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On the Regularity of Solutions to the Poisson Equation in Musielak-Orlicz Spaces

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ABSTRACT: In this paper, we study some regularity results of solutions of the Poisson equation $\triangle u = f$, in Musielak-Orlicz spaces.

Key Words: Musielak-Orlicz spaces, Distributions, Poisson equation, Newtonian potential.

Contents

1. Introduction

Let Ω be an open subset of $\mathbb{R}^N$ with $N \geq 2$ and let f be a distribution on Ω. We consider the Poisson equation

$$\triangle u = f \text{ in } \Omega. \tag{1.1}$$

The regularity of solution u of the equation(1.1), in relation to the regularity of the second member f is one of the classical questions concerning this equation (cf .[6-9]).
In particular, it's well known that if u is a solution de (1.1) then:
(R_1) If f is a distribution of order 1 (resp. 0), then the solution $u \in L^p_{loc}(\Omega)$ for all $p < \frac{N}{N-1}$ (resp. $u \in L^p_{loc}(\Omega)$ for all $p < \frac{N}{N-2}$ and $\frac{\partial u}{\partial x_i} \in L^q_{loc}(\Omega)$ for all $q < \frac{N}{N-1}$).
(R_2) If $f \in L^r_{loc}(\Omega)$ with $r > \frac{N}{2}$ (resp. $r > N$), then u is continuous (resp. continuously differentiable) on Ω.
(R_3) If $f \in L^p(\Omega)$, then $\frac{\partial^2 u}{\partial x_i \partial x_j} \in L^p(\Omega)$ for all $1 < p < \infty$.
In the case where $p = 1$ (resp. $p = \infty$) the second derivatives of u are in general not integrable (resp. bounded) in Ω.
These results are been generalized by E.Azroul,A.Benkirane and M.Tienari in the setting of Orlicz-spaces([3])$L_M(\Omega)$, where the exponent function t^p is replaced by an N-function M convex even and nondecreasing .

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In this work, we propose to generalize the previous results ([3]) to the case of the spaces of Musielak-Orlicz $L_\varphi(\Omega)$,where $\varphi(x,t)$ is an N-function with respect to t and measurable function with respect to x.
The main difficulty in our generalization lies in the fact that the function of Musielak-Orliczφ depends also on the space variable, which makes impossible the use of classical operators of translation and convolution. Our results allow us, in particular, to obtain results of regularity in the case of Lebesgue spaces with variable exponents, see corollaries (3.1) and (3.2).

This work is organized as follows: In Section 2 we recall some will-know preliminaries ,and results of Orlicz-spaces, Musielak -Orlicz Sobolev Spaces. In Section 3 we prove the analogous regularity result of $(R1)$ and $(R2)$ in the general setting of Musielak-Orlicz spaces. Final section is devoted to obtain in the radial case some regularity results on the second derivatives u,

2. Preliminary

2.1- Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function (i.e. M is continuous, strictly increasing, convex with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ (resp. $+\infty$) as $t \to 0^+$ (resp. $t \to +\infty$).
Equivalently, M admits the representation $M(t) = \int_0^t a(s)ds$ where the function $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \to +\infty$ as $t \to +\infty$.

2.2- Let Ω be an open subset of $\mathbb{R}^N$ and M an N-function. The Orlicz class $K_M(\Omega)$ is defined as the set of real-valued measurable function u on Ω such that

$$\int_\Omega M(u(x))dx < \infty.$$

The Orlicz-space $L_M(\Omega)$ is the set of (equivalence classes of) real valued measurable functions u such that $\frac{u}{\lambda} \in K_M(\Omega)$, for $\lambda = \lambda(u) > 0$.

Lemma 2.1. *[10] Let M be an N-function and u be an element of $L_M(\mathbb{R}^N)$ with $2u \in K_M(\mathbb{R}^N)$ and $T_y u$ the translation of u, i.e. $T_y u(x) = u(x - y)$, then*

$$\int_{R^N} M(T_y u - u)dx \to 0 \ \ as \ \ |y| \to 0$$

2.3- Musielak-Orlicz function
Let Ω be an open set in $\mathbb{R}^N$ and let φ be a real-valued function defined on $\Omega \times \mathbb{R}_+$ and satisfying the following conditions

a) $\varphi(x, .)$ is an N-function for all $x \in \Omega$.

b) $\varphi(., t)$ is a measurable function for all $t \geq 0$.

A function φ which satisfies the conditions $a)$ and $b)$ is called Musielak-Orlicz function.

In this section we define Lebesgue spaces with variable exponents, $L^{p(.)}$.
They differ classical L^p spaces in that the exponent p is not constant but a function from Ω to $[1, +\infty]$
We define $\mathcal{P}(\Omega)$ to be the set of all measurable function $p : \Omega \to (1, +\infty)$
$p \in \mathcal{P}(\Omega)$ are called variable exponents on Ω.
We define $p_- = ess\inf_{x\in\Omega} p(x)$ and $p^+ = ess\sup_{x\in\Omega} p(x)$.If$p^+ < +\infty$,then we can p a bounded variable exponent .

For a Musielak-Orlicz function φ we put $\varphi_x(t) = \varphi(x, t)$ and we associate its nonnegative reciprocal function φ_x^{-1}, with respect to t that is

$$\varphi_x^{-1}(\varphi(x, t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

The Musielak-Orlicz function φ is said to satisfy the Δ_2-condition if for some $k > 0$ and a non negative function h integrable in Ω, we have

$$\varphi(x, 2t) \leq k\varphi(x, t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{2.1}$$

When (2.1) holds only for $t \geq t_0 > 0$, then φ is said to satisfy Δ_2 near infinity.

Let φ and γ be two Musielak-Orlicz functions, we say that φ dominate γ, and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for almost all $x \in \Omega$

$$\gamma(x, t) \leq \varphi(x, ct) \text{ for all } t \geq t_0, \quad (\text{ resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity), and we write $\gamma \prec\prec \varphi$, if for every positive constant c we have

$$\lim_{t\longrightarrow 0}\left(\sup_{x\in\Omega}\frac{\gamma(x, ct)}{\varphi(x, t)}\right) = 0, \quad (\text{resp. } \lim_{t\longrightarrow\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x, ct)}{\varphi(x, t)}\right) = 0).$$

2.4- Musielak-Orlicz spaces
We define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx$$

where $u : \Omega \longrightarrow \mathbb{R}$ is a Lebesgue measurable function.

The set

$$K_\varphi(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \Big/ \rho_{\varphi,\Omega}(u) < +\infty\right\}$$

is called the generalized Orlicz class.
The Musielak-Orlicz space (or the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing

the set $K_\varphi(\Omega)$.
Equivalently

$$L_\varphi(\Omega)=\Big\{u:\Omega\longrightarrow\mathbb{R}\ \Big/\rho_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big)<+\infty,\ \text{for some } \lambda>0\Big\}.$$

If $p\in\mathcal{P}(\Omega)$,the spaces $L^{p(.)}$ fit into the framework of Musielak-Olicz spaces ,then we have:

$$L^{p(.)}(\Omega)=\Big\{u:\Omega\to\mathbb{R}/\int_\Omega(\frac{|f|}{\lambda})^{p(x)}dx<+\infty,\text{for some } \lambda>0\Big\}$$

The Musielak-Orlicz function complementary to φ in the sense of Young with respect to the variable t, is defined by

$$\varphi^*(x,s)=\sup_{t\geq 0}\{st-\varphi(x,t)\}.$$

We define in the space $L_\varphi(\Omega)$ the following norm

$$\|u\|_{\varphi,\Omega}=\inf\Big\{\lambda>0/\int_\Omega\varphi\Big(x,\frac{|u(x)|}{\lambda}\Big)dx\leq 1\Big\}.$$

Let $E_\varphi(\Omega)$ denote the closure in $L_\varphi(\Omega)$ of the space of function u, which are bounded on Ω and have bounded support in $\overline{\Omega}$, it is a separable space and $(E_{\varphi^*}(\Omega))^*=L_\varphi(\Omega)$.(see [15])
We say that a sequence of functions $u_n\in L_\varphi(\Omega)$ is modular convergent to $u\in L_\varphi(\Omega)$ if there exists a constant $\lambda>0$ such that

$$\lim_{n\to\infty}\rho_{\varphi,\Omega}\Big(\frac{u_n-u}{\lambda}\Big)=0.$$

For any fixed nonnegative integer m we define

$$W^mL_\varphi(\Omega)=\Big\{u\in L_\varphi(\Omega):\forall|\alpha|\leq m,\ D^\alpha u\in L_\varphi(\Omega)\Big\}.$$

and

$$W^mE_\varphi(\Omega)=\Big\{u\in E_\varphi(\Omega):\forall|\alpha|\leq m,\ D^\alpha u\in E_\varphi(\Omega)\Big\}.$$

where $\alpha=(\alpha_1,...,\alpha_n)$ with nonnegative integers α_i, $|\alpha|=|\alpha_1|+...+|\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^mL_\varphi(\Omega)$ is called the Musielak-Orlicz-Sobolev space.
Let

$$\overline{\rho}_{\varphi,\Omega}(u)=\sum_{|\alpha|\leq m}\rho_{\varphi,\Omega}\Big(D^\alpha u\Big)\text{ and }\|u\|^m_{\varphi,\Omega}=\inf\Big\{\lambda>0:\overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big)\leq 1\Big\}$$

for $u \in W^mL_\varphi(\Omega)$, these functionals are a convex modular and a norm on $W^mL_\varphi(\Omega)$, respectively, and the pair $\left(W^mL_\varphi(\Omega), \|\|^m_{\varphi,\Omega}\right)$ is a Banach space(see [15]) if φ satisfies the following condition :

$$\text{there exist a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \geq c. \tag{2.2}$$

