Bol. Soc. Paran. Mat. (3s.) v. 38 5 (2020): 131–145.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v38i5.33978

On a Nonlinear PDE Involving Weighted p-Laplacian

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ABSTRACT: In the present paper, we study the nonlinear partial differential equation with the weighted p-Laplacian operator

$$-\operatorname{div}(w(x)|\nabla u|^{p-2}\nabla u) = \frac{f(x)}{(1-u)^2},$$

on a ball $B_r \subset \mathbb{R}^N (N \geq 2)$. Under some appropriate conditions on the functions f, w and the nonlinearity $\frac{1}{(1-u)^2}$, we prove the existence and the uniqueness of solutions of the above problem. Our analysis mainly combines the variational method and critical point theory. Such solution is obtained as a minimizer for the energy functional associated with our problem in the setting of the weighted Sobolev spaces.

Key Words: Weighted p-Laplacian operator, Sobolev spaces, Muckenhoupt Weighted, Existence, Uniqueness of solutions.

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1. Introduction

Differential equations and variational problems have many applications in mathematical physics such as in the Micro-Electro Mechanical Systems(MEMS), in thin film theory, nonlinear surface diffusion on solids, interface dynamics, flow in Hele-Shaw cells, phase field models of multi-phase systems and the deformation of an elastic beam, see for instance ([10], [11]) or [16] and the references therein.

The most important linear partial differential equations of the second odrer are governed by the celebrated Laplacian operator Δ. It is less well-known that is also a nonlinear counterpart, the so called p-Laplacian defined by $\Delta_p u = \nabla(|\nabla u|^{p-2}u)$. At the critical points ($\nabla u = 0$), this prototype of nonlinear operator for $p \neq 2$ is degenerate for $p > 2$ and singular for $p < 2$. For $p = 2$, we just get the usual Laplacian operator. During the last quarter of cycle the Partial differential

2010 *Mathematics Subject Classification:* 47J05, 46E35, 74H20, 74H25.
Submitted October 23, 2016. Published January 10, 2018

equations governed by p-Laplacian have been much studied and its theory is by now rather developed. The purpose of this paper is to established the existence and uniqueness of the solutions for the following nonlinear elliptic equation with the weighted p-Laplacian operator

$$\begin{cases} -\operatorname{div}(w(x)|\nabla u|^{p-2}\nabla u) = \frac{f(x)}{(1-u)^2} & \text{in } B_r, \\ 0 < u < 1 & \text{in } B_r, \\ u \in W_0^{1,p}(B_r, w), \end{cases} \tag{1.1}$$

where B_r is an open ball in $\mathbb{R}^N (N \geq 2)$, of radius $r > 0$ and centered at the origin, $1 < p < \infty$, w is a positive weight function locally integrable in $\mathbb{R}^N$, i.e., $w \in L^1_{loc}(\mathbb{R}^N))$, f is a positive nonzero bounded continuous function. Notice that the nonlinearity $F(u) = \frac{1}{(1-u)^2}$ is differentiable, increasing and convex on the interval [0,1) with $F(0) = 1$ and $\lim\limits_{u \nearrow 1} F(u) = +\infty$.

Our methode is more direct and is mainly based on the critical point theory. For degenerate partial differential equations, i.e., equations with various types of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces ([3], [4], [5], [8], [15]). The type of a weight depends on the equation type.

A class of weights, which is particularly well understood, is the class of $\mathcal{A}_p$-weights (or Muckenhoupt class) that was introduced by Muckenhoupt [8]. These classes have found many useful applications in harmonic analysis in the linear case [13] and [14]. Another reason for studying $\mathcal{A}_p$-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^N$ often belong to $\mathcal{A}_p$ [1] and [15]. There are, in fact, many interesting examples of weights [5].

In the particular case $p = 2$ with $w(x) \equiv 1$ and [2] studied the problem:

$$\begin{cases} -\Delta u = \lambda \frac{f(x)}{(1-u)^2} & \text{in } \Omega, \\ 0 < u < 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

The authors established the existence of a regular as well as a singular solution to (1.2). This simple model, which lends itself to the vast literature on second order quasi-linear eigenvalue problems, is already a rich source of interesting mathematical problems.

In the degenerate case, the weighted p-Laplacian operator has been studied [1] and references cited there in.

Inspired by the above-mentioned papers, we study the existence and uniqueness of solutions of problem (1.1) in a neighborhood of the origin. More precisely, under some appropriate conditions on the functions f, w and the nonlinearity $\frac{1}{(1-u)^2}$.

