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Martindale's Like Results in Inverse Semirings

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ABSTRACT: The purpose of this paper is to determine extended centroid of an inverse semiring. We also generalize a few striking results of W.S Martindale on extended centroid of rings to inverse semirings.

Key Words: Inverse semiring, Prime Semiring, Right S-semimodule, Extended Centroid.

Contents

1. Introduction

W. S Martindale[12] introduced ring of quotients and extended centroid, as a key tool to study prime rings satisfying polynomial identities. His concept later generalized for semiprime rings[1]. The notion of extended centroid has significant role in various branches of algebra. For example, in the study of functional identities[3], Galois theory ([9],[10],[13]) and additive mappings ([2],[7],[11]). Recently, extended centroid of multiplicatively cancellative semiring considered in [16]. Our aim is to investigate Martindale's work for inverse semirings. We generalize some results concerning extended centroid of rings to inverse semirings. These results might be fruitful in enriching the theory of semiring in various other directions of algebra.

By S, we mean a semiring $(S, +, .)$ with commutative addition and an absorbing zero. S is called an inverse semiring[8] if for every $a \in S$ there exists a unique element $a' \in S$ such that $a + a' + a = a$ and $a' + a + a' = a'$, where a' is called pseudo inverse of a. If X is a nonempty set and S is an inverse semiring then the set of all mappings $Map(X, S)$ from X into S is also an inverse semiring, where for every $f \in Map(X, S)$, the pseudo inverse f' is defined as $f'(x) = f(x)'$, $x \in X$. Throughout this paper, S will denote an inverse semiring such that $a+a'$ is in center of S. This class of semiring is known as MA semiring which has been studied in several directions([8],[14],[15]). S is prime if $aSb = 0$ implies that either $a = 0$ or $b = 0$. It is observed that center of a prime inverse semiring is zero divisor free. Let S be a semiring then a right S-semimodule is a commutative monoid $(M, +)$, with additive identity 0_M, for which we have the function $M \times S \to M$, denoted by $(s, m) \mapsto sm$, which satisfy the following conditions, for all elements s_1 and s_2 of S and all elements m_1 and m_2 of M :

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(i) $m(s_1s_2) = (ms_1)s_2$
(ii) $(m_1 + m_2)s = m_1s + m_2s$
(iii) $m1_S = m$
(iv) $0_Ms = 0_M = m0_S$.
Let M and N be right S-semimodules then an additive mapping $\alpha : M \to N$ is right S-semimodule homomorphism if $\alpha(ms) = \alpha(m)s$ for all $m \in M$ and $s \in S$.

We will need the following lemma.

Lemma 1{Lemma 1.1, [14]}. Let S be an inverse semiring, $a, b \in S$. Then $a+b = 0$ implies that $a = b'$.

Construction of Right Martindale Semiring of Quotients

Let $S \neq 0$ be a prime semiring and Ω be the set of all non-zero ideals of S which is closed under finite intersection and product of ideals. Let $\Delta = \{(f, I); f : I \to S$ is right S-semimodule homomorphism, where $I \in \Omega\}$. Define a relation $\sim$ on Δ as follows; $(f, I) \sim (g, J)$ iff f coincides with g on some $K \in \Omega$ such that $K \subseteq I \cap J$ then $\sim$ is an equivalence relation. Let $[f, I]$ be the equivalence class determined by (f, I). Denote the set of all equivalence classes with $Q_r(S)$. Then $Q_r(S)$ forms a semiring with respect to the following operations

$$[f, I] + [g, J] = [f + g, I \cap J]$$
$$[f, I].[g, J] = [fg, JI]$$

Here fg is defined on JI because $g(JI) = g(J)I \subseteq I$. These operations are well-defined, indeed if, $[f_1, I_1] = [g_1, J_1]$ and $[f_2, I_2] = [g_2, J_2]$, that is; $f_i = g_i$ on some $K_i \in \Omega$ such that $K_i \subseteq I_i \cap J_i$, $i = 1, 2$. Then $f_1 + f_2 = g_1 + g_2$ on $K_1 \cap K_2 \in \Omega$ and $f_1f_2 = g_1g_2$ on $K_2K_1 \in \Omega$. Thus $[f_1 + f_2, I_1 \cap I_2] = [g_1 + g_2, J_1 \cap J_2]$ and $[f_1f_2, I_2I_1] = [g_1g_2, J_2J_1]$. It is easy to calculate that $Q_r(S)$ forms a semiring with $[id_S, S]$ as identity element and $[0, S]$ as absorbing zero.