The space $W^mL_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_{\varphi^*})$ closed.
We denote by $D(\Omega)$ the space of infinitely smooth functions with compact support in Ω and by $D(\overline{\Omega})$ the restriction of $D(\mathbb{R}^N)$ on Ω.
Let $W_0^mL_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_{\varphi^*})$ closure of $D(\Omega)$ in $W^mL_\varphi(\Omega)$.
Let $W^mE_\varphi(\Omega)$ the space of functions u such that u and its distribution derivatives up to order m lie to $E_\varphi(\Omega)$, and $W_0^mE_\varphi(\Omega)$ is the (norm) closure of $D(\Omega)$ in $W^mL_\varphi(\Omega)$.
The following spaces of distributions will also be used :

$$W^{-m}L_{\varphi^*}(\Omega) = \left\{ f \in D'(\Omega);\ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_{\varphi^*}(\Omega) \right\}.$$

and

$$W^{-m}E_{\varphi^*}(\Omega) = \left\{ f \in D'(\Omega);\ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_{\varphi^*}(\Omega) \right\}.$$

We say that a sequence of functions $u_n \in W^mL_\varphi(\Omega)$ is modular convergent to $u \in W^mL_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

For a Musielak-Orlicz function φ, the following inequality is called the Young inequality:

$$ts \leq \varphi(x, t) + \varphi^*(x, s), \quad \forall t, s \geq 0, x \in \Omega. \tag{2.3}$$

For a Musielak-Orlicz function φ, let $u \in L_\varphi(\Omega)$ and $v \in L_{\varphi^*}(\Omega)$, then we have the Hölder inequality

$$\left| \int_\Omega u(x)v(x)dx \right| \leq 2\|u\|_{\varphi,\Omega}\|v\|_{\varphi^*,\Omega}. \tag{2.4}$$

Let us define:

$$E^\varphi(\Omega) = \left\{ f \in L_\varphi(\Omega) : \int_\Omega \varphi(x, \lambda|f|)dx < +\infty, \forall\lambda > 0 \right\}$$

Remark 2.2. *[7] The set E^φ is a closed subset of L_φ.*

Definition 2.3. *A Musielak-Orlicz function φ is called locally integrable on Ω if: $\int_\Omega \varphi(x, t\chi_E)dx < +\infty$, for all $t \geq 0$ and all measurable set $E \subset \Omega$ with $mes(E) < +\infty$.*

Theorem 2.4. *[7] Let φ be a Musielak-Orlicz function locally integrable and let S be the set of simple functions then $\overline{S}^{\|\|_\varphi} = E^\varphi(\Omega)$.*

Remark 2.5. *If Ω is of finite measure and φ a Musielak-Orlicz function locally integrable then*

$$E^\varphi(\Omega) = E_\varphi(\Omega)$$

Indeed: let f be a bounded function with compact support in $\overline{\Omega}$ then:
$\forall\lambda > 0, \int_\Omega \varphi(x, \lambda|f|)dx \leq \int_{Supp f} \varphi(x, \lambda\|f\|_\infty)dx < +\infty$, thus $f \in E^\varphi(\Omega)$
So $E_\varphi(\Omega) \subset E^\varphi(\Omega)$ (Remark 2.2).
Since $mes(\Omega) < +\infty$ then $S \subset E_\varphi(\Omega)$ (S be the set of simple functions)
$\Longrightarrow \overline{S} \subset E_\varphi(\Omega)$
According to the theorem 2.1, we have $E^\varphi \subset E_\varphi$. So,

$$E^\varphi(\Omega) = E_\varphi(\Omega)$$

We put: $C_0(\Omega)$={the continuous function with compact support contained in Ω}

Corollary 2.6. *If Ω is of finite measure and φ is a Musielak-Orlicz function satisfies the condition:*

$$there\, existe\, M\, an\, N-function\, such\, that \quad \varphi(x,t) \leq M(t) \quad \forall x \in \Omega, \forall t \geq 0 \quad (2.5)$$

then

$$\overline{C_0(\Omega)} = E_\varphi(\Omega)$$

Indeed: Firsty by (2.5) ,we get:
Let $t \geq 0$,for all$K$ measurable set,with $mes(K) < +\infty$,

$$\int_\Omega \varphi(x, t\chi_K)dx \leq \int_K M(t)dx = t \cdot mes(K) < +\infty$$

So φ locally integrable ,then $E_\varphi(\Omega) = E^\varphi(\Omega) = \overline{S}$, let $u \in E_\varphi(\Omega)$ and let $\varepsilon > 0$, $\exists s \in S$ such that $\|u - s\|_\varphi < \frac{\varepsilon}{2}$. Since $mes(\Omega) < +\infty$, then $mes(Supp \quad s) < +\infty$, we may also assume that $s(x) = 0$, for all $x \in \Omega^c$. Applying Lusin's Theorem, we obtain a function $\phi \in C_0(\Omega)$ such that :

$$|\phi(x)| \leq \|s\|_\infty \quad \text{for all } x \in \Omega$$

and

$$mes\{x \in \Omega, \phi(x) \neq s(x)\} < \frac{1}{M(\frac{4\|s\|_\infty}{\varepsilon})}$$

Then

$$\begin{aligned}\int_\Omega \varphi(x, \frac{2|s(x)-\phi(x)|}{\varepsilon})dx &= \int_{\{x\in\Omega, s(x)\neq\phi(x)\}} \varphi(x, \frac{4\|s\|_\infty}{\varepsilon})dx \\ &\leq \int_{\{x\in\Omega, s(x)\neq\phi(x)\}} M(\frac{4\|s\|_\infty}{\varepsilon})dx < 1\end{aligned}$$

Consequently $\|s-\phi\|_\varphi < \frac{\varepsilon}{2}$. It follows that

$$\|u-\phi\|_\varphi < \varepsilon.$$

In the papier we assume that φ^* satisfies the condition:

$$\text{there exists } N \text{ an } N-\text{function such that} \quad \varphi^*(x,t) \leq N(t) \quad \forall x\in\Omega, \forall t\geq 0 \tag{2.6}$$

Generalized Hölder's inequality:

A function v locally integrable on Ω with $mes(\Omega) < +\infty$, belong to L_φ iff there exists $c > 0$ such that

$$|\int_\Omega v\xi dx| \leq c\|\xi\|_{\varphi^*,\Omega} \quad for\ all \quad \xi\in C_0(\Omega) \tag{2.7}$$

2.5- We define the Newtonian $P_N(.)$ on $\mathbb{R}^N - \{0\}$ by

$$P_N(x) = P_N(|x|) = \begin{cases} \frac{|x|}{2} \text{ if } N=1 \\ \frac{\log(|x|)}{2\pi} \text{ if } N=2 \\ \frac{1}{k_N|x|^{N-2}} \text{ if } N\geq 3\end{cases}$$

where $K_N = (2-N)\sigma_N$ and σ_N is the measure of $\Sigma = \Big\{\sigma = (\sigma_1, \sigma_2, ..., \sigma_N) \in \mathbb{R}^N; |\sigma| = (\sigma_1^2+\sigma_2^2+...+\sigma_N^2)^{\frac{1}{2}} = 1\Big\}$.

The function $P_N(.)$ which is locally integrable, is the elementary solution of the Poisson equation

$$\triangle P_N = \delta \quad \text{on} \quad \mathbb{R}^N$$

where δ is the Dirac measure. It's easy to prove that :

$$\frac{\partial P_N}{\partial x_i} = \frac{x_i}{\sigma_N|x|^N} \ (1\leq i\leq N)$$

and

For $N=1$, $P_1(x)$ and $\frac{dP_1(x)}{dx}$ lie $L^p_{loc}(\mathbb{R})$, for $1<p<\infty$;

For $N=2$, $P_2(x)\in L^p_{loc}(\mathbb{R}^2)$ (resp. $\frac{\partial P_2(x)}{\partial x_i} \in L_{loc}(\mathbb{R}^2)$), for all $p<\infty$ (resp. $p<2$);

For $N\geq 3$, $P_N(x) \in L^p_{loc}(\mathbb{R}^N)$ (resp. $\frac{\partial P_N(x)}{\partial x_i} \in L^p_{loc}(\mathbb{R}^\mathbb{N})$), for all $p<\frac{N}{N-2}$ (resp. $p<\frac{N}{N-1}$).

Definition 2.7. *[6] Let f be a distribution on $\mathbb{R}^N$ with compact support. The distribution $P.N(f) = P_N * f$ is called the Newtonian potential of f.*

In particular, P_N is the Newtonian potential of δ and each distribution u with compact support is the Newtonian potential of its Laplacian.