The paper is organized as follow. First, in Section 2, we recall and we prove some preliminary results which will be used later. In Section 3, we establish the existence and uniqueness of solutions for problem (1.1). Finally, the last Section, we give an application illustrating our main results.

2. Preliminary Results

Before we discuss some results concerning the problem (1.1), let us recall some various definitions and basic properties of the weighted Sobolev spaces.

For convenience, let both dx and $|.|$ stand for the(N-dimensional) Lebesgue measure in $\mathbb{R}^N$. As we shall always a positive weight a locally integrable function on $\mathbb{R}^N$. Every weight w gives rise to a measure on the measurable subsets of $\mathbb{R}^N$through integration. This measure will be denoted by μ, That is:

$$\mu(E) = \int_E w(x)dx,$$

for a measurable set $E \subset \mathbb{R}^N$.

2.1. Muckenhoupt weights.

We briefly recall some fundamentals on Muckenhoupt classes $\mathcal{A}_p$.

Definition 2.1. *Let w be a positive, locally integrable function on $\mathbb{R}^N$*

- ***(i)*** *Let $1 < p < \infty$,Then w belongs to the Muckenhoupt class $\mathcal{A}_p$, if there exists a positive constant $C = C(p, w)$ such that, for all balls $B \subset \mathbb{R}^N$,*

$$\Big(\frac{1}{|B|}\int_B w(x)\,dx\Big)\Big(\frac{1}{|B|}\int_B w(x)^{-1/p-1}\,dx\Big)^{p-1} \leq C,$$

- ***(ii)*** *The Muckenhoupt class $\mathcal{A}_\infty$ is given by $\mathcal{A}_\infty = \bigcup\limits_{p>1} \mathcal{A}_p$.*

Remark 2.2. *Since the pioneering works of Muckenhoupt [7] and [9] these classes of weight functions have been studied in great detail. In present paper, we are only concerned with the case $p > 1$.*

Definition 2.3. *(space of functions of bounded mean oscillation (BMO)). Suppose that f is integrable over compact sets in $\mathbb{R}^N$ and that for any ball $B \subset \mathbb{R}^N$, with volume denoted by $|B|$, the mean of f over B will be*

$$f_B = \frac{1}{|B|}\int_B f(t)dt.$$

We say that f belongs to BMO if

$$||f||_* = \sup_B \frac{1}{|B|}\int_B |f(t) - f_B|dt < \infty,$$

where the supremum is taken over all balls B. Here, $||f||_*$ is called the BMO-norm of f , and it becomes a norm on BMO after dividing out the constant functions.

Remark 2.4. *(i) Functions of bounded mean oscillation were introduced by F. John and L. Nirenberg [6] and Simon [12].*

(ii) (the monotonicity) If $1 < p_1 \leq p_2$*, then* $\mathcal{A}_{p_1} \subset \mathcal{A}_{p_2}$ *[4,5] and [15] for more information about* $\mathcal{A}_p$*-weights).*

Example 2.5. *(i). One of the most prominent examples of an* $\mathcal{A}_p$*-weight is* $w(x) = |x|^\alpha, x \in \mathbb{R}^N, -N < \alpha < N(p-1)$ *given by [14]. An other example is given by* $w(x) = e^{\alpha\varphi(x)}$ *which belongs in* $\mathcal{A}_2$*, whenever* $\varphi \in BMO(\mathbb{R}^N)$ *and the real* $\alpha > 0$ *[12].*

Lemma 2.6. *Let* $w \in \mathcal{A}_p$*,where* $1 < p < \infty$*,and let* E *be a measurable subsets of a ball* B*. then*

$$\Big(\frac{|E|}{|B|}\Big)^P \leq C\frac{\mu(E)}{\mu(B)}$$

where C *is the* $\mathcal{A}_p$ *constant of* w*.*

Proof: By writing $1 = w^{1/p}w^{-1/p}$,Hölder's inequality implies that

$$\begin{aligned}
|E| &\leq \Big(\int_E w dx\Big)^{1/p}\Big(\int_E w^{-1/p-1}dx\Big)^{p-1/p} \\
&\leq \mu(E)^{1/p}|B|^{p-1/p}\Big(\frac{1}{|B|}\int_E w^{-1/p-1}dx\Big)^{p-1/p} \\
&\leq C^{1/p}\mu(E)^{1/p}|B|^{p-1/p}\Big(\frac{1}{|B|}\int_E w dx\Big)^{-1/p} \\
&= C^{1/p}\Big(\frac{\mu(E)}{\mu(B)}\Big)^{1/p}|B|
\end{aligned} \tag{2.1}$$

