressions, we can get the equation (12). Hence d is a derivation of C. As V is a field and d takes values in V, d is a derivation with invertible values. □

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