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Stability Analysis of Linear Conformable Fractional Differential Equations System with Time Delays

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ABSTRACT: In this paper, we first study stability analysis of linear conformable fractional differential equations system with time delays. Some sufficient conditions on the asymptotic stability for these systems are proposed by using properties of the fractional Laplace transform and fractional version of final value theorem. Then, we employ conformable Euler's method to solve conformable fractional differential equations system with time delays to illustrate the effectiveness of our theoretical results.

Key Words: Stability analysis, Conformable fractional derivative, Conformable Euler's method, Delay.

Contents

1. Introduction

Fractional derivative is generalized from the ordinary derivative whose source refers to the late seventeenth century when the basics of differential and integral calculus were being developed by Newton and Leibniz. Although this subject has a history of over 300 years, it is about forty years that it has been used by engineering communities as a powerful tool in solving engineering problems. Because of the freedom on the order of the derivative, physical and engineering system such as electromagnetic waves, viscoelastic systems, etc. can be described with very high accuracy. In literature there are many definitions on fractional derivatives but the most frequently used are as below.

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i) The Riemann-Liouville fractional derivative is defined by [1,2,3]

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-\varepsilon)^{-\alpha} f(\varepsilon) d\varepsilon, \quad 0 \leq \alpha < 1. \tag{1.1}$$

ii) The Caputo fractional derivative is defined by [1]

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\varepsilon)^{-\alpha} f'(\varepsilon) d\varepsilon, \quad 0 \leq \alpha < 1. \tag{1.2}$$

iii) The Hilfer fractional derivative of order α and type β is defined by [4]

$$D_t^{\alpha,\beta} f(t) = I_t^{\beta(1-\alpha)} \frac{d}{dt} I_t^{(1-\beta)(1-\alpha)} f(t), \quad t > 0, \quad 0 < \alpha < 1, \; 0 \leq \beta \leq 1, \tag{1.3}$$

where $I_t^{\beta(1-\alpha)}$ is the Riemann-Liouville fractional integral operator of order $\beta(1-\alpha)$.

iv) The Ji-Huan He's fractal derivative [5]

$$\frac{Df(t)}{Dt^{\alpha}} = \Gamma(\alpha+1) \lim_{\Delta t = t_1 - t_2 \to L} \frac{f(t_1) - f(t_2)}{(t_1 - t_2)^{\alpha}}, \quad t > 0, \quad 0 < \alpha < 1. \tag{1.4}$$

where Δt does not tend to zero, it can be the thickness (L) of a porous medium.

v) The conformable fractional derivatives [6]

$$D_t^{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \quad t > 0, \quad 0 < \alpha \leq 1. \tag{1.5}$$

Since investigation of the stability of differential equations system with fractional derivatives is important in control engineering, numerous articles are provided in this field in recent years. Matignon in [7] firstly studied the stability of linear fractional differential systems with the Caputo derivative. Since then, many researchers have done further studies on the stability of fractional differential systems. For example,Qian et al. [8] studied the case of linear fractional differential equations with Riemann-Liouville derivative and the same fractional order α , where $\alpha \in (0,1)$. Then, in [9,10] authors derived the same conclusion as [7] for the case $\alpha \in (1,2)$. Aminikhah et al. examined the stability of fractional differential equations system of distribution order with non-negative density function [11,12]. Deng et al. investigated the stability of fractional differential equations system with multiple delays and provided the results that ensure the stability of these systems [13]. Rezazadeh et al. recently examined the stability of Hilfer fractional differential equations system by using the properties of Mittag-Leffler functions [14]. In [15], the authors studied the stability for the fractional Floquet system and shown the fractional

Floquet system is asymptotically stable if all multipliers have real parts between -1 and 1.
In this paper, we first introduce linear conformable fractional differential equations system with time delay and then, state some results about the stability of these systems which are consistent with the results presented about the stability of the delay system with ordinary derivatives in particular occasions. Then, we offer a numerical method namely conformable Euler's method to solve the conformable fractional differential equations system with time delay to show our claim about the assessment of the stability of such systems on the graph.
The rest of the paper is organized as follows: Section 2 give some definitions and theorems of conformable fractional derivative. Section 3 describes the stability analysis of conformable fractional order systems with time delay, Section 4 explains the numerical method of conformable fractional delays systems and in Section 5 analytical results of an example with the conclusion in Section 6.