□

Remark 2.7. *If* $\mu(E) = 0$*, then* $|E| = 0$*.*

2.2. Weighted Lebesgue and Sobolev spaces.

Definition 2.8. *Let* w *be a positive weight. We shall denote by* $L^p(\Omega, w); (1 \leq p < \infty)$ *the Banach space of all measurable functions* f *defined in* Ω *for which*

$$||f||_{p,\Omega} = \Big(\int_\Omega |f(x)|^p w(x)dx\Big)^{\frac{1}{p}} < \infty.$$

Note that the dual space of $L^p(\Omega, w)$ is the space $[L^p(\Omega, w)]^* = L^{p'}(\Omega, w^*)$ where $w^* = w^{-1/p-1}$ and the conjugate index of p will be denoted by p' in such a way that $(1/p + 1/p' = 1)$.

Lemma 2.9. *[15] If* $w \in \mathcal{A}_p$*,* $1 < p < \infty$*, then since* $w^{-1/(p-1)}$ *is locally integrable, we have* $L^p(\Omega, w) \subset L^1_{loc}(\Omega)$*.*

Proof: Suppose that $f \in L^p(\Omega, w)$, and let $B \subset \Omega$ be a ball. Thus Hölder's inequality implies that,

$$\begin{aligned}
\int_B |f(x)|dx &= \int_B |f(x)|w(x)^{1/p}w(x)^{-1/p}dx \\
&\leq \Big(\int_B |f(x)|^p w(x)dx\Big)^{1/p}\Big(\int_B (w(x)^{1/(1-p)})dx\Big)^{1/p'}
\end{aligned}$$

It follows that $L^p(\Omega, w) \subset L^1_{loc}(\Omega)$ and that convergence in $L^p(\Omega, w)$ implies local convergence in $L^1(\Omega)$. □

Remark 2.10. *(i) if Ω is bounded,one obtains in the same way that $L^p(\Omega, w)$ is continuously embedded in $L^1(\Omega)$.*

(ii) by lemma 2.9, we make sense to talk about weak derivatives of functions in $L^p(\Omega, w)$.

Definition 2.11. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, for $1 < p < \infty$, $w \in \mathcal{A}_p$ and a positive integer k, the weighted Sobolev spaces $W^{k,p}(\Omega, w)$ is defined by*

$$W^{k,p}(\Omega, w) := \{u \in L^p(\Omega, w) : D^\alpha u \in L^p(\Omega, w), 1 \leq |\alpha| \leq k\} \tag{2.2}$$

endowed by the weighted norm

$$||u||_{k,p,\Omega} = \Big(\int_\Omega |u(x)|^p w(x) dx + \sum_{1 \leq |\alpha| \leq k} \int_\Omega |D^\alpha u(x)|^p w(x) dx \Big)^{1/p}. \tag{2.3}$$

We also define the space $W_0^{k,p}(\Omega, w)$ as the closure in $W^{k,p}(\Omega, w)$ of $C_0^\infty(\Omega)$ with respect to the norm $||.||_{k,p,\Omega}$.

Proposition 2.12. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set, $1 < p < \infty, k$ be a nonnegative integer, suppose that $w \in \mathcal{A}_p$. Then The spaces $W^{k,p}(\Omega, w)$ and $W_0^{k,p}(\Omega, w)$ are Banach spaces.*

Remark 2.13. *It is evident that a weight function w which satisfies $0 < C_1 \leq w(x) \leq C_2$,for $x \in \Omega$, gives nothing new (the space $W^{k,p}(\Omega, w)$)is then identical with the classical Sobolev space $W^{k,p}(\Omega)$. Consequently, we shall be interested above all in such weight functions w which either vanish somewhere in $\Omega \cup \partial\Omega$ or increase to infinity (or both).*

In order to avoid too many suffices, at each step, a generic constant is denoted by C_{B_r} or C. we need the following basic result.