3. Regularity of the solution and its first derivatives

Lemma 3.1. *Let Ω be an open of $\mathbb{R}^N$ and let φ be a Musielak-Orlicz function. If $u_n \longrightarrow u$ (mod) in $L_\varphi(\Omega)$ then*

$$u_n \cdot f \to u \cdot f \quad \text{in } L^1(\Omega) \quad \text{for all } f \in L_{\varphi^*}(\Omega).$$

Proof. Let $f \in L_{\varphi^*}(\Omega)$, then there exists a constant $\lambda_1 > 0$ such that

$$\varphi^*(x, \frac{|f(x)|}{\lambda_1}) \in L^1(\Omega)$$

also, there exists a constant $\lambda_2 > 0$ such that

$$\varphi(x, \frac{2(u_n - u)}{\lambda_2}) \longrightarrow 0 \text{ in } L^1(\Omega) \text{ as } n \to +\infty$$

which implies that there exists a subsequence (u_{n_k}) such that

$$\varphi(x, \frac{2(u_{n_k} - u)}{\lambda_2}) \longrightarrow 0$$

as $n_k \to +\infty$, (i.e $u_{n_k} \longrightarrow u$ a.e in Ω as $n_k \to +\infty$) and

$$\varphi(x, \frac{2(u_{n_k} - u)}{\lambda_2}) \le h_1(x)$$

a.e in Ω for some $h_1(x) \in L^1(\Omega)$.
On the other hand, using the convexity of φ we get

$$\varphi(x, \frac{u_{n_k}}{\lambda}) \le \frac{1}{2}\varphi(x, \frac{2(u_{n_k} - u)}{\lambda}) + \frac{1}{2}\varphi(x, \frac{2u}{\lambda})$$

for $\lambda > \lambda_2$ such that $\varphi(x, \frac{2u}{\lambda}) \in L^1(\Omega)$ we have $\varphi(x, \frac{u_{n_k}}{\lambda}) \le \frac{1}{2}h_1(x) + \frac{1}{2}\varphi(x, \frac{2u}{\lambda})$. We put $h(x) = \frac{1}{2}[h_1(x) + \varphi(x, \frac{2u}{\lambda})]$ then

$$u_{n_k} \cdot \frac{|f(x)|}{\lambda_1} \le \lambda \cdot \varphi^{-1}(x, h(x)) \cdot \frac{|f(x)|}{\lambda_1}$$

$$\Rightarrow u_{n_k} \cdot \frac{|f(x)|}{\lambda_1} \le \lambda \cdot [h(x) + \varphi^*(x, \frac{|f(x)|}{\lambda_1})] \in L^1(\Omega)$$

By virtue of Lebesgue theorem we get

$$u_{n_k} \cdot f \longrightarrow u \cdot f \text{ in } L^1(\Omega).$$

Finally, we conclude the result for the original sequence (u_n) by a standard contradiction argument. In fact: If we assume that $u_n\cdot$ f does not converge to $u \cdot f$ in $L^1(\Omega)$ (i.e) there exists $\delta > 0$ such that for all m, there exists $n_m > m$ such that $\|u_{n_m} \cdot f - u \cdot f\|_1 > \delta$, then we can extract a subsequence (u_n) such that

$$\|u_{n_m} \cdot f - u \cdot f\|_1 > \delta, \ \forall m > 0$$

on the other hand, we have

$$\varphi(x, \frac{2(u_{n_m} - u)}{\lambda_2}) \to 0 \text{ in } L^1(\Omega) \text{ as } m \to +\infty.$$

Similarly, we can extract a subsequence of u_{n_m} (still denoted by u_{n_m}) such that $u_{n_m} \cdot f \to u \cdot f$ in $L^1(\Omega)$ as $m \to +\infty$ i.e. there exists $m_0 > 0$ such that $\forall m \geq m_0$ we have $\|u_{n_m} \cdot f - u \cdot f\|_1 < \delta$ (contradiction). □

Definition 3.2. *Let φ be a Musielak-Orlicz function defined on $\Omega \times \mathbb{R}^+$, we say that φ satisfies the locally constant condition if for almost every $x \in \Omega$, $\exists \vartheta_x$(Neighborhood of x) such that $\varphi(z,t) = \varphi(x,t)$, $\forall z \in \vartheta_x$ and $\forall t \geq 0$.*

Definition 3.3. *Let φ be a Musielak-Orlicz function defined on $\Omega \times \mathbb{R}^+$. We say that φ satisfies the conditions $(*)$ (resp $(**)$) if there is a function φ_1 defined on $\mathbb{R}^+ \times \mathbb{R}^+$ such that: $\varphi(x,t) = \varphi_1(\|x\|, t)$, $\forall x \in \Omega$ and $\forall t \geq 0$. (resp condition $(*)$ and $\forall x_1, x_2 \in \Omega$ such that $\|x_1\| \leq \|x_2\| \Rightarrow \varphi_1(\|x_1\|, t) \geq \varphi_1(\|x_2\|, t)$, $\forall t \geq 0$).*

Lemma 3.4. *Let φ be a Musielak-Orlicz function and let $g \in L^{loc}_{\varphi}(\mathbb{R}^N)$*
1- If f is a Radon measure on $\mathbb{R}^N$ with compact support, φ satisfies the condition $(**)$ and g is radiale, decreasing with respect to $\|x\|$, then the convolution $g * f$ is a function which lies in $L^{loc}_{\varphi}(\mathbb{R}^{\mathbb{N}})$.
2- If $f \in L_{\varphi^*}(\mathbb{R}^N)$ with compact support and φ satisfying the locally constant condition then $g * f$ is continuous.

Proof. 1)- Let $r_0 > 0$ such that $supp f \subset \overline{B}(0, r_0)$. We denote by μ the variation of f, using the Lebesgue decomposition theorem or theorem of Rdon-Nikodym we can write $f = \varepsilon \cdot \mu$, where ε is a Borel function taking the values $+1$ or -1. Let $r > 0$, we have

$$\begin{aligned}
\int_{B(0,r)} \Big| g * f(x) \Big| dx &= \int_{B(0,r)} (\Big| \int_{\mathbb{R}^N} g(x-y) df(y) \Big|) dx \\
&\leq \int_{B(0,r)} dx \int_{\mathbb{R}^N} \Big| g(x-y)\varepsilon(y) \Big| d\mu(y) \\
&\overset{Fubini}{=} \int_{\overline{B}(0,r_0)} d\mu(y) \int_{B(y,r)} \Big| g(x) \Big| dx \\
&\leq (\int_{\overline{B}(0,r_0)} d\mu(y))(\int_{\overline{B}(0,r+r_0)} \Big| g(x) \Big| dx) < +\infty
\end{aligned}$$

Taking ξ a continuous function on $\mathbb{R}^N$ with a support in $B(0,r)$, we have

$$\begin{aligned}
\Big\langle g * f, \xi \Big\rangle &= \int_{B(0,r)} g * f(x) \xi(x) dx \\
&= \int_{B(0,r)} (\int_{\mathbb{R}^N} g(x-y) df(y)) \xi(x) dx
\end{aligned}$$

then

$$\begin{aligned}
\Big| < g*f,\xi > \Big| &\leq \int_{B(0,r)} \Big(\int_{\mathbb{R}^N} \Big| g(x-y)\varepsilon(y) \Big| d\mu(y) \Big) \Big|\xi(x)\Big| dx \\
&\overset{Fubini}{=} \int_{\overline{B}(0,r_0)} d\mu(y) \int_{B(0,r)} \Big| g(x-y) \Big| \Big|\xi(x)\Big| dx \\
&\leq c\int_{\overline{B}(0,r_0)} d\mu(y) \Big(\int_{B(0,r)} \varphi(x, \frac{|g(x-y)|}{\|g\|_{\varphi,\overline{B}(0,r+r_0)}}) dx \\
&\quad + \int_{B(0,r)} \varphi^*(x, \frac{|\xi(x)|}{\|\xi\|_{\varphi^*,B(0,r)}}) dx \Big) \\
&= c\int_{\overline{B}(0,r_0)} d\mu(y) \Big(\int_{B(-y,r)} \varphi_1(\|x+y\|, \frac{|g(x)|}{\|g\|_{\varphi,\overline{B}(0,r+r_0)}}) dx \\
&\quad + \int_{B(0,r)} \varphi^*(x, \frac{|\xi(x)|}{\|\xi\|_{\varphi^*,B(0,r)}}) dx \Big) \\
&\leq c\int_{\overline{B}(0,r_0)} d\mu(y) \Big(\int_{Z^+} \varphi_1(\|x+y\|, \frac{|g(x)|}{\|g\|_{\varphi,\overline{B}(0,r+r_0)}}) dx \\
&\quad + \int_{Z^-} \varphi_1(\|x+y\|, \frac{|g(x)|}{\|g\|_{\varphi,\overline{B}(0,r+r_0)}}) dx \\
&\quad + \int_{B(0,r)} \varphi^*(x, \frac{|\xi(x)|}{\|\xi\|_{\varphi^*,B(0,r)}}) dx \Big) \\
&\leq c\int_{\overline{B}(0,r_0)} d\mu(y) \Big(2\int_{B(0,r+r_0)} \varphi_1(\|x\|, \frac{|g(x)|}{\|g\|_{\varphi,\overline{B}(0,r+r_0)}}) dx \\
&\quad + \int_{B(0,r)} \varphi^*(x, \frac{|\xi(x)|}{\|\xi\|_{\varphi^*,B(0,r)}}) dx \Big) \\
&\leq 3\Big(\int_{\overline{B}(0,r_0)} d\mu(y) \Big) \Big\| g \Big\|_{\varphi,\overline{B}(0,r+r_0)} \Big\| \xi \Big\|_{\varphi^*,B(0,r)} \qquad (3.1)
\end{aligned}$$

where

$$c = \Big\| g \Big\|_{\varphi,\overline{B}(0,r+r_0)} \Big\| \xi \Big\|_{\varphi^*,B(0,r)}$$

and

$$Z^+ = B(-y,r) \bigcap \{\|x+y\| \leq \|x\|\},\ Z^- = B(-y,r) \bigcap \{\|x+y\| > \|x\|\}.$$