2. Conformable fractional derivative

Conformable fractional derivatives were stated by Khalil et al. (2015) [6] , and developed by Abdeljawad [16] . Moreover, Abdeljawad gave the fractional chain rule, the fractional integration by parts formulas, the fractional power series expansion and the fractional Laplace transform definition.Then in short time, other professionals provided mathematical models in structure of which conformable fractional derivatives have been used [17,18,19,20,21,22,23,24,25,26] . Since the stability of such systems is very important, Rezazadeh et al. (2016) studied the stability of conformable fractional linear differential equations system for the first time [27]
.
In this Section, we briefly recall some definitions, notations and results from the conformable fractional which will be used in our main results.
If the limit (1.5) exist then we say is differentiable and if is α-differentiable in $(0, a)$, $a > 0$ and $\lim\limits_{t\to 0^+} D_t^\alpha f(t)$ exist then we have

$$D_t^\alpha f(0) = \lim_{t\to 0^+} D_t^\alpha f(t). \tag{2.1}$$

Theorem 2.1. *Let $\alpha \in (0,1)$, and f and g be α-differentiable at a point t, then the conformable derivative satisfies the following properties:*

i) $D_t^\alpha(af(t) + bg(t)) = a\, D_t^\alpha f(t) + b\, D_t^\alpha g(t), \quad \forall\, a, b \in \mathbb{R}.$

ii) $D_t^\alpha(t^\mu) = \mu t^{\mu-\alpha}, \;\; \forall\, \mu \in \mathbb{R}.$

iii) $D_t^\alpha(f(t)g(t)) = f(t)D_t^\alpha g(t) + g(t)\, D_t^\alpha f(t).$

iv) $D_t^\alpha\left(\frac{f(t)}{g(t)}\right) = \frac{g(t)\; D_t^\alpha f(t) - f(t)\; D_t^\alpha g(t)}{g^2(t)}.$

Furthermore,if f is differentiable, then $D_t^\alpha f(t) = t^{1-\alpha}\frac{df}{dt}.$

In [16] T. Abdeljawad established the chain rule for conformable fractional derivatives as following theorem.

Theorem 2.2. *[16] Suppose f, $g : (0,\infty) \to \mathbb{R}$ be α-differentiable functions and $\alpha \in (0,1]$. Then fog is α-differentiable and for all $t \neq 0$, $g(t) \neq 0$ we have*

$$D_t^\alpha (fog)(t) = (D_t^\alpha f)(g(t))(D_t^\alpha g)(t)g(t)^{\alpha-1}. \tag{2.2}$$

Definition 2.3. *The conformable exponential function is defined by*

$$e^{\frac{1}{\alpha}t^\alpha} = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\alpha^k k!}. \tag{2.3}$$

As further result of the above formula

$$e^{\frac{1}{\alpha}(t+r)^\alpha} = e^{\frac{1}{\alpha}t^\alpha} e^{\frac{\beta}{\alpha}r^\alpha}, \qquad \beta \in \mathbb{R}. \tag{2.4}$$

Now, we list here the fractional derivatives of certain functions [6]

i) $D_t^\alpha (e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$

ii) $D_t^\alpha (\sin \frac{1}{\alpha}t^\alpha) = \cos \frac{1}{\alpha}t^\alpha$

iii) $D_t^\alpha (\cos \frac{1}{\alpha}t^\alpha) = -\sin \frac{1}{\alpha}t^\alpha$

iv) $D_t^\alpha (\frac{1}{\alpha}t^\alpha) = 1$

On letting $\alpha = 1$ in these derivatives, we get the corresponding ordinary derivatives.

Definition 2.4. ***(Fractional Integral)*** *Let $a \geq 0$ and $t \geq a$. Also, let f be a function defined on $(a,t]$ and $\alpha \in \mathbb{R}$. Then, the α-fractional integral of f is defined by*

$${}_aI_t^\alpha f(t) = \int_a^t \frac{f(t)}{t^{1-\alpha}} dt, \tag{2.5}$$

if the Riemann improper integral exists. If $f : (0,\infty) \to \mathbb{R}$ be a continuous function such that ${}_0I_t^\alpha$ exist then for all $t > 0$, $0 < \alpha \leq 1$ we have