Theorem 2.14. *(the weighted Imbedding theorem.)*
Given $1 < p < \infty$ and $w \in \mathcal{A}_p$.There exist constants C_{B_r} and δ positive such that for all balls B_r, all $u \in C_0^\infty(B_r)$ and all numbers k satisfying $1 \leq k \leq N/(N-1)+\delta$,

$$||u||_{kp,B_r} \leq C_{B_r} \|\nabla u\|_{p,B_r}$$

Proof: A proof of the above statement can be found in Theorem [3, Theorem1.2]. □

For $k = 1$ in the above inequality, we have

$$||u||_{p,B_r} \leq C_{B_r} ||u||_{0,1,p}. \tag{2.4}$$

where

$$||u||_{0,1,p} = ||\nabla u||_{p,B_r} := \Big(\int_{B_r} |\nabla u|^p \; w \; dx\Big)^{1/p} \tag{2.5}$$

Throughout this paper,we use the following definition:

Definition 2.15. *Let $B_r \subset \mathbb{R}^N$ be an open ball and $w \in \mathcal{A}_p$,$1 < p < \infty$. We denote by*

$$X := W_0^{1,p}(B_r, w).$$

Define a norm $||.||_X$ of X by

$$||u||_X = \Big(\int_{B_r} |\nabla u|^p w(x) dx\Big)^{1/p}.$$

Then, endowed with $||.||_X$, X is a separable and reflexive Banach space.

3. Main Result

We need the following assumption.

$$(H) \qquad \frac{f}{w(1-u)^2} \in L^{p'}(B_r, w).$$

Remark 3.1. *(a) Regarding condition (H) and the fact that $0 < u < 1$, for a.e. $x \in B_r$, $w(x) > 0$ and $f(x) > 0$, we have*

$$\frac{f(x)}{w(x)\Big(1-u(x)\Big)^2} \geq \frac{f(x)}{w(x)\Big(1-u(x)\Big)}$$

Then, for $w \in \mathcal{A}_p$(with $1 < p < \infty$), we have also

$$\frac{f}{w(1-u)} \in L^{p'}(B_r, w).$$

(b) One of the most prominent examples of $f(x)$, is given by the function

$$f(x) = \beta |x|^\alpha$$

where $\alpha \geq 0$ and $\beta > 0$.

Definition 3.2. *A function $u \in X$ is a **weak solution** of problem (1.1) if,and only if: $0 < u < 1$ almost everywhere in B_r and*

$$\int_{B_r} w(x)|\nabla u|^{p-2}\nabla u \nabla v \, dx = \int_{B_r} \frac{f(x)}{(1-u)^2} v \, dx \quad for\, all\, v \in X. \tag{3.1}$$

Now, we are ready to state our main theorem of this paper.

Theorem 3.3. *Assume that (H) holds. If $w \in \mathcal{A}_p$(with $1 < p < \infty$) Then the problem (1.1) has a unique solution $u \in X$.*

Let now define the energy functionals : $\Phi, \Psi, I : X \to \mathbb{R}$ by:

$$\Phi(u) = \frac{1}{p}\int_{B_r} |\nabla u|^p w(x)\, dx,$$

$$\Psi(u) = \int_{B_r} \frac{f(x)u}{(1-u)}\, dx,$$

$$I(u) = \Phi(u) - \Psi(u),$$

In order to prove Theorem 3.3, we need the following auxillary lemma.

Lemma 3.4. *We have the following statements*

(a) Ψ *is weakly lower semi-continuous,* $\Psi \in C^1(X, \mathbb{R})$*, and*

$$\langle \Psi'(u), v\rangle = \int_{B_r} \frac{f}{(1-u)^2} v\, dx$$

for all $u, v \in X$.

(b) Φ *is weakly lower semi-continuous,* $\Phi \in C^1(X, \mathbb{R})$*, and*

$$\langle \Phi'(u), v\rangle = \int_{B_r} |\nabla u|^{p-2}\nabla u \nabla v \;w(x)\, dx$$

for all $u, v \in X$.

Proof: We start first by showing that $\Psi \in C^1(X, \mathbb{R})$, that is, for all $h \in X$,

$$\lim_{t\to 0} \frac{\Psi(u+th) - \Psi(u)}{t} = \langle d\Psi(u), h\rangle,$$

and $d\Psi : X \to X^*$ is continuous, where we denote by X^* the dual space of X. For all $h \in X$, we have

$$\begin{aligned}
\lim_{t\to 0} \frac{\Psi(u+th) - \Psi(u)}{t} &= \frac{d}{dt}\Big(\Psi(u+th)\Big)\Big|_{t=0} \\
&= \Big(\frac{d}{dt}\int_{B_r} f(x)\frac{(u+th)}{(1-u-th)}dx\Big)\Big|_{t=0} \\
&= \int_{B_r} \frac{d}{dt}\Big(f(x)\frac{(u+th)}{(1-u-th)}\Big)\Big|_{t=0} dx \\
&= \int_{B_r} f(x)\Big(\frac{h[(1-u-th)+(u+th)]}{(1-u-th)^2}\Big)\Big|_{t=0} dx \\
&= \int_{B_r} \frac{hf(x)}{(1-u)^2}dx \\
&= \langle d\Psi(u), h\rangle
\end{aligned}$$