Hence by using (2.7), we deduce that $g*f \in L^{loc}_{\varphi}(\mathbb{R}^{\mathbb{N}})$. 2)- We will study the continuity of $g*f$ at x_1, let $r_0 > 0$ such that $suppf \subset \overline{B}(0,r_0)$
For $R = r_0 + |x_1| + 1$, we have

$$\begin{aligned}
\Big| \frac{g}{2\lambda^*} * f(x_1) - \frac{g}{2\lambda^*} * f(x_2) \Big| &\leq \int_{\overline{B}(0,R)} \big| \frac{g}{2\lambda^*}(x + x_2 - x_1) - \frac{g}{2\lambda^*}(x) \big| |f(x_1 - x)| dx \\
&\leq \int_{\overline{B}(0,R)} |T_{x_2 - x_1}(\frac{g}{2\lambda^*})(x) - \frac{g}{2\lambda^*}(x)| |f(x_1 - x)| dx.
\end{aligned}$$

$$(\text{ a.e } x_2 \in B(x_1,1))$$

In order to facilitate the proof, we can assume that φ is locally constant all over on $\overline{B}(0,R)$ (otherwise we take $\overline{B}(0,R)\setminus A$, with $mes(A)=0$). So, $\forall y \in \overline{B}(0,R)$, $\exists\ r_y>0$, such that $\varphi(z,t)=\varphi(y,t)$, $\forall z \in B(y,r_y)$ and $\forall\ t\geq 0$, then we have $\overline{B}(0,R)\subset \bigcup_{y\in\overline{B}(0,R)} B(y,r_y)$.
Since $\overline{B}(0,R)$ is compact then there exists y_1,y_2,...,y_n and r_1,r_2,...r_n such that:

$$\overline{B}(0,R)\subset \bigcup_{i=1}^{n} B(y_i,r_i)$$

(we denote r_i instead of r_{y_i}).
We put: $M_i(t)=\varphi(y_i,t)$ (it is an N-function), then $\forall i\in\{1,....,n\}$, $\exists\lambda_i>0$ such that

$$\int_{B(y_i,r_i)} M_i(|\frac{g(x)}{\lambda_i}|)dx=\int_{B(y_i,r_i)}\varphi(x,\frac{|g(x)|}{\lambda_i})dx<+\infty.$$

Taking $\lambda^*=\max_{i\in\{1,..,n\}}\lambda_i$ we have $\frac{g}{\lambda^*}\in K_{M_i}(B(y_i,r_i))$, $\forall i\in\{1,....,n\}$ and $\int_{\overline{B}(0,R)}\varphi(x,|T_{x_1-x_2}(\frac{g}{2\lambda^*})(x)-\frac{g}{2\lambda^*}(x)|)dx\leq\sum_{i=1}^{n}\int_{B(y_i,r_i)} M_i(|T_{x_1-x_2}(\frac{g}{2\lambda^*})(x)-\frac{g}{2\lambda^*}(x)|)dx$
Hence by Lemma 2.1 we have

$$\int_{\overline{B}(0,R)}\varphi(x,|T_{x_1-x_2}(\frac{g}{2\lambda^*})(x)-\frac{g}{2\lambda^*}(x)|)dx\longrightarrow 0 \quad as \quad x_2\to x_1$$

since $f(x_1-.)\in L_{\varphi^*}(\overline{B}(0,R))$, then by Lemma 3.1, we get

$$\int_{\overline{B}(0,R)}|T_{x_1-x_2}(\frac{g}{2\lambda^*})(x)-\frac{g}{2\lambda^*}(x)||f(x_1-x)|dx\longrightarrow 0 \quad as \quad x_2\to x_1,$$

(i.e) $\forall\varepsilon>0$, $\exists\eta>0$ such that $|x_1-x_2|<\eta\Longrightarrow|\frac{g}{2\lambda^*}*f(x_1)-\frac{g}{2\lambda^*}*f(x_2)|\leq\frac{\varepsilon}{2\lambda^*}$
then $\forall\varepsilon>0$, $\exists\eta>0$ such that $|x_1-x_2|<\eta\Longrightarrow|g*f(x_1)-g*f(x_2)|\leq\varepsilon$. □

Example 3.5. *Let $\Omega=\bigcup_{i=1}^{n}\Omega_i$ such that Ω_i are disjoint opens and let M_i be an N-function $\forall i\in\{1,2,..,n\}$. Then the Musielak-Orlicz function defined by $\varphi(x,t)=\sum_{i=1}^{n}M_i(t)\chi_{\Omega_i}(x)$ is locally constant.*

Definition 3.6. *Let φ be a Musielak-Orlicz function defined on $\Omega\times\mathbb{R}^+$ by $\varphi(x,t)=\varphi_1(\|x\|,t)$ such that φ_1 defined on $\mathbb{R}^+\times\mathbb{R}+$. We say that φ satisfies the property (P_1) (resp (P_2)) if*

$$\int_1^{+\infty}\frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-2}},t)}{t^{1+\frac{N}{N-2}}}dt<+\infty$$

$$\Big(resp\int_1^{+\infty}\frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-1}},t)}{t^{1+\frac{N}{N-1}}}dt<+\infty\Big)$$

Remark 3.7. $(P_2)\Longrightarrow(P_1)$

Indeed: we put $r = (\frac{1}{t})^{\frac{1}{N-2}}$, then $t = \frac{1}{r^{N-2}}$ and $dt = -\frac{N-2}{r^{N-1}}dr$. We have

$$\int_1^{+\infty} \frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-2}}, t)}{t^{1+\frac{N}{N-2}}} dt = (N-2) \int_0^1 r^{N-1} \varphi_1(r, \frac{1}{r^{N-2}}) dr$$

and

$$\int_1^{+\infty} \frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-1}}, t)}{t^{1+\frac{N}{N-1}}} dt = (N-1) \int_0^1 r^{N-1} \varphi_1(r, \frac{1}{r^{N-1}}) dr,$$

since $0 < r \leq 1$ then $(\frac{1}{r})^{N-2} \leq (\frac{1}{r})^{N-1}$. On the other hand, since φ_1 is nondecreasing for the second variable then

$$\int_0^1 r^{N-1} \varphi_1(r, \frac{1}{r^{N-2}}) dr \leq \int_0^1 r^{N-1} \varphi_1(r, \frac{1}{r^{N-1}}) dr.$$

So

$$\int_1^{+\infty} \frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-2}}, t)}{t^{1+\frac{N}{N-2}}} dt \leq \frac{N-2}{N-1} \int_1^{+\infty} \frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-1}}, t)}{t^{1+\frac{N}{N-1}}} dt.$$

Theorem 3.8. *Let Ω be an open subset of $\mathbb{R}^N$ ($N \geq 3$), f be a distribution on Ω and u the solution of the equation (1.1). We assume that the Musielak-Orlicz function φ is satisfied the following condition:*

$$If\ D \subset \Omega \quad is\ an\ open\ bounded\ then, \quad \int_D \varphi(x, 1) dx < +\infty \tag{3.2}$$

1)- Assume that φ satisfies the condition $(**)$.
a)- If $ord(f) = 0$, then $u \in L_\varphi^{loc}(\Omega)$ (resp $\frac{\partial u}{\partial x_i} \in L_\varphi^{loc}(\Omega)$) if φ satisfying (P_1) (resp. if φ satisfying (P_2)).
b)- If $ord(f) = 1$, then $u \in L_\varphi^{loc}(\Omega)$ if φ satisfying (P_2).
2)- Assume that φ satisfies the locally constant condition
((i.e) a.e $x \in \Omega$ $\exists \vartheta_x$ (Neighborhood of x) such that $\varphi(z, t) = \varphi(x, t) = M_x(t)$,$\forall z \in \vartheta_x, \forall t \geq 0$)
c)- If $f \in L_{\varphi^*}^{loc}(\Omega)$ and $\int_1^{+\infty} \frac{M_x(t)}{t^{1+\frac{N}{N-2}}} dt < +\infty$ then u is continuous.
d)- If $f \in L_{\varphi^*}^{loc}(\Omega)$ and $\int_1^{+\infty} \frac{M_x(t)}{t^{1+\frac{N}{N-1}}} dt < +\infty$ then u is continuously differentiable.