$$D_t^\alpha \, {}_0I_t^\alpha f(t) = f(t). \tag{2.6}$$

Also let $f : (0,\infty) \to \mathbb{R}$ be a α-differentiable function and $0 < \alpha \leq 1$ then for all $t > 0$ we have

$${}_0I_t^\alpha D_t^\alpha f(t) = f(t) - f(0). \tag{2.7}$$

Theorem 2.5 (Fractional power series expansions [16])**.** *Assume f is an infinitely α-differentiable function, for some $0 < \alpha \leq 1$ at a neighborhood of a point zero, then f has the fractional power series expansion*

$$f(t) = \sum_{n=0}^{\infty} \frac{D_t^{\alpha n} f(0)\, t^{n\alpha}}{\alpha^n n!}, \qquad 0 < t < R^{\frac{1}{\alpha}}, \quad R > 0, \tag{2.8}$$

where $D_t^{\alpha n} f(0)$ means the application of the fractional derivative n times.

Theorem 2.6. *[27] Let $f, g : (0, b] \to \mathbb{R}$ are two real function such that $f.g$ is differentiable then*

$$\int_0^b f(t)\, D_t^\alpha g(t)\, d\alpha(t) = f.g|_0^b - \int_0^b g(t)\, D_t^\alpha f(t)\, d\alpha(t), \tag{2.9}$$

where $d\alpha(t) = t^{\alpha-1}dt$.

Definition 2.7. ***(Fractional Laplace transform** [16] **)*** *Let $0 < \alpha \leq 1$ and $f : (0, \infty) \to \mathbb{R}$ be a function with real values then the fractional Laplace transform of order α is defined as follow*

$$L_\alpha\{f(t)\} = F_\alpha(s) = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}} f(t)\, d\alpha(t), \tag{2.10}$$

where $d\alpha(t) = t^{\alpha-1}dt$.

The fractional Laplace transform of some crucial function is as follow [16]

1) $L_\alpha\{1\} = \frac{1}{s}, \quad s > 0,$
2) $L_\alpha\{t^\alpha\} = \frac{1}{s}, \quad s > 0,$
3) $L_\alpha\{e^{a\frac{t^\alpha}{\alpha}}\} = \frac{1}{s-a}, \quad s > a,$
4) $L_\alpha\{\sin \frac{1}{\alpha} t^\alpha\} = \frac{1}{s^2+1}, \quad s > 1,$
5) $L_\alpha\{\cos \frac{1}{\alpha} t^\alpha\} = \frac{s}{s^2+1}, \quad s > 1.$

Furthermore from properties of the fractional exponential function and theorem 2.6 we get

$$L_\alpha\{D_t^\alpha f(t)\} = s\, F_\alpha(s) - f(0), \quad 0 < \alpha \leq 1. \tag{2.11}$$

Theorem 2.8. ***(Fractional final value Theorem** [27]**)*** *If $f(t)$ and $D_t^\alpha f(t)$ have the fractional Laplace transform and $F_\alpha(s)$ be the fractional Laplace transform of $f(t)$ and $sF_\alpha(s)$ has no poles in the area $\Re(s) \geq 0$,then*

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF_\alpha(s). \tag{2.12}$$

3. Stability analysis of linear conformable fractional differential systems with time delays

In this Section we evaluate the stability of linear conformable fractional differential systems with time delays.

The general form of a linear conformable fractional differential system with time delays is as follows

$$\begin{cases} D_t^{\alpha_1} x_1(t) = a_{11}x_1(t) + \cdots + a_{1n}x_n(t) + b_{11}x_1(t-\tau_{11}) + \cdots + b_{1n}x_n\left(t-\tau_{1n}\right), \\ D_t^{\alpha_2} x_2(t) = a_{21}x_1(t) + \cdots + a_{2n}x_n(t) + b_{21}x_1(t-\tau_{21}) + \cdots + b_{2n}x_n\left(t-\tau_{2n}\right), \\ \quad\vdots \qquad\qquad\qquad\qquad \vdots \\ D_t^{\alpha_n} x_n(t) = a_{n1}x_1(t) + \cdots + a_{nn}x_n(t) + b_{n1}x_1\left(t-\tau_{n1}\right) + \cdots + b_{nn}x_n\left(t-\tau_{nn}\right), \end{cases} \tag{3.1}$$

with the initial conditions $x_i(t) = x_{i,0}(t)$ for $-\max\tau_{ij} = -\tau_{\max} \le t \le 0$ and $0 < \alpha_i \le 1$, $i,j = 1,...,n$. In this system $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n} \in \mathbb{R}^{n\times n}$, coefficient matrix $x_i(t), x_i(t-\tau_{ij}) \in \mathbb{R}$ state variables are given.