Moreover, If S is an inverse semiring so is $Q_r(S)$ such that for every element $[f, I] \in Q_r(S)$, $[f', I]$ is pseudo inverse of $[f, I]$. Thus $[f, I]+[f', I]+[f, I] = [f, I]$ and $[f', I] + [f, I] + [f', I] = [f', I]$, where $f' : I \to S$ is defined as $f'(x) = f(x)', x \in I$.

In what follows, S will be a prime inverse semiring, $E(S)$, set of all additively idempotent elements in S such that $Ann_l(E(S)) \neq 0$, where $Ann_l(E(S))$ is left annihilator of $E(S)$. As an example, consider the inverse semiring, $S = M_2(\mathbb{N}) = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathbb{N}\}$, in which $Ann_l(E(S)) \neq 0$, where $\mathbb{N}$ is prime inverse semiring with addition $x + y = max(a, b)$ and usual multiplication.

Theorem 2. Let S be a prime inverse semiring and Ω be the set of all non-zero ideals of S. Then $Q_r(S)$ has following properties.

a) $Q_r(S)$ is a unital semiring containing S as subsemiring.
b) For every $q \in Q_r(S)$ there exists $I \in \Omega$ such that $qI \subseteq S$.
c) For every $q \in Q_r(S)$ and $I \in \Omega, qI = 0 \Rightarrow q = 0$.
d) If $I \in \Omega$ and $f : I \to S$ a right S-semimodule homomorphism then there exists $q \in Q_r(S)$ such that $f(x) = qx, \forall x \in I$.

Moreover, these properties characterize $Q_r(S)$ upto isomorphism.

Proof: We have seen that $Q_r(S)$ is a unital semiring. Define $\rho : S \to Q_r(S)$ by $\rho(s) = [L_s, S], s \in S$, where L_s is a left multiplication map. Clearly, ρ is a well-defined homomorphism. To see ρ is $(1-1)$, let $\rho(s) = \rho(t)$, so we have, $L_s(x) = L_t(x)$ for all $x \in I, I \in \Omega$. Thus $sx = tx, x \in I$. It follows that $(s+s')x = (t+s')x, x \in I$. But $Ann_l(E(S)) \neq 0$, let $0 \neq x_1 \in Ann_l(E(S))$ then we get, $0 = x_1(t+s')x, x \in I$. Since $(t+s')x$ is in center of S and center of prime inverse semiring is zero divisor free so we arrive at $(t+s')x = 0$. Thus $t + s' = 0$. By lemma 1, it follows that ρ is (1-1). Hence, ρ is an embedding of S into $Q_r(S)$. Identifying S with its isomorphic copy $\rho(S)$, we may consider S as subsemiring of $Q_r(S)$. To prove (c), let $q = [f, I] \in Q_r(S)$ then for every $x \in I$, we have,

$$qx = [f, I][L_x, S] = [fL_x, SI] = [L_{f(x)}, S] = f(x) \in S$$

This establishes (c). If $qI = 0$ then $f(I) = 0$ hence $q = [f, I] = 0$. Also, if $qJ = 0$, for any other $J \in \Omega$, then $qIJ \subseteq qJ = 0$. But S is prime thus $qI = 0$ implies $q = 0$, it proves (d). As we have seen above, for $I \in \Omega$ and a right S-semimodule homomorphism $f : I \to S$ we can select $q = [f, I] \in Q_r(S)$, as required by (d).

Let Q be arbitrary semiring which satisfies conditions (a)-(d). Let $q \in Q$ then by (b) there exists $I \in \Omega$ such that $qI \subseteq S$. Thus we can define right S-semimodule homomorphism $f : I \to S$ such that $f(x) = qx, x \in I$. If there is another $J \in \Omega$ such that $qJ \subseteq S$ then $g : J \to S$ defined as $g(x) = qx, x \in J$ coincides with f on $I \cap J$. Thus we have a map $\phi : Q \to Q_r(S)$ defined by $\phi(q) = [f, I]$ which is well-defined. ϕ is also a homomorphism. Injectivity and ontoness follows from (c) and (d). Hence ϕ is an isomorphism.

Lemma 3. Let $q_1, q_2 \in Q_r(S)$ and $I \in \Omega$ then $q_1 I q_2 = 0$ implies either $q_1 = 0$ or $q_2 = 0$.

Proof: Using (b), there exists $I_i \in \Omega$ such that $q_i I_i \subseteq S$, i= 1,2. Thus

$$(q_1 I_1) I (q_2 I_2) \subseteq q_1 I q_2 I_2 = 0,$$

that is; $(q_1 I_1) x (q_2 I_2) = 0, x \in I$. Replacing x by sx, $s \in S$ and using primeness of S and (c), we obtain the required result.