Proof. Given an open Ω_1 such that $\overline{\Omega_1} \subset \Omega$ and $\varrho \in D(\Omega)$ such that $\varrho \equiv 1$ on Ω_1. Consider the functions $F_0 = \varrho f$ and $F_1 = \Delta(\varrho u) - F_0$. Let u_0 and u_1 be the Newtonian potential of F_0 and F_1 respectively. We have $u_0 = P_N * F_0$ and $u_1 = P_N * F_1 = P_N * (\Delta(\varrho u)) - P_N * F_0$, then

$$u_0 + u_1 = P_N * (\Delta(\varrho u)) = u \text{ on } \Omega_1.$$

Moreover u_1 is harmonic on Ω_1 (because $F_1 = 0$ on Ω_1), and so u_0 and u have similar regularity (locally), then we can assume that: $\Omega = \mathbb{R}^N$ and f is a distribution with compact support and $u = P \cdot N(f)$.

If $ord(f)\leq 1$ then $f=f_0+\sum_{k=1}^{N}\frac{\partial f_k}{\partial x_k}$ where $f_k(0\leq k\leq N\ )$ are measures with compact support.
1-a)- Assume that $Ord(f)=0$ i.e. $f=f_0$. If φ satisfies (P_1), then

$$P_N\in L_{\varphi}^{loc}(\mathbb{R}^{\mathbb{N}})$$

Indeed, for all real $R>0$, we have

$$\begin{aligned}\int_{B(0,R)}\varphi(x,k_NP_N(x))dx &= \int_{|x|<R}\varphi(x,\frac{1}{|x|^{N-2}})dx\\ &= \int_0^R r^{N-1}\Big(\int_{\Sigma}\varphi(r\sigma,\frac{1}{r^{N-2}})d\sigma\Big)dr\\ &= \sigma_N\int_0^R r^{N-1}\varphi_1(r,\frac{1}{r^{N-2}})dr\end{aligned}$$

we put $t=\frac{1}{r^{N-2}}$ and $r=R$ then $t=\frac{1}{R^{N-2}}$ and $dt=-\frac{N-2}{r^{N-1}}dr$. We have

$$\int_{B(0,R)}\varphi(x,K_NP_N(x))dx=\frac{\sigma_N}{N-2}\int_{\frac{1}{R^{N-2}}}^{+\infty}\frac{\varphi_1\Big((\frac{1}{t})^{\frac{1}{N-2}},t\Big)}{t^{1+\frac{N}{N-2}}}dt$$

If $R\leq 1$, then

$$\begin{aligned}\int_{B(0,R)}\varphi(x,K_NP_N(x))dx &= \frac{\sigma_N}{N-2}\int_{\frac{1}{R^{N-2}}}^{+\infty}\frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-2}},t)}{t^{1+\frac{N}{N-2}}}dt\\ &\leq \frac{\sigma_N}{N-2}\int_1^{+\infty}\frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-2}},t)}{t^{1+\frac{N}{N-2}}}dt<+\infty\end{aligned}$$

If $R>1$, we have

$$\begin{aligned}\int_{\frac{1}{R^{N-2}}}^{+\infty}\frac{\varphi_1\Big((\frac{1}{t})^{\frac{1}{N-2}},t\Big)}{t^{1+\frac{N}{N-2}}}dt &= \int_{\frac{1}{R^{N-2}}}^{1}\frac{\varphi_1\Big((\frac{1}{t})^{\frac{1}{N-2}},t\Big)}{t^{1+\frac{N}{N-2}}}dt+\int_1^{+\infty}\frac{\varphi_1\Big((\frac{1}{t})^{\frac{1}{N-2}},t\Big)}{t^{1+\frac{N}{N-2}}}dt\\ &= I_1+I_2\end{aligned}$$

We have $I_1=\int_{\frac{1}{R^{N-2}}}^{1}\frac{\varphi_1\Big((\frac{1}{t})^{\frac{1}{N-2}},t\Big)}{t^{1+\frac{N}{N-2}}}dt\leq\int_{\frac{1}{R^{N-2}}}^{1}\frac{\varphi_1\Big((\frac{1}{t})^{\frac{1}{N-2}},1\Big)}{t^{1+\frac{N}{N-2}}}dt$

We put: $r=\Big(\frac{1}{t}\Big)^{\frac{1}{N-2}}$ then $t=\frac{1}{r^{N-2}}$ ($r=0\Rightarrow t\to+\infty$) and $dt=-\frac{N-2}{r^{N-1}}dr$. So

$$\begin{aligned}\int_{\frac{1}{R^{N-2}}}^{1}\frac{\varphi_1\Big((\frac{1}{t})^{\frac{1}{N-2}},1\Big)}{t^{1+\frac{N}{N-2}}}dt &= (N-2)\int_1^R r^{N-1}\varphi_1(r,1)dr\\ &= \frac{N-2}{\sigma_N}\int_{B(1,R)}\varphi(x,1)dx<+\infty\end{aligned}$$

where $B(1,R)=\Big\{x\in\mathbb{R}^{\mathbb{N}}/1\leq\|x\|\leq R\Big\}$ then we find,

$$\begin{aligned}\int_{B(0,r)}\varphi(x,K_NP_N(x))dx &\leq \int_{B(1,R)}\varphi(x,1)dx+\frac{\sigma_N}{N-2}\int_1^{+\infty}\frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-2}},t))}{t^{1+\frac{N}{N-2}}}dt\\ &< +\infty\end{aligned}$$

by using Lemma 3.4, we are getting $u=P_N*f=P_N*f_0\in L_\varphi^{loc}(\mathbb{R}^N)$. Similarly, if φ satisfies (P_2) then

$$\frac{\partial P_N}{\partial x_i}\in L_\varphi^{loc}(\mathbb{R}^N)\quad\text{for all}\quad 1\leq i\leq N$$

Indeed, for all real $R>0$ we have

$$\begin{aligned}\int_{B(0,R)}\varphi(x,\sigma_N|\frac{\partial P_N}{\partial x_i}(x)|)dx &\leq \int_{B(0,R)}\varphi(x,\frac{1}{|x|^{N-1}})dx\\ &= \sigma_N\int_0^R r^{N-1}\varphi_1(r,\frac{1}{r^{N-1}})dr.\end{aligned}$$

Put $t=\frac{1}{r^{N-1}}$ then $r=R\Rightarrow t=\frac{1}{R^{N-1}}$,$(r\to 0\Rightarrow t\to+\infty)$ and $dt=-\frac{N-1}{r^N}dr$

$$\sigma_N\int_0^R r^{N-1}\varphi_1(r,\frac{1}{r^{N-1}})dr=\frac{\sigma_N}{N-1}\int_{\frac{1}{R^{N-1}}}^{+\infty}\frac{\varphi_1\Big((\frac{1}{t})^{\frac{1}{N-1}},t\Big)}{t^{1+\frac{N}{N-1}}}dt$$

If $R\leq 1$,then

$$\begin{aligned}\int_{B(0,R)}\varphi(x,\sigma_N|\frac{\partial P_N}{\partial x_i}(x)|)dx &\leq \frac{\sigma_N}{N-1}\int_{\frac{1}{R^{N-1}}}^{+\infty}\frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-1}},t)}{t^{1+\frac{N}{N-1}}}dt\\ &\leq \frac{\sigma_N}{N-1}\int_1^{+\infty}\frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-1}},t)}{t^{1+\frac{N}{N-1}}}dt<+\infty.\end{aligned}$$

If $R>1$, we get

$$\begin{aligned}\int_{B(0,R)}\varphi(x,\sigma_N|\frac{\partial P_N}{\partial x_i}(x)|)dx &\leq \frac{\sigma_N}{N-1}\int_{R^{\frac{1}{N-1}}}^1\frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-1}},t)}{t^{1+\frac{N}{N-1}}}dt\\ &+ \frac{\sigma_N}{N-1}\int_1^{+\infty}\frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-1}},t)}{t^{1+\frac{N}{N-1}}}dt\\ &\leq \int_{B(1,R)}\varphi(x,1)dx\\ &\quad+\frac{\sigma_N}{N-1}\int_1^{+\infty}\frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-1}},t)}{t^{1+\frac{N}{N-1}}}dt\\ &< +\infty\end{aligned}$$

On the other hand: $u = P_N * f \Longrightarrow \frac{\partial u}{\partial x_i} = P_N * \frac{\partial f}{\partial x_i} = \frac{\partial P_N}{\partial x_i} * f$. By Lemma 3.4, we have

$$\frac{\partial u}{\partial x_i} \in L^{loc}_{\varphi}(\mathbb{R}^{\mathbb{N}}).$$

b)- Assume that: $ord(f) = 1$ and φ satisfies (P_2) we have

$$f = f_0 + \sum_{k=1}^{N} \frac{\partial f_k}{\partial x_k}$$

and

$$\frac{\partial P_N}{\partial x_k} \in L^{loc}_{\varphi}(\mathbb{R}^N), \text{ for all } k \in \{1, ..., N\}$$

and

$$u_k = P_N * \frac{\partial f_k}{\partial x_k} = \frac{\partial P_N}{\partial x_k} * f_k, \text{ for all } k \in \{1, ..., N\}.$$