Definition 3.1. *Linear conformable fractional derivative system with time delays (3.1)*

- *i) is said to be stable if for any initial value x_0 , there exist an $\delta > 0$ such that $\|x(t)\| \le \delta$ for all $t \ge 0$.*
- *ii) is asymptotically stable if at first it is stable and $\|x(t)\| \to 0$ as $t \to \infty$.*

Remark 3.2. *Denoting by $X_\alpha(s)$ the fractional Laplace transform of the function $x(t)$, we can evaluate*

$$L_\alpha[x(t-\tau)] = \int_0^\infty e^{-s\frac{t^\alpha}{\alpha}}\, x(t-\tau)\, d\alpha(t) = \int_{-\tau}^\infty e^{-s\frac{(t+\tau)^\alpha}{\alpha}}\, x(t)\; d\alpha(t)$$

$$= e^{-s\frac{\beta\tau^\alpha}{\alpha}} X_\alpha(s) + e^{-s\frac{\beta\tau^\alpha}{\alpha}} \int_{-\tau}^0 e^{-s\frac{t^\alpha}{\alpha}}\, x(t)\, d\alpha(t),$$

where $\beta \in \mathbb{R}$.

Now we consider system (3.1) and express the main theorem for analysis of stability by applying the fractional Laplace transform and fractional final value theorem.

Theorem 3.3. *If all roots of*

$$\det\begin{pmatrix} s-a_{11}-b_{11}e^{-s\frac{\beta_{11}\tau_{11}^{\alpha_1}}{\alpha_1}} & -a_{12}-b_{12}e^{-s\frac{\beta_{12}\tau_{12}^{\alpha_1}}{\alpha_1}} & \dots & -a_{1n}-b_{1n}e^{-s\frac{\beta_{1n}\tau_{1n}^{\alpha_1}}{\alpha_1}} \\ -a_{21}-b_{21}e^{-s\frac{\beta_{21}\tau_{21}^{\alpha_2}}{\alpha_2}} & s-a_{22}-b_{22}e^{-s\frac{\beta_{22}\tau_{22}^{\alpha_2}}{\alpha_2}} & \dots & -a_{2n}-b_{2n}e^{-s\frac{\beta_{2n}\tau_{2n}^{\alpha_2}}{\alpha_2}} \\ \vdots & \vdots & \ddots & \\ -a_{n1}-b_{n1}e^{-s\frac{\beta_{n1}\tau_{n1}^{\alpha_n}}{\alpha_n}} & -a_{n2}-b_{n2}e^{-s\frac{\beta_{n2}\tau_{n2}^{\alpha_n}}{\alpha_n}} & \dots & s-a_{nn}-b_{nn}e^{-s\frac{\beta_{nn}\tau_{nn}^{\alpha_n}}{\alpha_n}} \end{pmatrix} = 0,$$

have negative real parts, then system (3.1) is asymptotically stable.

proof: We get the fractional Laplace transform of both sides of system (3.1) and obtain

$$\begin{aligned} s\,X_{\alpha_1}(s) - x_{1,0}(0) &= \sum_{j=1}^n a_{1j}X_{\alpha_j}(s) + \sum_{j=1}^n b_{1j}e^{-s\frac{\beta_{1j}\tau_{1j}{}^{\alpha_1}}{\alpha_1}} X_{\alpha_j}(s) \\ &\quad + \sum_{j=1}^n b_{1j}e^{-s\frac{\beta_{1j}\tau_{1j}{}^{\alpha_1}}{\alpha_1}} \int_{-\tau_{1j}}^0 e^{-s\frac{t^{\alpha_1}}{\alpha_1}} x_{j,0}(t)\, d\alpha_1(t), \\ s\,X_{\alpha_2}(s) - x_{2,0}(0) &= \sum_{j=1}^n a_{2j}X_{\alpha_j}(s) + \sum_{j=1}^n b_{2j}e^{-s\frac{\beta_{2j}\tau_{2j}{}^{\alpha_2}}{\alpha_2}} X_{\alpha_j}(s) \\ &\quad + \sum_{j=1}^n b_{2j}e^{-s\frac{\beta_{2j}\tau_{2j}{}^{\alpha_2}}{\alpha_2}} \int_{-\tau_{2j}}^0 e^{-s\frac{t^{\alpha_2}}{\alpha_2}} x_{j,0}(t)\, d\alpha_2(t), \\ &\;\;\vdots \\ s\,X_{\alpha_n}(s) - x_{n,0}(0) &= \sum_{j=1}^n a_{nj}X_{\alpha_j}(s) + \sum_{j=1}^n b_{nj}e^{-s\frac{\beta_{nj}\tau_{nj}{}^{\alpha_n}}{\alpha_n}} X_{\alpha_j}(s) \\ &\quad + \sum_{j=1}^n b_{nj}e^{-s\frac{\beta_{nj}\tau_{nj}{}^{\alpha_n}}{\alpha_n}} \int_{-\tau_{nj}}^0 e^{-s\frac{t^{\alpha_n}}{\alpha_n}} x_{j,0}(t)\, d\alpha_n(t), \end{aligned} \tag{3.2}$$