Using condition (H) and Hölder's inequality, we obtain

$$\begin{aligned}\int_{B_r} \frac{hf(x)}{(1-u)^2}dx &\leq \Big(\int_{B_r} |\frac{f}{w(1-u)^2}|^{p'} wdx\Big)^{\frac{1}{p'}} \Big(\int_{B_r} |h|^p wdx\Big)^{\frac{1}{p}} \\ &= \Big|\Big|\frac{f}{w(1-u)^2}\Big|\Big|_{p',B_r} ||h||_{p,B_r} \\ &= \Big|\Big|\frac{f}{w(1-u)^2}\Big|\Big|_{p',B_r} ||h||_X \\ &< +\infty.\end{aligned}$$

Then

$$\begin{aligned}|\langle d\Psi(u), h\rangle| &= \Big|\int_{B_r} \frac{hf(x)}{(1-u)^2}dx\Big| \\ &\leq \Big|\Big|\frac{f}{w(1-u)^2}\Big|\Big|_{p',B_r} ||h||_{p,B_r} \\ &\leq C||h||_{p,B_r}.\end{aligned}$$

Using the linearity of $d\Psi(u)$ and the above inequality, we deduce that $d\Psi(u) \in X^*$. Note that the function $u \mapsto \frac{1}{(1-u)^2}$ is continuous. So, we conclude that Ψ is Fréchet differentiable. Furthermore,

$$\langle \Psi'(u), v\rangle = \int_{B_r} \frac{f(x)}{(1-u)^2} v\, dx,$$

for all $u, v \in X$.
By the continuity and the convexity of Ψ, we deduce that Ψ is weakly lower semi-continuous .
(b) Similarly, we can also prove that $\Phi \in C^1(X, \mathbb{R})$. Moreover since $1 < p < \infty$ and in view of the weakly lower semi-continuity of the norm , we deduce that Φ is lower semi-continuous for the weak convergence. Furthermore,

$$\langle \Phi'(u), v\rangle = \int_{B_r} |\nabla u|^{p-2} \nabla u \nabla v \ w(x)\, dx$$

for all $u, v \in X$. Which gives the Fréchet differentiability of Φ. □

Proof: of Theorem 3.3
Our aim is to obtain a minimizer as the limit of a minimizing sequence $\{u_n\}$ of the Euler-Lagrange functional I_p, which is a weak solution of problem (1.1).

We will divide the proof into five steps.
Step 1. We shall prove that

$$\inf\{I(u) | u \in X\} > -\infty.$$

Thanks to hypothesis (H) and Hölder's inequality, we have

$$\begin{aligned}|\Psi(u)| &\leq \Big(\int_{B_r} |\frac{f}{w(1-u)}|^{p'} w dx\Big)^{\frac{1}{p'}} \Big(\int_{B_r} |u|^p w dx\Big)^{\frac{1}{p}} \\ &= \Big|\Big|\frac{f}{w(1-u)}\Big|\Big|_{p',B_r} ||u||_{p,B_r}.\end{aligned} \tag{3.2}$$

By Theorem 2.14, there exists a constant $C_{B_r} > 0$ such that

$$||u||_{p,B_r} \leq C_{B_r} \|\nabla u\|_{p,B_r}.$$

Bearing in mind our definition of norm, then (3.2) implies

$$|\Psi(u)| \leq C_{B_r} \Big|\Big|\frac{f}{w(1-u)}\Big|\Big|_{p',B_r} ||u||_X.$$

Young's inequality yields

$$|\Psi(u)| \leq \frac{1}{p}||u||_X^p + \frac{1}{p'}[C_{B_r}\Big|\Big|\frac{f}{w(1-u)}\Big|\Big|_{p',B_r}]^{p'}. \tag{3.3}$$

