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Some Properties of a Class of Analytic Functions Involving a New Generalized Differential Operator

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ABSTRACT: In the present paper, we introduce a new generalized differential operator $D^m_{\mu,\lambda,\sigma}(\alpha,\beta)$ defined on the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. A novel subclass $\Omega^*_m(\delta,\lambda,\alpha,\beta,b)$ by means of the operator $D^m_{\mu,\lambda,\sigma}(\alpha,\beta)$ is also introduced. Coefficient estimates, growth and distortion theorems, closure theorems, and class preserving integral operators for functions in the class $\Omega^*_m(\delta,\lambda,\alpha,\beta,b)$ are discussed. Furthermore, sufficient conditions for close-to-convexity, starlikeness, and convexity for functions in the class $\Omega^*_m(\delta,\lambda,\alpha,\beta,b)$ are obtained.

Key Words: Analytic functions, Close-to-convex functions, Differential operator, Integral operator.

Contents

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* and *normalized* in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. For functions f in $\mathcal{A}$, we define the following new generalized differential operator as follows:

$$D^0_{\mu,\lambda,\sigma}(\alpha,\beta)f(z) = f(z),$$

$$D^1_{\mu,\lambda,\sigma}(\alpha,\beta)f(z) = \left(\frac{\mu+\lambda-(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right) f(z) + \left(\frac{(\beta-\sigma)(\lambda-\alpha)}{\mu+\lambda}\right) zf'(z),$$

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and

$$D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z) = D_{\mu,\lambda,\sigma}(\alpha,\beta)(D^{m-1}_{\mu,\lambda,\sigma}(\alpha,\beta)f(z)), \tag{1.2}$$

where $\alpha,\sigma \geq 0$, $\beta,\lambda,\mu > 0$, $\lambda \neq \alpha$ and $m \in \mathbb{N}$.

If f is given by (1.1), then from (1.2) we see that

$$D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z) = z + \sum_{n=2}^{\infty}\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_n z^n, \quad (m \in \mathbb{N}_0). \tag{1.3}$$

We observe that the generalized differential operator $D^m_{\mu,\lambda,\sigma}(\alpha,\beta)$ reduces to several interesting many other differential operators considered earlier for different choices of $\mu,\lambda,\sigma,\alpha$ and β:

(i) $D^m_{1-\lambda,\lambda,\sigma}(\alpha,\beta)f(z) = z + \sum\limits_{n=2}^{\infty}[1+(n-1)(\lambda-\alpha)(\beta-\sigma)]^m a_n z^n$ was introduced and studied by Ramadan and Darus [8];

(ii) $D^m_{1-\lambda,\lambda,0}(\alpha,\beta)f(z) = z + \sum\limits_{n=2}^{\infty}[1+(n-1)(\lambda-\alpha)\beta]^m a_n z^n$ was introduced and studied by Darus and Ibrahim [7];

(iii) $D^m_{\mu,\lambda,0}(0,1)f(z) = z + \sum\limits_{n=2}^{\infty}\left[\frac{\mu+\lambda n}{\mu+\lambda}\right]^m a_n z^n$ was introduced and studied by Swamy [10];

(iv) $D^m_{1-\lambda,\lambda,0}(0,1)f(z) = z + \sum\limits_{n=2}^{\infty}[1+(n-1)\lambda]^m a_n z^n$ was introduced and studied by Al-Oboudi [2];

(v) $D^m_{0,1,0}(0,1)f(z) = z + \sum\limits_{n=2}^{\infty} n^m a_n z^n$ was introduced and studied by Sălăgean [9].

With the aid of the differential operator $D^m_{\mu,\lambda,\sigma}(\alpha,\beta)$ we define the class

$$\Omega_m(\delta,\lambda,\alpha,\beta,b)$$

as follows:

A function f in $\mathcal{A}$ is said to be in the class $\Omega_m(\delta,\lambda,\alpha,\beta,b)$ if it satisfies the condition:

$$\operatorname{Re}\left\{1+\frac{1}{b}\left[(1-\delta)\frac{D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z)}{z} + \delta(D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z))' - 1\right]\right\} > 0. \tag{1.4}$$

Or, equivalently:

$$\left|\frac{(1-\delta)\frac{D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z)}{z} + \delta(D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z))' - 1}{(1-\delta)\frac{D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z)}{z} + \delta(D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z))' - 1 + 2b}\right| < 1, \tag{1.5}$$

where $z \in U$, $\delta \geq 0$, $m \in \mathbb{N}_0$ and $b \in \mathbb{C} - \{0\}$.