By Lemma 3.4 $u_k \in L^{loc}_{\varphi}(\mathbb{R}^N), \forall k \in \{1, ..., N\}$. Since $(P_2) \Longrightarrow (P_1)$, then $P_N \in L^{loc}_{\varphi}(\mathbb{R}^N) \Longrightarrow u_0 \in L^{loc}_{\varphi}(\mathbb{R}^N)$. So $u = u_0 + \sum_{k=1}^{N} u_k \in L^{loc}_{\varphi}(\mathbb{R}^N)$.
2)-We assume that φ is locally constant all over on $\overline{B}(0, R)$ (otherise,we take $\overline{B}(0, R) \setminus A$such that $mes(A) = 0$). Since $\overline{B}(0, R)$ is compact then $\exists\ y_1$,y_2,...,$y_m \in \overline{B}(0, R)$ and $r_1 > 0$,$r_2 > 0$,....,$r_m > 0$ such that:

$$B(0, R) \subset \overline{B}(0, R) \subset \bigcup_{i=1}^{m} B(y_i, r_i).$$

c)-Since $\int_1^{+\infty} \frac{M_x(t)}{t^{1+\frac{N}{N-2}}} dt < +\infty$ then $P_N \in L^{loc}_{\varphi}(\mathbb{R}^N)$. Indeed, for all real $R > 0$ we have

$$\begin{aligned}
\int_{B(0,R)} \varphi(x, K_N P_N(x))dx &\leq \sum_{i=1}^{m} \int_{B(y_i,r_i)} M_{y_i}(K_N P_N)dx \\
&= \sigma_N \sum_{i=1}^{m} \int_{|y_i|}^{|y_i|+r_i} r^{N-1} M_{y_i}(\frac{1}{r^{N-2}})dr \\
&\leq \frac{\sigma_N}{N-2} \sum_{i=1}^{m} \int_{\frac{1}{(|y_i|+r_i)^{N-2}}}^{+\infty} \frac{M_{y_i}(t)}{t^{1+\frac{N}{N-2}}} dt \\
&< +\infty
\end{aligned}$$

moreover $u = P_N * f$ then by Lemma 3.4, we conclude that u is continuous.
d)- Since $\int_1^{+\infty} \frac{M_x(t)}{t^{1+\frac{N}{N-1}}} dt < +\infty$ then $\frac{\partial P_N}{\partial x_i} \in L^{loc}_{\varphi}(\mathbb{R}^N)$, $\forall i \in \{1, 2, ..., N\}$. Indeed,for

all real $R > 0$ we have:

$$\begin{aligned}
\int_{B(0,R)} \varphi(x, \sigma_N |\frac{\partial P_N}{\partial x_i}(x)|)dx &\leq \sum_{i=1}^{m} \int_{B(y_i,r_i)} M_{y_i}(\sigma_N |\frac{\partial P_N}{\partial x_i}(x)|)dx \\
&\leq \sum_{i=1}^{m} \int_{B(y_i,r_i)} M_{y_i}(\frac{1}{|x|^{N-1}})dx \\
&= \sigma_N \sum_{i=1}^{m} \int_{|y_i|}^{|y_i|+r_i} r^{N-1} M_{y_i}(\frac{1}{r^{N-1}})dr \\
&\leq \frac{\sigma_N}{N-1} \sum_{i=1}^{m} \int_{\frac{1}{(|y_i|+r_i)^{N-1}}}^{+\infty} \frac{M_{y_i}(t)}{t^{1+\frac{N}{N-1}}}dt \\
&< +\infty
\end{aligned}$$

we have

$$\frac{\partial u}{\partial x_i} = P_N * \frac{\partial f}{\partial x_i} = \frac{\partial P_N}{\partial x_i} * f, \ \forall i \in (1, 2, .., N)$$

then by using Lemma 3.4, we have $\frac{\partial u}{\partial x_i}$ is continuous, $\forall i \in \{1.2, ..N\}$,
On the other hand:

$$\int_1^{+\infty} \frac{M_x(t)}{t^{1+\frac{N}{N-1}}}dt < +\infty \Rightarrow \int_1^{+\infty} \frac{M_x(t)}{t^{1+\frac{N}{N-2}}}dt < +\infty$$

then by Lemma 3.4 $u = P_N * f$ is continuous consequently u is continuously differentiable. □

Corollary 3.9. *Let Ω be an open subest of $\mathbb{R}^{\mathbb{N}}$ $(N \geq 3)$, f be a distribution on Ω and u the solution of the equation (1.1). Let $p \in \mathcal{P}(\Omega)$ is radial decreasing with resp to $\|x\|$ (i.e): There is a function $v : \mathbb{R}^+ \longrightarrow (1, +\infty)$ such that $p(x) = v(\|x\|)$ and $p_- = ess \inf_{x\in\Omega} p(x) > 1$ then:*
1-if $ord(f) = 0$ then $u \in L^{p(.)}_{loc}(\Omega)$ (resp. $\frac{\partial u}{\partial x_i} \in L^{p(.)}_{loc})(\Omega)$) if
$\int_1^{+\infty} \frac{t^{v\left((\frac{1}{t})^{\frac{1}{N-2}}\right)}}{t^{1+\frac{N}{N-2}}}dt < +\infty$
(resp $\int_1^{+\infty} \frac{t^{v\left((\frac{1}{t})^{\frac{1}{N-1}}\right)}}{t^{1+\frac{N}{N-1}}}dt < +\infty$*)*
2-if $ord(f) = 1$ then $u \in L^{p(.)}_{loc}(\Omega)$ if $\int_1^{+\infty} \frac{t^{v\left((\frac{1}{t})^{\frac{1}{N-1}}\right)}}{t^{1+\frac{N}{N-1}}}dt < +\infty$

Example 3.10. *We consider the Musielak-Orlicz functions defined on $(\mathbb{R}^+)^* \times \mathbb{R}^+$:*

$$\varphi_1(r, t) = \frac{t^{\frac{N}{N-2}+1}}{t^\alpha}(\frac{1}{r^{N-2}} + 1), \quad 1 < \alpha < \frac{N}{N-2}, \quad N > 2$$

$$\phi_1(r, t) = \frac{t^{\frac{N}{N-1}+1}}{t^\alpha}(\frac{1}{r^{N-1}} + 1), \quad 1 < \alpha < \frac{N}{N-1}, \quad N > 1$$

the functions φ_1(resp. ϕ_1) satisfies the property (P_1) (resp. (P_2)). We put $\varphi(x,t) = \varphi_1(\|x\|,t)$ and $\phi(x,t) = \phi_1(\|x\|,t)$ defined on $\Omega \times \mathbb{R}^+$ then : φ^* and ϕ^* are locally integrable. Indeed:

$$\begin{aligned}
\int_1^{+\infty} \frac{\varphi_1((\frac{1}{t})^{\frac{1}{N-2}},t)}{t^{1+\frac{N}{N-2}}}dt &= \int_1^{+\infty} \frac{t^{\frac{N}{N-2}+1}(t+1)}{t^{1+\frac{N}{N-2}}t^{\alpha}}dt \\
&\leq 2\int_1^{+\infty} \frac{1}{t^{\alpha-1}}dt \\
&\leq 2\lim_{c\to+\infty}\left[\frac{-1}{(\alpha-1)t^{\alpha}}\right]_1^c \\
&= \frac{2}{\alpha-1} < +\infty
\end{aligned}$$

and

$$\begin{aligned}
\int_1^{+\infty} \frac{\phi_1((\frac{1}{t})^{\frac{1}{N-1}},t)}{t^{1+\frac{N}{N-1}}}dt &= \int_1^{+\infty} \frac{t^{\frac{N}{N-1}+1}(t+1)}{t^{1+\frac{N}{N-1}}t^{\alpha}}dt \\
&\leq 2\int_1^{+\infty} \frac{1}{t^{\alpha-1}}dt \\
&\leq 2\lim_{c\to+\infty}\left[\frac{-1}{(\alpha-1)t^{\alpha}}\right]_1^c \\
&\leq \frac{2}{\alpha-1} < +\infty
\end{aligned}$$

Let $N(t) = \frac{t^{\frac{N}{N-2}+1}}{t^{\alpha}}$ then $N(t) \leq \varphi(x,t)$, for all $x \in \Omega$ and for all $t \geq 0$. Then, $\varphi^*(x,t) \leq N^*(t)$, for all $x \in \Omega$ and $t \geq 0$. We have for all $t \geq 0$ and all measurable $E \subset \Omega$ such that $mes(E) < +\infty$,

$$\int_\Omega \varphi^*(x, t\chi_E(x))dx \leq \int_\Omega N^*(t\chi_E(x))dx \leq N^*(t).mes(E) < +\infty.$$

Similarly, ϕ^* is locally integrable.