On the other hand, we have

$$\Phi(u) = \frac{1}{p}\int_{B_r} |\nabla u|^p w(x)\, dx = \frac{1}{p}||u||_X^p. \tag{3.4}$$

In view of (3.3) and (3.4), we deduce that

$$\begin{aligned} I(u) &\geq \Phi(u) - |\Psi(u)|. \\ &\geq \frac{1}{p}||u||_X^p - \frac{1}{p}||u||_X^p - \frac{1}{p'}[C_{B_r}\Big|\Big|\frac{f}{w(1-u)}\Big|\Big|_{p',B_r}]^{p'} \\ &\doteq -\frac{1}{p'}[C_{B_r}\Big|\Big|\frac{f}{w(1-u)}\Big|\Big|_{p',B_r}]^{p'}, \end{aligned}$$

that is, I is bounded from below. This completes step 1.
Step 2. We shall prove that any minimizing sequence is bounded in X
Let $\{u_n\}$ be a minimizing sequence, that is, a sequence such that

$$I(u_n) \to \inf_{\varphi\in X} I(\varphi)$$

and satisfies (H).
Then for n large enough, we obtain that

$$0 \geq I(u_n) = \frac{1}{p}\int_{B_r} |\nabla u_n|^p w\, dx - \int_{B_r} \frac{f u_n}{(1-u_n)}\, dx,$$

and we get by applying Theorem 2.14

$$\begin{aligned}||u_n||_X^p &\leq p\ (\int_{B_r} \frac{f u_n}{(1-u_n)} dx)\\ &\leq p\ \Big|\Big|\frac{f}{w(1-u_n)}\Big|\Big|_{p',B_r} ||u_n||_{p,B_r}\\ &\leq p\ C_{B_r}\Big|\Big|\frac{f}{w(1-u_n)}\Big|\Big|_{p',B_r} ||u_n||_X.\end{aligned}$$

Therefore

$$||u_n||_X \leq \Big[p\ C_{B_r}||\frac{f}{w(1-u_n)}||_{p',B_r}\Big]^{\frac{1}{p-1}}.$$

Hence u_n is bounded in X. By the reflexivity of the space X , there exists a function $u \in X$ such that $u_n \rightharpoonup u \quad in \quad X$ (for a subsequence if necessary).

Step 3. We shall prove that I is weakly lower semi-continuous.

By (a) and (b) of Lemma 3.4, I is weakly lower semi-continuous, $I \in C^1(X, \mathbb{R})$. It follows that

$$I(u) \leq \liminf_n I(u_n) = \inf_{u\in X} I(u),$$

and thus u is a minimizer of I on X.

Step 4. We shall prove that u is a minimizer of I and it's equivalently a weak solution of problem (1.1).

For any $\varphi \in X$, the function

$$\theta \mapsto \frac{1}{p}\int_{B_r} |\nabla(u+\theta v)|^p\, w(x)dx - \int_{B_r} \frac{(u+\theta v)f}{(1-u-\theta v)}\, dx.$$

has a minimum at $\theta = 0$. Hence

$$\frac{d}{d\theta} I(u+\theta v)|_{\theta=0} = 0, \quad \forall v \in X.$$

By expanding the terms of the functional $\frac{d}{d\theta}I(u+\theta v)|_{\theta=0} = 0$, we get

$$\begin{aligned}\frac{d}{d\theta}\Big(\Phi(u+\theta v)\Big)\Big|_{\theta=0} &= \Big(\frac{d}{d\theta}\frac{1}{p}\int_{B_r}(|\nabla(u+\theta v)|^p w(x)dx\Big)\Big|_{\theta=0}\\ &= \frac{1}{p}\int_{B_r}\frac{d}{d\theta}\Big(|\nabla(u+\theta v)|^p w(x)\Big)\Big|_{\theta=0} dx\\ &= \frac{1}{p}\int_{B_r} p\Big(|\nabla(u+\theta v)|^{p-2}(\nabla u+\theta\nabla v)\nabla v\ w(x)\Big)\Big|_{\theta=0} dx\\ &= \int_{B_r} |\nabla u|^{p-2}\nabla u\nabla v\ w(x)dx\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}\frac{d}{d\theta}\Big(\Psi(u+\theta v)\Big)\Big|_{\theta=0} &= \Big(\frac{d}{d\theta}\int_{B_r} f(x)\frac{(u+\theta v)}{(1-u-\theta v)}dx\Big)\Big|_{\theta=0}\\ &= \int_{B_r}\frac{d}{d\theta}f(x)\frac{(u+\theta v)}{(1-u-\theta v)}\Big)\Big|_{\theta=0}dx\\ &= \int_{B_r} f(x)\Big(\frac{v[(1-u-\theta v)+(u+\theta v)]}{(1-u-\theta v)^2}\Big)|_{\theta=0}dx\\ &= \int_{B_r}\frac{vf(x)}{(1-u)^2}dx.\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6), we obtain that