Let $\mathcal{A}^*$ denote the subclass of $\mathcal{A}$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \qquad a_n \geq 0. \tag{1.6}$$

Further, we shall define the class $\Omega_m^*(\delta, \lambda, \alpha, \beta, b)$ by:

$$\Omega_m^*(\delta, \lambda, \alpha, \beta, b) = \Omega_m(\delta, \lambda, \alpha, \beta, b) \cap \mathcal{A}^*. \tag{1.7}$$

In our present paper, we obtain some interesting properties for functions in the class $\Omega_m^*(\delta, \lambda, \alpha, \beta, b)$. We employ techniques similar to these used earlier by Al-Hawary et al. [1], Darus and Faisal [6], and Amourah et al. [3,4,5,11].

2. Coefficient Inequalities

In this section we find the coefficient estimates for the functions in the class $\Omega_m^*(\delta, \lambda, \alpha, \beta, b)$. Our main characterization theorem for this function class is stated as Theorem 2.1 below.

Theorem 2.1. *A function $f \in \mathcal{A}$ given by (1.1) is in the class $\Omega_m^*(\delta, \lambda, \alpha, \beta, b)$ if and only if*

$$\sum_{n=2}^{\infty} [1 + \delta(n-1)] \left[\frac{\mu + \lambda + (n-1)(\lambda - \alpha)(\beta - \sigma)}{\mu + \lambda} \right]^m a_n \leq |b|, \tag{2.1}$$

where $\alpha, \sigma \geq 0$, $\beta, \lambda, \mu > 0$, $\lambda \neq \alpha$ and $m \in \mathbb{N}_0$.

Proof: By definition, $f \in \Omega_m^*(\delta, \lambda, \alpha, \beta, b)$ if and only if the condition (1.5) is satisfied.

Suppose that $f \in \Omega_m^*(\delta, \lambda, \alpha, \beta, b)$, then for $z \in U$ we have

$$\begin{aligned} &\left| (1-\delta) \frac{D_{\mu,\lambda,\sigma}^m(\alpha, \beta) f(z)}{z} + \delta (D_{\mu,\lambda,\sigma}^m(\alpha, \beta) f(z))' - 1 \right| \\ &\quad - \left| (1-\delta) \frac{D_{\mu,\lambda,\sigma}^m(\alpha, \beta) f(z)}{z} + \delta (D_{\mu,\lambda,\sigma}^m(\alpha, \beta) f(z))' - 1 + 2b \right| = \end{aligned}$$

$$\left|\sum_{n=2}^{\infty}[1+\delta(n-1)]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_n z^{n-1}\right|$$
$$-\left|2b-\sum_{n=2}^{\infty}[1+\delta(n-1)]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_n z^{n-1}\right|$$
$$\leq \sum_{n=2}^{\infty}[1+\delta(n-1)]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_n \left|z^{n-1}\right| - 2\left|b\right|$$
$$+\sum_{n=2}^{\infty}[1+\delta(n-1)]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_n \left|z^{n-1}\right|$$
$$\leq \sum_{n=2}^{\infty}[1+\delta(n-1)]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_n - |b| \leq 0.$$

This implies

$$\sum_{n=2}^{\infty}[1+\delta(n-1)]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_n \leq |b|\,.$$

Conversely, suppose the inequality (2.1) is satisfied then

$$\left|\frac{(1-\delta)\frac{D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z)}{z}+\delta(D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z))'-1}{(1-\delta)\frac{D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z)}{z}+\delta(D^m_{\mu,\lambda,\sigma}(\alpha,\beta)f(z))'-1+2b}\right|<1.$$

This completes the proof of Theorem 2.1. □

Corollary 2.2. *If $f \in \Omega^*_m(\delta,\lambda,\alpha,\beta,b)$ is given by (1.1), then*

$$a_n \leq \frac{|b|}{[1+\delta(n-1)]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m},\ n\geq 2.$$

3. Growth and Distortion Theorems

A growth and distortion property for function f to be in the class

$$\Omega^*_m(\delta,\lambda,\alpha,\beta,b)$$

is contained in the following theorem.