Lemma 4. Let $q_i \in Q_r(S)$, (i= 1, 2...n) then there exists $I \in \Omega$ such that $q_i I \subseteq S$.

Proof: By (b) there exists $I_i \in \Omega$ such that $q_i I_i \subseteq S$. Thus $I = \cap I_i$ has the desired property.

Lemma 5. If $q_1, q_2 \in Q_r(S)$ and $I \in \Omega$ such that $q_1 i = q_2 i$, for all $i \in I$ then $p = q$.

Proof: Let $q_1 = [f, I]$ and $q_2 = [g, J]$. As S can be considered as subsemiring of $Q_r(S)$, thus for every $i \in I$, $q_1 i = q_2 i$ implies that $[f, I][L_i, S] = [g, J][L_i, S]$ or $[fL_i, SI] = [gL_i, SJ]$. This implies that $fL_i = gL_i$ on some $K \in \Omega$ such that $K \subseteq SI \cap SJ$. That is, $f(ik) = g(ik), k \in K$. From this, we can conclude that $f = g$ on $IK \in \Omega$, where $IK \subseteq I \cap J$. Thus $q_1 = [f, I] = [g, J] = q_2$.

Definition 6. The set $C = \{g \in Q_r(S) : gf = fg, \forall f \in Q_r(S)\}$ is called the extended centroid of a semiring S. It is easily seen that C is a subsemiring of $Q_r(S)$.

Lemma 7. Let $f : I \to S$ be S-bisemimodule homomorphism(a map which is both right and left S-semimodule homomorphism), where $I \in \Omega$ then there exists $q \in C$ such that $f(x) = qx, x \in I$.

Proof: Let q be an element of $Q_r(S)$ determined by f that is; $q = [f, I]$. As in theorem 2, for $x \in I$, $f(x) = qx$. Thus we only need to show that $q \in C$. Let $p = [g, J]$ be arbitrary element of $Q_r(S)$ then $pq = [gf, JI]$ and $qp = [gf, IJ]$. Consider JK where $IJ \cap JI = K$ then $JK \in \Omega$ such that $JK \subseteq IJ \cap JI$. Let $x = \sum_{n=1}^{n} a_i b_i$ be arbitrary element of JK then we have, $gf(x) = gf(a_1b_1 + ... + a_nb_n) = g(f(a_1b_1) + ... + f(a_nb_n)) = g(a_1f(b_1) + ... + a_nf(b_n)) = g(a_1f(b_1)) + ... + g(a_nf(b_n)) = f(g(a_1)b_1) + ... + f(g(a_n)b_n) = f(g(a_1b_1) + ... + f(g(a_nb_n) = fg(x)$.
Thus gf and fg coincides on JK. Hence, $pq = qp$ or $q \in C$. This completes the proof.

Lemma 8. The extended centroid C of a non-zero prime semiring S is semifield.

Proof: Let $(0 \neq \lambda) \in C$ be arbitrary element. Let $I \in \Omega$ such that $\lambda I \subseteq S$. If $\lambda I = 0$ then by (c), $\lambda = 0$ therefore, $\lambda I \in \Omega$. Define a map $f : \lambda I \to S$ by $f(\lambda x) = x$, then f is well-defined. Indeed, if for some $x \in I$, $\lambda x = 0$ then $\lambda I_x = 0$, where I_x is ideal of S generated by S. From this and (c), we obtain that $x = 0$. Hence, f is well-defined. It is easy to see that f is S-bisemimodule homomorphism. Thus from above lemma there exists $\beta \in C$ such that $f(y) = \beta y$, $y \in \lambda I$. Put $y = \lambda x$ we obtain $\beta\lambda x = x$. By lemma 5, we can conclude that C is semifield.

Theorem 9. Let S be prime inverse semiring, $a, b \in Q_r(S)$ and $I \in \Omega$ such that $axb + bxa' = 0$, for all $x \in I$, then there exists $\lambda \in C$ such that $b = \lambda a$.

Proof: By lemma 4, there exists $J \in \Omega$ such that $aJ \subseteq S$ and $bJ \subseteq S$. Let $K = I \cap J$ then $aK \subseteq S$ and $bK \subseteq S$. Let $a \neq 0$ then $aK \neq 0$, thus $V = KaK \in \Omega$. Define a map $f : V \to S$ by $f(\sum_i x_i a y_i) = \sum_i x_i b y_i, x_i, y_i \in K$. Then f is well-defined. If $\sum_i x_i a y_i = 0$, then $\sum_i x_i a(y_i z)b = 0$, where $z \in S$. From above lemma and $axb + bx\acute{a} = 0$, we obtain $\sum_i x_i b y_i = 0$. Moreover, f is S-bisemimodule homomorphism. Thus by lemma 7, there exists $\lambda \in C$ such that $f(v) = \lambda v, v \in V$. In particular, $xby = \lambda xay$, for all $x, y \in K$. Post multiplying $axb + b'xa = 0$ by y and using the last relation we have, $(a\lambda + b')xay = 0$. Primeness of S and lemma 1 implies that $a\lambda = b$.