$$\begin{aligned}0 &= \frac{d}{d\theta}I(u+\theta v)|_{\theta=0}\\ &= \int_{B_r}|\nabla u|^{p-2}\nabla u\nabla v w(x)dx - \int_{B_r}\frac{vf(x)}{(1-u)^2}dx\end{aligned}$$

Therefore

$$\int_{B_r}|\nabla u|^{p-2}\nabla u\nabla v w(x)dx = \int_{B_r}\frac{vf(x)}{(1-u)^2}dx, \forall\ v\in X.$$

In other words u is a weak solution of problem (1.1).
Step 5. We claim that the limit function u is unique.

It is clear that u is a weak solution of problem (1.1) if and only if u is a critical point of I, thus $\Phi'(u)=\Psi'(u)$. The operator $\Phi' : X \longrightarrow X^*$ defined as :

$$\langle\Phi',\varphi\rangle = \int_{B_r}|\nabla u|^{p-2}\nabla u\nabla\varphi w(x)\,dx,\ \ for\ any\ u,\varphi\in X.$$

We shall prove that Φ' is strictly monotone. Let $u, v\in X$ are two weak solutions of problem (1.1), with $u\neq v$ in X.

We recall the following well-known inequalities [12], which hold for every $a, b\in \mathbb{R}^N$

$$\Big(a|a|^{p-2}-b|b|^{p-2}\Big)(a-b)\geq c(p)\begin{cases}|a-b|^p, & \text{if } p\geq 2\\ \dfrac{|a-b|^2}{(|a|+|b|)^{2-p}}, & \text{if } 1<p<2,\end{cases} \tag{3.7}$$

where the constant $c(p)=\min(2^{2-p}, p-1)$.

We can easily obtain that

$$\begin{aligned}\langle \Phi'(u)-\Phi'(v),u-v\rangle &= \langle \Phi'(u),u-v\rangle-\langle \Phi'(v),u-v\rangle \\ &= \int_{B_r}|\nabla u|^{p-2}\nabla u\Big(\nabla u-\nabla v\Big)w(x)dx \\ &- \int_{B_r}|\nabla v|^{p-2}\nabla v\Big(\nabla u-\nabla v\Big)w(x)dx \\ &= \int_{B_r}\Big(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v\Big)\Big(\nabla u-\nabla v\Big)w(x)dx \\ &\geq c(p)\begin{cases}\displaystyle\int_{B_r}|\nabla u-\nabla v|^{p}w(x)dx, & \text{if } p\geq 2\\ \displaystyle\int_{B_r}\frac{|\nabla u-\nabla v|^{2}}{(|\nabla u|+|\nabla v|)^{2-p}}w(x)dx, & \text{if } 1<p<2,\end{cases} \\ &\geq 0\end{aligned} \tag{3.8}$$

Which means that Φ' is monotone. In fact,Φ' is strictly monotone.
Indeed,if $\langle \Phi'(u)-\Phi'(v),u-v\rangle=0$, then we have $\nabla u=\nabla v\ \mu-a.e.$, Thus, we obtain

$$\begin{aligned}\langle \Phi'(u)-\Phi'(v),u-v\rangle &= \langle \Phi'(u),u-v\rangle-\langle \Phi'(v),u-v\rangle \\ &= \int_{B_r}|\nabla u|^{p-2}\Big((\nabla u)^2-\nabla u\nabla v\Big)w(x)dx \\ &- \int_{B_r}|\nabla v|^{p-2}\Big(\nabla u\nabla v-(\nabla v)^2\Big)w(x)dx \\ &= \int_{B_r}|\nabla u|^{p-2}\Big((\nabla u)^2+(\nabla v)^2-2\nabla u\nabla v\Big)w(x)dx \\ &= \int_{B_r}|\nabla u|^{p-2}\Big(\nabla u-\nabla v\Big)^2 w(x)dx \\ &= 0\end{aligned}$$

Now we distinguish two cases with respect to the values of the exponent p which related to the singularity of the p-Laplacian at the points where the divergence vanishes.