Theorem 3.1. *If the function f defined by (1.6) is in the class $\Omega^*_m(\delta,\lambda,\alpha,\beta,b)$, then for $|z|=r<1$, we have*

$$r-\frac{|b|}{[1+\delta]\left[\frac{\mu+\lambda+(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m}r^2 \leq |f(z)| \leq r+\frac{|b|}{[1+\delta]\left[\frac{\mu+\lambda+(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m}r^2$$

and

$$1-\frac{2\,|b|}{[1+\delta]\left[\frac{\mu+\lambda+(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m}r \leq \left|f^{'}(z)\right| \leq 1+\frac{2\,|b|}{[1+\delta]\left[\frac{\mu+\lambda+(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m}r.$$

Proof: Since $f \in \Omega_m^*(\delta, \lambda, \alpha, \beta, b)$, from Theorem 2.1 readily yields the inequality

$$\sum_{n=2}^{\infty} a_n \leq \frac{|b|}{[1+\delta]\left[\frac{\mu+\lambda+(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m}. \tag{3.1}$$

Thus, for $|z| = r < 1$, and making use of (3.1) we have

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} a_n |z^n| \leq r + r^2 \sum_{n=2}^{\infty} a_n \leq r + \frac{|b|}{[1+\delta]\left[\frac{\mu+\lambda+(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m} r^2$$

and

$$|f(z)| \geq |z| - \sum_{n=2}^{\infty} a_n |z^n| \geq r - r^2 \sum_{n=2}^{\infty} a_n \geq r - \frac{|b|}{[1+\delta]\left[\frac{\mu+\lambda+(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m} r^2.$$

Also from Theorem 2.1, it follows that

$$\frac{[1+\delta]\left[\frac{\mu+\lambda+(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m}{2} \sum_{n=2}^{\infty} n a_n \leq$$

$$\sum_{n=2}^{\infty} [1+\delta(n-1)] \left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_n \leq |b|.$$

Hence

$$\left|f'(z)\right| \leq 1 + \sum_{n=2}^{\infty} n a_n |z^n| \leq 1 + r \sum_{n=2}^{\infty} n a_n \leq 1 + \frac{2|b|}{[1+\delta]\left[\frac{\mu+\lambda+(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m} r.$$

and

$$\left|f'(z)\right| \geq 1 - \sum_{n=2}^{\infty} n a_n |z^n| \geq 1 - r \sum_{n=2}^{\infty} n a_n \geq 1 - \frac{2|b|}{[1+\delta]\left[\frac{\mu+\lambda+(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m} r.$$

This completes the proof of Theorem 3.1. □

4. Closure Theorems

Let the functions $f_j(z)$, $j = 1, 2, \cdots, I$, be defined by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, \quad a_{n,j} \geq 0 \tag{4.1}$$

for $z \in U$.

Closure theorems for the class $\Omega_m^*(\delta, \lambda, \alpha, \beta, b)$ are given by the following theorems.

Theorem 4.1. *Let the functions $f_j(z)$ defined by (4.1) be in the class*

$$\Omega_m^*(\delta, \lambda, \alpha, \beta, b),$$

$\alpha, \sigma \geq 0$, $\beta, \lambda, \mu > 0$, $\lambda \neq \alpha$ and $m \in \mathbb{N}_0$, for every $j = 1, 2, \cdots, I$. Then the function $G(z)$ defined by

$$G(z) = z - \sum_{n=2}^{\infty} p_n z^n, \quad p_n \geq 0 \tag{4.2}$$

is a member of the class $\Omega_m^*(\delta, \lambda, \alpha, \beta, b)$, where

$$p_n = \frac{1}{I} \sum_{j=1}^{I} a_{n,j} \quad (n \geq 2).$$

Proof: Since $f_j(z) \in \Omega_m^*(\delta, \lambda, \alpha, \beta, b)$, it follows from Theorem 2.1 that

$$\sum_{n=2}^{\infty} [1 + \delta(n-1)] \left[\frac{\mu + \lambda + (n-1)(\lambda - \alpha)(\beta - \sigma)}{\mu + \lambda} \right]^m a_{n,j} \leq |b|$$

for every $j = 1, 2, \cdots, I$.