where $X_{\alpha_i}(s)$ is the fractional Laplace transform of $x_i(t)$, $i = 1, \ldots, n$.
We can rewrite (3.2) as follows

$$\Delta(s).\begin{bmatrix} X_{\alpha_1}(s) \\ X_{\alpha_2}(s) \\ \vdots \\ X_{\alpha_n}(s) \end{bmatrix} = \begin{bmatrix} h_1(s) \\ h_2(s) \\ \vdots \\ h_n(s) \end{bmatrix}, \tag{3.3}$$

in which

$$\Delta(s) = \begin{bmatrix} \Delta_{11}(s) & \Delta_{12}(s) & \cdots & \Delta_{1n}(s) \\ \Delta_{21}(s) & \Delta_{22}(s) & \cdots & \Delta_{2n}(s) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta_{n1}(s) & \Delta_{n2}(s) & \cdots & \Delta_{nn}(s) \end{bmatrix},$$

where

$$\Delta_{ij}(s) = \begin{cases} s - a_{ii} - b_{ii}e^{-s\frac{\beta_{ii}\tau_{ii}{}^{\alpha_i}}{\alpha_i}}, & i = j, \\ -a_{ij} - b_{ij}e^{-s\frac{\beta_{ij}\tau_{ij}{}^{\alpha_i}}{\alpha_i}}, & i \neq j. \end{cases}$$

$$\begin{cases} h_1(s) = x_{1,0} + \sum\limits_{j=1}^{n} b_{1j}e^{-s\frac{\beta_{1j}\tau_{1j}{}^{\alpha_1}}{\alpha_1}} \int_{-\tau_{1j}}^{0} e^{-s\frac{t^{\alpha_1}}{\alpha_1}} x_{j,0}(t)\, d\alpha_1(t), \\ \qquad\vdots \\ h_2(s) = x_{2,0} + \sum\limits_{j=1}^{n} b_{2j}e^{-s\frac{\beta_{2j}\tau_{2j}{}^{\alpha_2}}{\alpha_2}} \int_{-\tau_{2j}}^{0} e^{-s\frac{t^{\alpha_2}}{\alpha_2}} x_{j,0}(t)\, d\alpha_2(t), \\ \qquad\vdots \\ h_n(s) = x_{n,0} + \sum\limits_{j=1}^{n} b_{nj}e^{-s\frac{\beta_{nj}\tau_{nj}{}^{\alpha_n}}{\alpha_n}} \int_{-\tau_{nj}}^{0} e^{-s\frac{t^{\alpha_n}}{\alpha_n}} x_{j,0}(t)\, d\alpha_n(t)), \end{cases}$$

multiplying s on both sides of (3.3) gives

$$\Delta(s).\begin{bmatrix} sX_{\alpha_1}(s) \\ sX_{\alpha_2}(s) \\ \vdots \\ sX_{\alpha_n}(s) \end{bmatrix} = \begin{bmatrix} sh_1(s) \\ sh_2(s) \\ \vdots \\ sh_n(s) \end{bmatrix}. \tag{3.4}$$