Theorem 3.11. *Let Ω be an open subset of $\mathbb{R}^2$ and φ be a Musielak-Orlicz function. We assume that φ satisfies the condition$(*)$and:*

$$If\ D \subset \Omega \quad is\ an\ open\ bounded\ then, \quad \int_D \varphi(x,1)dx < +\infty \tag{3.3}$$

$$\exists \lambda > 0 \quad such\ that \quad \int_1^{+\infty} \frac{\varphi_1(e^{-\lambda t},t)}{e^{2\lambda t}}dt < +\infty \tag{3.4}$$

If f is a measure on Ω then, every solution u of the equation (1.1) lies in $L_\varphi^{loc}(\Omega)$;

Proof. As in the proof of theorem 3.1, we can assume that $\Omega = \mathbb{R}^2$, $supp(f)$ is compact and $u = P.N(f)$. First, we have $P_2 \in L_{\varphi}^{loc}(\mathbb{R}^2)$. Indeed: for all real $R > 0$, we have

$$\begin{aligned}\int_{|x|<R} \varphi(x, \frac{2\pi|P_2(x)|}{\lambda})dx &= \int_0^R rdr \int_{\Sigma} \varphi_1(r, \frac{|\log r|}{\lambda})d\sigma \\ &= \sigma_N \int_0^R r\varphi_1(r, \frac{|\log r|}{\lambda})dr\end{aligned}$$

We put $m = \frac{1}{r}$ then $dr = -\frac{1}{m^2}dm$, we have

$$\int_{|x|<R} \varphi(x, \frac{2\pi|P_2(x)|}{\lambda})dx = \sigma_N \int_{\frac{1}{R}}^{+\infty} \frac{1}{m^3}\varphi_1(\frac{1}{m}, \frac{|\log(\frac{1}{m})|}{\lambda})dm$$

we put $t = \frac{\log m}{\lambda}$ then $dm = \lambda e^{\lambda t}dt$
We get

$$\int_{|x|<R} \varphi(x, \frac{2\pi|P_2(x)|}{\lambda})dx = \lambda\sigma_N \int_{-\frac{\log R}{\lambda}}^{+\infty} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt$$

We distinguish two cases.
First case: $R \leq 1$.
If $-\frac{\log R}{\lambda} < 1 \Longrightarrow e^{-\lambda} < R$

$$\int_{-\frac{\log R}{\lambda}}^{+\infty} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt = \int_{-\frac{\log R}{\lambda}}^{1} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt + \int_1^{+\infty} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt$$

$$\int_{-\frac{\log R}{\lambda}}^{1} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt \leq \int_{-\frac{\log R}{\lambda}}^{1} \frac{\varphi_1(e^{-\lambda t}, 1)}{e^{2\lambda t}}dt$$

Put: $r = e^{-\lambda t}$ then $dt = -\frac{1}{\lambda r}dr$

$$\begin{aligned}\int_{-\frac{\log R}{\lambda}}^{1} \frac{\varphi_1(e^{-\lambda t}, 1)}{e^{2\lambda t}}dt &= \frac{1}{\lambda}\int_{e^{-\lambda}}^{R} r\varphi_1(r, 1)dr \\ &= \frac{1}{\lambda\sigma_N}\int_{B(e^{-\lambda}, R)} \varphi(x, 1)dx \\ &< +\infty.\end{aligned}$$

where $B(e^{-\lambda}, R) = \left\{x \in \mathbb{R}^{\mathbb{N}}/e^{-\lambda} \leq \|x\| \leq R\right\}$

So

$$\int_{|x|<R} \varphi(x, \frac{2\pi}{\lambda}|P_2(x)|)dx \leq \int_{B(e^{-\lambda}, R)} \varphi(x, 1)dx + \lambda\sigma_N \int_1^{+\infty} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt < +\infty.$$

If :$-\frac{\log R}{\lambda} > 1$ we have

$$\begin{aligned}\int_{|x|<R} \varphi(x, \frac{2\pi|P_2(x)|}{\lambda})dx &= \lambda\sigma_N \int_{-\frac{\log R}{\lambda}}^{+\infty} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt \\ &\leq \lambda\sigma_N \int_{1}^{+\infty} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt < +\infty.\end{aligned}$$

Second case: $R > 1$ we have

$$\int_{|x|<R} \varphi(x, \frac{2\pi|P_2(x)|}{\lambda})dx = \lambda\sigma_N \int_{-\frac{\log R}{\lambda}}^{+\infty} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt \leq \lambda\sigma_N \int_{0}^{+\infty} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt$$

$$\int_{0}^{+\infty} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt = \int_{0}^{1} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt + \int_{1}^{+\infty} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt$$

$$\int_{0}^{1} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt = \int_{0}^{1} \frac{\varphi_1(e^{-\lambda t}, 1)}{e^{2\lambda t}}dt.$$

Put: $r = e^{-\lambda t}$ then $dt = -\frac{1}{\lambda r}dr$

$$\begin{aligned}\int_{0}^{1} \frac{\varphi_1(e^{-\lambda t}, 1)}{e^{2\lambda t}}dt &= \frac{1}{\lambda}\int_{e^{-\lambda}}^{1} r\varphi_1(r, 1)dr \\ &= \frac{1}{\lambda\sigma_N}\int_{B(e^{-\lambda},1)} \varphi(x, 1)dx \\ &< +\infty.\end{aligned}$$

We get

$$\int_{|x|<R} \varphi(x, \frac{2\pi}{\lambda}|P_2(x)|)dx \leq \int_{B(e^{-\lambda},1)} \varphi(x, 1)dx + \lambda\sigma_N \int_{1}^{+\infty} \frac{\varphi_1(e^{-\lambda t}, t)}{e^{2\lambda t}}dt < +\infty.$$

So we conclude by the Lemma 3.4. □

Corollary 3.12. *let Ω be open subset of $\mathbb{R}^2$ and $p \in \mathcal{P}(\Omega)$ is radial decreasing with resp to $\|x\|$ (i.e): There is a function $\omega : \mathbb{R}^+ \longrightarrow (1, +\infty)$ such that $p(x) = \omega(\|x\|)$ we assume that p satisfies the conditions:*

$$p_- = ess \inf_{x\in\Omega} p(x) > 1$$

$$p^+ = ess \sup_{x\in\Omega} p(x) < +\infty$$

$$\exists\lambda > 0, \int_{1}^{+\infty} \frac{t^{\omega(e^{-\lambda t})}}{e^{2\lambda t}}dt < +\infty$$

if f is measure on Ω Then every solution u of the equation (1.1) lies in $L^{p(.)}_{loc}(\Omega)$

4. Maximal regularity of the radiale solution

Denote by $B(0,R_0)$ an open ball of $\mathbb{R}^N$ with radius $R_0>0$, which is equal to $\mathbb{R}^N$ for $R_0=+\infty$

Theorem 4.1. *Let φ be a Musielak-Orlicz function satisfying the locally constant condition and $f\in L_\varphi(B(0,R_0))$. Then every radiale solution u of the equation*

$$\Delta u = f \quad in\, B(0,R_0) \tag{4.1}$$

satisfies

$$\frac{\partial^2 u}{\partial x_i \partial x_j}\in L_\varphi^{loc}(B(0,R_0)-\{0\})$$

If in addition $\varphi(x,|f(x)|)\log|x|$ is integrable then

$$\frac{\partial^2 u}{\partial x_i \partial x_j}\in L_\varphi^{loc}(B(0,R_0))$$

The proof uses the following Lemma.

Lemma 4.2. *[6] Let f be a radial and integrable function on $\mathbb{R}^N$ with compact support. The Newtonian potential $P.N(f)$ is continuously differentiable on $\mathbb{R}^{\mathbb{N}}-\{0\}$ and*

$$P.N(f)(x)=P_N(x)\int_{B(0,|x|)} f(y)dy+\int_{\mathbb{R}^{\mathbb{N}}-B(0,|x|)} P_N(y)f(y)dy$$

$$\nabla P.N(f)(x)=\frac{x}{\sigma_N|x|^N}\int_{B(0,|x|)} f(y)dy$$

The second derivatives of $P.N(f)$ are locally integrable on $\mathbb{R}^{\mathbb{N}}-\{0\}$ and

$$\frac{\partial^2 P.N(f)}{\partial x_i\partial x_j}=\frac{x_ix_j}{|x|^2}f(x)+(\frac{\delta_{ij}}{N}-\frac{x_ix_j}{|x|^2})\frac{N}{\sigma_N|x|^N}\int_{B(0,|x|)} f(y)dy$$