Hence,

$$\begin{aligned}
&\sum_{n=2}^{\infty} [1 + \delta(n-1)] \left[\frac{\mu + \lambda + (n-1)(\lambda - \alpha)(\beta - \sigma)}{\mu + \lambda} \right]^m p_n \\
&\quad = \sum_{n=2}^{\infty} [1 + \delta(n-1)] \left[\frac{\mu + \lambda + (n-1)(\lambda - \alpha)(\beta - \sigma)}{\mu + \lambda} \right]^m \left\{ \frac{1}{I} \sum_{j=1}^{I} a_{n,j} \right\} \\
&\quad = \frac{1}{I} \sum_{j=1}^{I} \left(\sum_{n=2}^{\infty} [1 + \delta(n-1)] \left[\frac{\mu + \lambda + (n-1)(\lambda - \alpha)(\beta - \sigma)}{\mu + \lambda} \right]^m a_{n,j} \right) \\
&\quad \leq \frac{1}{I} \sum_{j=1}^{I} |b| = |b|
\end{aligned}$$

which implies that $G(z) \in \Omega_m^*(\delta, \lambda, \alpha, \beta, b)$. □

Theorem 4.2. *The class $\Omega_m^*(\delta, \lambda, \alpha, \beta, b)$ is closed under convex linear combination, where $\alpha, \sigma \geq 0$, $\beta, \lambda, \mu > 0$, $\lambda \neq \alpha$ and $m \in \mathbb{N}_0$.*

Proof: Suppose that the functions $f_j(z)$ $(j = 1, 2)$ defined by (4.1) are in the class $\Omega_m(\delta, \lambda, \alpha, \beta, b)$. It is suffices to prove that the function

$$H(z) = \varphi f_1(z) + (1 - \varphi) f_2(z) \quad (0 \leq \varphi \leq 1) \tag{4.3}$$

is also in the class $\Omega_m(\delta,\lambda,\alpha,\beta,b)$.

Since, for $0 \le \varphi \le 1$,

$$H(z) = z + \sum_{n=2}^{\infty} \{\varphi a_{n,1} + (1-\varphi)a_{n,2}\}\, z^n,$$

we observe that

$$\begin{aligned}
&\sum_{n=2}^{\infty} [1+\delta(n-1)] \left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m \{\varphi a_{n,1} + (1-\varphi)a_{n,2}\} \\
&\qquad = \varphi \sum_{n=2}^{\infty} [1+\delta(n-1)] \left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_{n,1} \\
&\qquad + (1-\varphi) \sum_{n=2}^{\infty} [1+\delta(n-1)] \left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_{n,2} \\
&\qquad \le \varphi\, |b| + (1-\varphi)\, |b| = |b|\,.
\end{aligned}$$

Hence $H(z) \in \Omega_m(\delta,\lambda,\alpha,\beta,b)$. This completes the proof of Theorem 4.2. □

5. Integral Operators

In this section, we consider integral transforms of functions in the class

$$\Omega_m^*(\delta,\lambda,\alpha,\beta,b).$$

Theorem 5.1. *If the function f defined by (1.6) is in the class $\Omega_m^*(\delta,\lambda,\alpha,\beta,b)$, where $\alpha,\sigma \ge 0$, $\beta,\lambda,\mu > 0$, $\lambda \ne \alpha$, $m \in \mathbb{N}_0$. Then the function $F(z)$ defined by*

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)dt, \quad (c > -1) \tag{5.1}$$

also belongs to the class $\Omega_m^*(\delta,\lambda,\alpha,\beta,b)$.