If all roots of $\det(\Delta(s)) = 0$ lie in open left half complex plain,i.e.,$\Re(s) < 0$,then we consider (3.4) in $\Re(s) \geq 0$ .In this restricted area, (3.4) has a unique solution $sX_\alpha(s) = (sX_{\alpha_1}(s), sX_{\alpha_2}(s), \ldots, sX_{\alpha_n}(s))$. Since $\lim_{s\to 0} sh_i(s) = 0$ for $i = 1, 2, \ldots, n$, so we have

$$\lim_{s\to 0, \Re(s)\geq 0} sX_{\alpha_i}(s) = 0, \quad i = 1, 2, \ldots, n. \tag{3.5}$$

Therefore $sX_{\alpha_i}(s)$ has no poles in area $\Re(s) \geq 0$ which from the fractional final value theorem of fractional Laplace transform we get

$$\begin{aligned}\lim_{t\to\infty} x(t) &= \lim_{t\to\infty}(x_1(t), x_2(t), \dots, x_n(t)) \\ &= \lim_{s\to 0, \Re(s)\geq 0}(sX_{\alpha_1}(s), sX_{\alpha_2}(s), \dots, sX_{\alpha_n}(s)) = 0,\end{aligned}$$

so system (3.1) is asymptotically stable.

For simplicity we name $\Delta(s)$ a characteristic matrix of system (3.1) and $\det(\Delta(s)) = 0$ a characteristic equation of system (3.1).

Corollary 3.4. *If $\tau_{ij} = 0$ for $i, j = 1, ..., n$ system (3.1) is changed to:*

$$D_t^\alpha\, x(t) = Cx(t), \quad t > 0, \quad x(0) = x_0, \tag{3.6}$$

where $C = A + B$. Hence characteristic equation becomes $\det(sI - C) = 0$. Then $\lambda = s$ where λ is eigenvalue of C .If all roots of $\det(sI - C) = 0$ have negative real parts then system (3.1) is as asymptotically stable.

Remark 3.5. *If $\alpha_1 = \ldots = \alpha_n = 1$ system (3.1) is changed to a linear common differential equation.*

Definition 3.6. *The inertia of system (3.1) is the triple*

$$I_{n(con)}(A, B) = (\pi_{n(con)}(A, B), \nu_{n(con)}(A, B), \delta_{n(con)}(A, B)),$$

where $\pi_{n(con)}(A, B)$,$\nu_{n(con)}(A, B)$ and $\delta_{n(con)}(A, B)$ are the numbers of roots of $\det(\Delta(s)) = 0$ with positive,negative and zero real parts,respectively.

Theorem 3.7. *If $\pi_{n(con)}(A, B) = \delta_{n(con)}(A, B) = 0$ then system (3.1) is asymptotically stable.*

4. Numerical method

Since most of the conformable fractional differential equations systems do not have exact analytic solutions, so approximation and numerical techniques must be used. Several analytical and numerical methods have been proposed to solve the conformable fractional differential equations [26,28,29] . Hence, in the simulations of this paper we employ conformable Euler's method to solve conformable fractional differential equations systemby applying the fractional power series expansions.
Consider the following initial value problem

$$D_t^\alpha y(t) = f(t, y(t)), \quad a \leq t \leq b, \quad y(a) = c \tag{4.1}$$

where $a = t_0, t_1, \dots, t_N = b$ such that $t_i = a + ih$,$i = 0, 1, \cdots, N$ and step length $h = \frac{b-a}{N}$ are given.
Suppose that $D_t^\alpha y(t), D_t^{2\alpha} y(t) \in C^0[a, b]$. According to the fractional power series expansion,for $i = 0, 1, \cdots, N - 1$ we can write

$$y(t_{i+1}) = y(t_i + h) = y(t_i) + \frac{h^\alpha}{\alpha}(D_t^\alpha y)(t_i) + \frac{h^{2\alpha}}{2\alpha^2}(D_t^{2\alpha} y)(\xi_i), \tag{4.2}$$

where $t_i < \xi_i < t_{i+1}$.
Since $h = t_{i+1} - t_i$, there exist θ_i where $0 < \theta_i < 1$ such that

$$y(t_{i+1}) = y(t_i) + \frac{h^\alpha}{\alpha}(D_t^\alpha y)(t_i) + \frac{h^{2\alpha}}{2\alpha^2}(D_t^{2\alpha} y)(t_i + \theta_i h), \tag{4.3}$$