Proof. Let B_1 be an open ball such that $\overline{B_1}\subset B(0,R_0))$ and let ϱ be a radial smooth function with compact support in $B(0,R_0)$ such that $\varrho\equiv 1$ on B_1. We consider the function $f_0=\varrho f$, $f_1=\triangle(\varrho u)-f_0$. We have

$$P.N(f_0)+P.N(f_1)=P_N * \triangle(\varrho u)=u \quad \text{on} \quad B_1$$

Since $P.N(f_1)$ is harmonic on B_1, then $P.N(f_0)$ and u have the same regularity on B_1, so we can assume that f is radial with compact support and $u=P.N(f)$.
Let $0<R<R_0$ and $\varepsilon>0$, we shall show that

$$\frac{\partial^2 u}{\partial x_i \partial x_j}\in L_\varphi(B(\varepsilon,R))$$

where $B(\varepsilon, R) = \Big\{x \in \mathbb{R}^{\mathbb{N}}, \varepsilon < |x| < R\Big\}$
It follows directly from Lemma 4.1 that

$$|\frac{\partial^2 u}{\partial x_i \partial x_j}(x)| \leq |f(x)| + \frac{c_1 N}{\sigma_N |x|^N} \int_{B(0,|x|)} |f(y)| dy$$

where $c_1 = |\frac{\delta_{i,j}}{N} - \frac{x_i x_j}{|x|^2}| < 1$
By virtue of convexity of $\varphi(x,.)$ we have

$$\int_{B(\varepsilon,R)} \varphi(x, \frac{1}{2\lambda} |\frac{\partial u}{\partial x_i \partial x_j}|) dx \leq I_1 + I_2$$

where $I_1 = \int_{B(\varepsilon,R)} \varphi(x, \frac{|f(x)|}{\lambda}) dx$ and $I_2 = \int_{B(\varepsilon,R)} \varphi(x, \frac{c_1 N}{\sigma_N |x|^N} \int_{B(0,|x|)} \frac{|f(y)|}{\lambda} dy) dx$.
We have $I_1 < +\infty$ (because $f \in L_\varphi(B(0, R_0))$). It's sufficient to show that $I_2 < \infty$.
We have

$$\int_{B(0,|x|)} dy = \sigma_N \int_0^{|x|} r^{N-1} dr = \frac{\sigma_N}{N} |x|^N \Longrightarrow \int_{B(0,|x|)} \frac{N}{\sigma_N |x|^N} dy = 1.$$

By Jensen's inequality

$$\begin{aligned} I_2 &\leq \int_{B(\varepsilon,R)} dx \int_{B(0,|x|)} \frac{c_1 N}{\sigma_N |x|^N} \varphi(x, \frac{|f(y)|}{\lambda}) dy \\ &= \int_{B(\varepsilon,R)} dx \int_{B(0,R)} \frac{c_2}{|x|^N} \varphi(x, \frac{|f(y)|}{\lambda}) \chi_{|y|<|x|} dy \end{aligned}$$

We assume that φ is locally constant all over on $B(0, R_0)$ (otherwise, we take $B(0, R_0) \setminus A$ such that $mes(A) = 0$), then: $\forall x \in B(0, R_0)$, $\exists r_x > 0$ such that $\varphi(z, t) = \varphi(x, t)$, $\forall z \in B(x, r_x)$.
Since $\overline{B}(0, R)$ is compact then $\exists$ y_1,y_2,....,$y_n \in \overline{B}(0, R)$ and r_1,r_2,...,r_n such that

$$B(0, R) \subset \bigcup_{i=1}^{n} B(y_i, r_i)$$

Denote by $r_i = r_{y_i}$ and put $M_i(t) = \varphi(y_i, t)$ then

$$\begin{aligned}
I_2 &\leq c_2 \sum_{i=1}^{n} \int_{B(\varepsilon,R)} dx \int_{B(y_i,r_i) \bigcap B(0,R_0)} M_i(\frac{|f(y)|}{\lambda})\chi_{|y|<|x|} dy \\
&\overset{Fubini}{=} c_2 \sum_{i=1}^{n} \int_{B(y_i,r_i) \bigcap B(0,R_0)} M_i(\frac{|f(y)|}{\lambda}) dy \int_{\max(\varepsilon,|y|)<|x|<R} \frac{1}{|x|^N} dx \\
&= c_3 \sum_{i=1}^{n} \int_{B(y_i,r_i) \bigcap B(0,R_0)} M_i(\frac{|f(y)|}{\lambda}) \log(\frac{R}{\max(\varepsilon, |y|)}) dy \\
&= c_3 \int_{\bigcup_{i=1}^{n} B(y_i,r_i) \bigcap B(0,R_0)} \varphi(y, \frac{|f(y)|}{\lambda}) \log(\frac{R}{\max(\varepsilon, |y|)}) dy \\
&= c_3 \int_{B(0,R_0)} \varphi(y, \frac{|f(y)|}{\lambda}) \log(\frac{R}{\max(\varepsilon, |y|)}) dy \\
&< +\infty
\end{aligned}$$

Now, assume that $\varphi(x, |f(x)|) \log(|x|)$ is integrable on $B(0, R_0)$:
Similarly, Let $0 < R < R_0$

$$\int_{B(0,R)} \varphi(x, \frac{1}{2\lambda} |\frac{\partial^2 u}{\partial x_i \partial x_j}|) dx < \frac{1}{2} \int_{B(0,R)} \varphi(x, \frac{|f(x)|}{\lambda}) dx + I_2$$

where $I_2 = \frac{1}{2} \int_{B(0,R)} dx \varphi(x, \frac{c_1 N}{\sigma_N |x|^N} \int_{B(0,|x|)} \frac{|f(y)|}{\lambda}) dy$

$$\begin{aligned}
I_2 &\leq \frac{1}{2} \int_{B(0,R)} \varphi(x, \frac{|f(x)|}{\lambda}) dx + \frac{1}{2} \int_{B(0,R)} dx \varphi(x, \frac{c_1 N}{\sigma_N |x|^N} \int_{B(0,|x|)} \frac{|f(y)|}{\lambda}) dy \\
&\leq I_1 + \frac{c_2}{2} \int_{B(0,R)} dx \int_{B(0,R)} \frac{1}{|x|^N} \varphi(x, \frac{|f(y)|}{\lambda}) \chi_{|y|<|x|} dy \\
&\leq I_1 + \frac{c_2}{2} \sum_{i=1}^{n} \int_{B(0,R)} dx \int_{B(y_i,r_i) \bigcap B(0,R_0)} \frac{1}{|x|^N} M_i(\frac{|f(y)|}{\lambda}) \chi_{|y|<|x|} dy \\
&\overset{fubini}{=} I_1 + \frac{c_2}{2} \sum_{i=1}^{n} \int_{B(y_i,r_i) \bigcap B(0,R_0)} M_i(\frac{|f(y)|}{\lambda}) dy \int_{|y|<|x|<R} \frac{1}{|x|^N} dx \\
&= I_1 + \frac{c_3}{2} \sum_{i=1}^{n} \int_{B(y_i,r_i) \bigcap B(0,R_0)} M_i(\frac{|f(y)|}{\lambda}) \log(\frac{R}{|y|}) dy \\
&= I_1 + \frac{c_3}{2} \int_{\bigcup_{i=1}^{n} B(y_i,r_i) \bigcap B(0,R_0)} \varphi(y, \frac{|f(y)|}{\lambda}) \log(\frac{R}{|y|}) dy \\
&= \frac{1}{2} \int_{B(0,R)} \varphi(x, \frac{|f(x)|}{\lambda}) dx + \frac{c_3}{2} \int_{B(0,R_0)} \varphi(y, \frac{|f(y)|}{\lambda}) \log(\frac{R}{|y|}) dy \\
&< +\infty
\end{aligned}$$

□

References

1. R. Adams, Sobolev spaces, Acad.press (1975).
2. M.L. Ahmed Oubeid, A.Benkirane and M. Sidi El vally, Strongly nonlinear parabolic problems in Musielak-Orlicz-Sobolev spaces, to appear in Bol. Soc. Parana. Mat.
3. E. Azroul, A. Benkirane, M. Tienari: On the regularity of solution to the poisson equation in Orlicz-spaces,Bull.Belg.Math.soc .(2000)
4. A. Benkirane, Potentiel de Riesz et problemes elliptiques dans les espaces d'Orlicz, thése de Doctorat, Université libre de Bruxelles,(1988).
5. A. Benkirane and J. P. Gossez, An approximation theorem in higher Orlicz-Sobolev spaces and application, Studia Math. 92(1989), pp. 231-255.
6. R. Dautray et J. L. Lions, Analyse Mathématique et calcul numérique, volume 2, Masson (1987).
7. L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka, Lebesgue and Sobolev Spaces With Variable Exponents, Lecture Notes in Mathematics (2017).
8. - Elias and M. Stein"Singular Integrals and Differentiability Properties of Functions" Princeton University press
9. D. Gilbarg and N. S. Trudinger, Elliptic partial differential equation of second order, Springer Verlag (1983).
10. J. P. Gossez, Some appoximation properties in Orlicz-Sobolov spaces, studia Math. 74(1982) pp. 17-24.
11. J. P. gossez, Non linear elliptic boundary value problems for equation with rapidly (or slowly) increasing coefficients, Trans. Amer. Math. Soc. 190(1974) pp. 163-205.
12. Grter and Wildman, Manuscripa Math 37(1982), pp. 303-342.
13. M. Krasnosel'skii and Ya. Rutickii, Convex functions and Orlicz spaces, Noord-hoof, (1961).
14. A. Kufner, O. John and S. Fucik, Function spaces, Academia, (1977).
15. J. Musielak: Modular spaces and Orlicz spaces, Lecture Notes in Math. 1034, (1983).
16. M. Tienari, A degree theory for a class of mappings of monotone type in Orlicz-Sobolev spaces, Ann. acad. Scientiarum Fennice Helsinki(1994).

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