Proof: From (5.1), it follows that $F(z) = z - \sum_{n=2}^{\infty} k_n z^n$, where $k_n = \left(\frac{c+1}{c+n}\right) a_n$. Therefore

$$\begin{aligned}
&\sum_{n=2}^{\infty} [1+\delta(n-1)] \left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m k_n \\
&\qquad = \sum_{n=2}^{\infty} [1+\delta(n-1)] \left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m \left(\frac{c+1}{c+n}\right) a_n \\
&\qquad \le \sum_{n=2}^{\infty} [1+\delta(n-1)] \left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_n \le |b|\,,
\end{aligned}$$

since $f(z) \in \Omega_m^*(\delta,\lambda,\alpha,\beta,b)$. Hence by Theorem 2.1, $F(z) \in \Omega_m^*(\delta,\lambda,\alpha,\beta,b)$. □

6. Close-to-Convexity, Starlikeness and Convexity

A function $f \in \mathcal{A}$ is said to be close-to-convex of order η if it satisfies

$$\operatorname{Re}\left\{f^{'}(z)\right\} > \eta, \tag{6.1}$$

for some $\eta (0 \leq \eta \leq 1)$ and for all $z \in U$. Also a function $f \in \mathcal{A}$ is said to be starlike of order η if it satisfies

$$\operatorname{Re}\left\{\frac{zf^{'}(z)}{f(z)}\right\} > \eta, \tag{6.2}$$

for some $\eta (0 \leq \eta \leq 1)$ and for all $z \in U$. Further, a function $f \in \mathcal{A}$ is said to be convex of order η, if and only if $zf^{'}(z)$ is starlike of order η, that is if

$$\operatorname{Re}\left\{1 + \frac{zf^{''}(z)}{f^{'}(z)}\right\} > \eta, \tag{6.3}$$

for some $\eta (0 \leq \eta \leq 1)$ and for all $z \in U$.

Theorem 6.1. *If $f \in \Omega_m^*(\delta, \lambda, \alpha, \beta, b)$, then $f(z)$ is close-to-convex of order η in $|z| < h_1(\mu, \delta, b, \eta)$, where*

$$h_1(\mu, \delta, b, \eta) = \inf_n \left\{\frac{(1-\eta)\left[1+\delta(n-1)\right]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m}{n\,|b|}\right\}^{\frac{1}{n-1}}.$$

Proof: It is sufficient to show that

$$\left|f^{'}(z) - 1\right| < \sum_{n=2}^{\infty} n a_n \left|z\right|^{n-1} \leq 1 - \eta \tag{6.4}$$

and

$$\sum_{n=2}^{\infty}\left[1+\delta(n-1)\right]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m a_n \leq |b|\,.$$

Observe that (6.4) is true if

$$\frac{n\,|z|^{n-1}}{1-\eta} \leq \frac{\left[1+\delta(n-1)\right]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m}{|b|}. \tag{6.5}$$

Solving (6.5) for $|z|$, we obtain

$$|z| \leq \left\{\frac{(1-\eta)\left[1+\delta(n-1)\right]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m}{n\,|b|}\right\}^{\frac{1}{n-1}} \quad (n \geq 2).$$

$\square$

Theorem 6.2. *If $f \in \Omega_m^*(\delta, \lambda, \alpha, \beta, b)$, then $f(z)$ is starlike of order η in $|z| < h_2(\mu, \delta, b, \eta)$, where*

$$h_2(\mu, \delta, b, \eta) = \inf_n \left\{ \frac{(1-\eta)\left[1+\delta(n-1)\right]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m}{(n-\eta)\,|b|} \right\}^{\frac{1}{n-1}}.$$

Proof: We must show that $\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\eta$ for $|z| < h_2(\mu, \delta, b, \eta)$.

Since

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n\,|z|^{n-1}}{1-\sum\limits_{k=2}^{\infty}a_n\,|z|^{n-1}}$$

if $\frac{(n-\eta)|z|^{n-1}}{1-\eta} \leq \frac{[1+\delta(n-1)]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m}{|b|}$, $f(z)$ is starlike of order η. □

Corollary 6.3. *If $f \in \Omega_m^*(\delta, \lambda, \alpha, \beta, b)$, then $f(z)$ is convex of order η in $|z| < h_3(\mu, \delta, b, \eta)$, where*

$$h_3(\mu, \delta, b, \eta) = \inf_n \left\{ \frac{(1-\eta)\left[1+\delta(n-1)\right]\left[\frac{\mu+\lambda+(n-1)(\lambda-\alpha)(\beta-\sigma)}{\mu+\lambda}\right]^m}{n(n-\eta)\,|b|} \right\}^{\frac{1}{n-1}}.$$

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