From the equation (4.1) we obtain

$$y(t_{i+1}) = y(t_i) + \frac{h^\alpha}{\alpha} f(t_i, y(t_i)) + \frac{h^{2\alpha}}{2\alpha^2}(D_t^{2\alpha} y)(t_i + \theta_i h), \tag{4.4}$$

We can rewrite (4.4) as follows

$$\frac{\alpha(y(t_{i+1}) - y(t_i))}{h^\alpha} = f(t_i, y(t_i)) + \frac{h^\alpha}{2\alpha}(D_t^{2\alpha} y)(t_i + \theta_i h), \tag{4.5}$$

When step length h be small enough the second term in the right-hand side of the equation (4.5) is small and can be deleted.Hence

$$\frac{\alpha(y(t_{i+1}) - y(t_i))}{h^\alpha} \approx f(t_i, y(t_i)), \tag{4.6}$$

as

$$y(t_{i+1}) \approx y(t_i) + \frac{h^\alpha}{\alpha} f(t_i, y(t_i)). \tag{4.7}$$

Relation (4.7) is called the conformable Euler's method
Obviously if we set $\alpha = 1$ then the conformable Euler's method (4.7) reduces to the classical Euler's method.
Now, we will consider the general conformable fractional order equation with time delay as follows

$$\begin{cases} D_t^\alpha y(t) = f(t, y(t), y(t-\tau)), & t \in (0, T], \\ y(t) = \phi(t), & t \in [-\tau, 0], \end{cases} \tag{4.8}$$

where α is real and lies in $(0, 1]$, $\phi(t) \in C^0[-\tau, 0]$ is the initial condition associated to above equation and $\tau > 0$ represents the time delay.
We consider conformable Euler's method to earn numerical solution for the conformable fractional differential equation with time delay.
Suppose j and N are integer numbers so that $T = hN$ and $\tau = hj$. We use a uniform grid

$$t_k = kh, \quad k = -j, -j+1, \ldots, -1, 0, 1, \ldots, N, \tag{4.9}$$

Let $\tilde{y}(t_k)$ denote the approximation to $y(t_k)$ and also we consider $\tilde{y}(t_k) = \phi(t_k)$ for $k = -j, -j+1, \ldots, -1, 0$.
In addition, according to the assumptions we have

$$\tilde{y}(t_k - \tau) = \tilde{y}(kh - jh) = \tilde{y}(t_{k-j}), \quad k = 0, 1, \ldots, N. \tag{4.10}$$

Presume we have already calculated approximation $\tilde{y}(t_k) \approx y(t_k)$,$k = -j, -j + 1, \ldots, -1, 0, 1, \ldots, n-1 \le N$. Therefore, a general numerical solution of the conformable fractional differential equation with time delay in the form (4.8) can be expressed by using the relation (4.10) as

$$\tilde{y}(t_{n+1}) = \tilde{y}(t_n) + \frac{h^\alpha}{\alpha} f(t_n, \tilde{y}(t_n), \tilde{y}(t_{n-j})), \quad n = 1, \ldots, N-1. \tag{4.11}$$

5. Example

In this Section,we give example to confirm out result for asymptotic stability of conformable fractional order systems.
Consider the following linear conformable fractional differential system with multiple delays

$$\begin{cases} D_t^{\alpha_1} y_1(t) = -2y_1(t) - ay_1(t-\tau_1) + ay_2(t), \\ D_t^{\alpha_2} y_2(t) = -ky_1(t) + by_1(t-\tau_2) - y_2(t) - ky_3(t), \\ D_t^{\alpha_3} y_3(t) = gy_1(t) - cy_3(t-\tau_3), \end{cases} \tag{5.1}$$

where a, b, c, k and g are real numbers and $y_{1,0}(t) = y_{2,0}(t) = y_{3,0}(t) = 1$ for $t \in [-\tau, 0]$ and $\tau = \max\{\tau_1, \tau_2, \tau_3\}$.
The characteristic matrix for this system is as follow

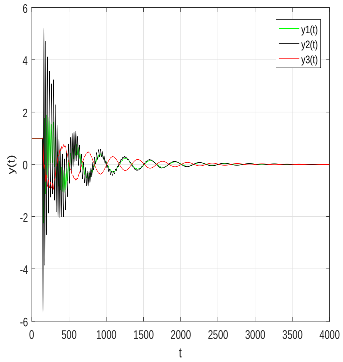
$$\Delta(s) = \begin{bmatrix} s + 2 + ae^{-s\frac{\beta_1 \tau_1{}^{\alpha_1}}{\alpha_1}} & -a & 0 \\ k - be^{-s\frac{\beta_2 \tau_2{}^{\alpha_2}}{\alpha_2}} & s+1 & k \\ -g & 0 & s + ce^{-s\frac{\beta_3 \tau_3{}^{\alpha_3}}{\alpha_3}} \end{bmatrix},$$