Theorem 10. Let $a_i, b_i \in Q_r(S)$, $(i = 1, 2, ..., n)$ and $I \in \Omega$ such that

$$\sum_1^n a_i x b_i = 0 \tag{1.1}$$

for all $x \in I$. If $a_1, ...a_n$ are linearly independent over C then each $b_i = 0$. Similarly, if $b_1, ..., b_n$ are linearly independent over C then each $a_i = 0$.

Proof: Suppose that $a_1, ..., a_n$ are linearly independent. The case $n = 1$ follows from lemma 3. Assume that the result holds if the number of summands is smaller than n. Choose $0 \neq b_n \in Q_r(S)$ then by (b), there exists $J \in \Omega$ such that $b_n J \subseteq S$. Thus $xb_n y \in I$, $x \in I$, $y \in J$ and therefore

$$\sum_1^n a_i x b_n y b_i = 0 \tag{1.2}$$

By using lemma 1 in (1), we have, $a_n x b_n = \sum_1^{n-1} a_i x \acute{b_i} = 0$. Thus from (2), we get

$$\sum_1^{n-1} a_i x(b_n y b_i + b_i' y b_n) = 0$$

for all $x \in I, y \in J$. By assumption $b_n y b_i + b_i' y b_n = 0$, $y \in J$ and $i = 1, ..., n-1$. By theorem 9, there exist $\lambda_i \in C$ such that $b_i = \lambda_i b_n, i = 1, ..., n-1$. Thus from (1) we have, $\sum_1^n a_i x \lambda_i b_n = 0$, where $\lambda_n = 1$ or $\sum_1^n (\lambda_i a_i) x b_n = 0$. By lemma 3, $\sum_1^n \lambda_i a_i = 0$. But $\lambda_n = 1$, contradicting the linear independence of the a_i's. Similarly, if b_i's are linearly independent then $a_i = 0$.

References

1. S.A Amitsur, *On rings of quotients*, Sympos. Math., 8 (1972), 149 − 164.
2. H. E. Bell and W.S Martindale, *Centralizing mappings of semiprime rings*, Cand. Math. Bull., 30(1987)1, 92 − 101.
3. M. Bresar, M. A. Chebotar, W. S. Martindale, *Functional identities*, Birkhauser Verlag, Boston (2007).
4. M.Bresar, *An Introduction to noncommutative Algebra*, Universitext, Springer International Publishing Switzerland, 2014.
5. J.S Golan, *The theory of semirings with applications in mathematics and theoretical computer science.*, John Wiley and Sons. Inc., New York, (1992).
6. U. Habisch, H.J. Weinert, *Semirings- Algebraic theory and applications in computer science*, World Scientific, 1993.

7. B.Hvala, *Generalized derivations in Rings*,Communication in Algebra, 26(1998)4, $1147-1166$.

8. P.H. Karvellas, *Inversive semirings*, J. Aust. Math. Soc. 18 (1974), $277-288$.

9. V.K Kharchenko, *Generalized identities with automorphisms Algebra* i Logika, 14 (1975), $215-237$.

10. V.K Kharchenko, *Galois theory of semiprime rings*, Algebra i Logika, 16 (1977), $313-363$.

11. C.Lanski, *Differential identities, lie ideals and Posner's theorem*, Pacific Journal of Math., 134(1988)2, $275-297$.

12. W.S Martindale, *Prime rings satisfying a generalized polynomial identity*, Journal of Algebra 12(1969), $576-584$.

13. S.Montgomery, D.S Passman, *Galois theory of prime rings*, J. Pure Appl. Algebra, 31 (1984), $139-184$.

14. S.Sara, M.Aslam and M.A. Javed *On Centralizer of semiprime inverse semirings*, Discussiones Mathematicae - General Algebra and Applications, 36 (2016), $71-72$.

15. S.Sara, M.Aslam, *On Jordan Mappings of inverse semirings*, Open Math., 15(2017), $1123-1131$.

16. H.Yazarli, M.A. Oztruk, *On the centroid of prime semirings*,Turk. J. Math. 37 (2013), $577-584$.

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