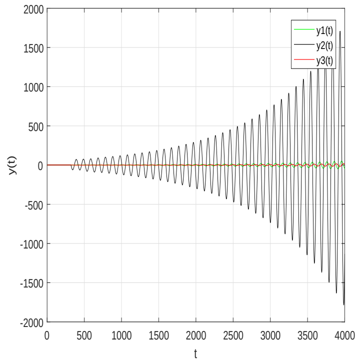
without loss of generality, we assume that $\beta_1 = \beta_2 = \beta_3 = 1$. Now we analyze the stability of this system by applying the previous theorems.In the table1 we compute $I_{n(con)}(A, B)$ for different values of a, b, c, k and g with time delays (τ_1, τ_2, τ_3) and fractional orders $(\alpha_1, \alpha_2, \alpha_3)$.

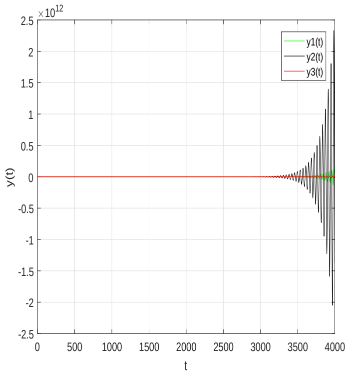
Table 1: Stability analysis of the system (5.1) for different values.

(a, b, c, k, g)	$(\alpha_1, \alpha_2, \alpha_3)$	(τ_1, τ_2, τ_3)	$I_{n(con)}$	stability
$(9, 5, 6, 2, 7)$	$(0.89, 1, 1)$	$(0, 0.6, 0.71)$	$(0, 3, 0)$	Yes
$(0.2, 3, 2, 2, 4)$	$(0.9, 0.95, 0.99)$	$(0.2, 1.6, 0.71)$	$(1, 2, 0)$	No
$(0.75, 1.8, 8, 1.5, 3)$	$(0.85, 0.9, 0.97)$	$(0.65, 0.8, 1.5)$	$(1, 2, 0)$	No

Figure 1 indicates that system (5.1) with parameters $(a, b, c, k, g) = (9, 5, 6, 2, 7)$ and orders $(\alpha_1, \alpha_2, \alpha_3) = (0.89, 1, 1)$, when $(\tau_1, \tau_2, \tau_3) = (0, 0.6, 0.71)$ is asymptotically stable. Figure 2 shows that system (5.1) with parameters $(a, b, c, k, g) = (0.2, 3, 2, 2, 4)$ is unstable when $(\alpha_1, \alpha_2, \alpha_3) = (0.9, 0.95, 0.99)$ and $(\tau_1, \tau_2, \tau_3) = (0.2, 1.6, 0.71)$. Figure 3 demonstrates that system (5.1) with parameters $(a, b, c, k, g) = (0.75, 1.8, 8, 1.5, 3)$ and orders $(\alpha_1, \alpha_2, \alpha_3) = (0.85, 0.9, 0.97)$, when $(\tau_1, \tau_2, \tau_3) = (0.65, 0.8, 1.5)$ is unstable. The final time of conformable Euler's method is $T = 20$ and the time step size $h = 0.05$.


y1(t)
y2(t)
y3(t)
y(t)
t


y1(t)
y2(t)
y3(t)
y(t)
t


$\times 10^{12}$
y1(t)
y2(t)
y3(t)
y(t)
t

6. Conclusion

In this paper, we have studied the asymptotic stability of the linear conformable fractional differential equations system with time delays. The results are obtained in terms of the fractional Laplace transform and the Fractional final value theorem. Further since system (5.1) needs to be solved numerically for the reconciling our results are given in Table 1, a suitable numerical method needs to be selected. Hence, we have employed conformable Euler's method to solve fractional conformable differential equations system. The linear conformable fractional differential system with multiple delays is given to demonstrate the effectiveness of the theorem and the proposed approach. All numerical results are obtained using Matlab 2010.

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