Case 1. If $1<p<2$,(3.8) follows that $\nabla u=\nabla v\ \mu-a.e.$ and since $u,v\in X$,then $u=v$ a.e. (by Lemma 2.6),which is a contradiction.

Case 2. If $p\geq 2$,(3.8) implies that $\nabla u=\nabla v\ \mu-a.e.$ which contradicts $u\neq v$ in X, or $|\nabla u|=0$.

Suppose that $|\nabla u|=0$, in view of $\nabla u=\nabla v\ \mu-a.e.$,we get $u=v\ a.e$,which is a contradiction again.
Therefore,$\langle \Phi'(u)-\Phi'(v),u-v\rangle>0$. It follows that Φ' is a strictly monotone operator on X and the problem (1.1) has a unique solution $u\in X$. Then, step 5 is verified and the proof of Theorem 3.3 is now acheived. □

4. Application

It is instructive to consider an example. We briefly present a typical leading applications of the our results.

$$\begin{cases} -\operatorname{div}(w(z)|\nabla u(z)|\nabla u(z)) = \frac{f}{(1-u(z))^2} & \text{in } \Omega, \\ 0 < u < 1 & \text{in } \Omega, \\ u \in W_0^{1,3}(\Omega, w), \end{cases} \tag{4.1}$$

Theorem 4.1. *Let Ω denote the unit ball in $\mathbb{R}^2$ and $p = 3$.Consider the radial functions w, f and S defined as*

$$w(x, y) = (x^2 + y^2)^{-1/4},$$

$$f(x, y) = 1/2(x^2 + y^2)^{1/4},$$

and

$$S(x, y) = 1 - (x^2 + y^2)^{1/2}$$

for all $z = (x, y) \in \Omega$.
Then u is a unique decreasing solution of problem (4.1) in $W_0^{1,3}(\Omega, w)$.

Proof: To simplity the notation, Let $r = |z|$.So

$$w(r) = r^{-1/2}$$

$$f(r) = 1/2\ r^{1/2}$$

and

$$S(r) = 1 - r.$$

Now, we can verify our required conditions: for $p = 3$ and $p' = 3/2$, we have by a simple calculation $w \in \mathcal{A}_3 - weight$.
(ii).

$$\begin{aligned} \int_0^{2\pi}\int_0^1 \Big|\frac{f(r)}{w(r)(1-S(r))^2}\Big|^{3/2} rdrd\theta &= 2\pi \int_0^1 |\frac{1/2\ r^{1/2}}{r^{-1/2}r^2}|^{3/2} rdr \\ &= 2\pi \frac{1}{2^{3/2}} \int_0^1 \frac{1}{r^{1/2}} dr \\ &< \infty. \end{aligned}$$

Then $\frac{f}{w(1-S(r))^2} \in L^{3/2}(\Omega, w)$ and (H) is satisfied.
(iii).

$$\begin{aligned} \int_0^{2\pi}\int_0^1 \Big|\frac{f}{w(x)(1-S(r))}\Big|^{3/2} rdrd\theta &= 2\pi \int_0^1 \frac{\left(1/2\ r^{1/2}\right)^{3/2} r}{\left(r^{-1/2}\ r\right)^{3/2}} dr \\ &= \frac{\pi}{2^{3/2}} < \infty. \end{aligned}$$

Then $\frac{f}{w(1-S(r))} \in L^2(\Omega, w)$ and(H) is verified, Consequently our Theorem 3.3 implies that S is a unique solution for problem (4.1).
Moreover, $S'(r) = -1 < 0$ then S is a unique radially decreasing solution of problem (4.1).

On the other hand, the divergence equation for radial function is given by:

$$\operatorname{div} \overrightarrow{V} = \frac{1}{r}\frac{d}{dr}\big[rV_r\big],$$

where $\overrightarrow{V} = V_r\overrightarrow{e_r}$.

Now, for the particular vector function $\overrightarrow{V} = w(r)|\nabla u(r)|\nabla u(r)\overrightarrow{e_r}$.Its divergence is defined as follows

$$\begin{aligned} \operatorname{div}(w(r)|\nabla u(r)|\nabla u(r)) &= \frac{1}{r}\frac{d}{dr}\big[-r\ r^{-1/2}\big] \\ &= \frac{1}{r}(-1/2\ r^{-1/2}) \\ &= -\frac{1}{2\ r^{3/2}}. \end{aligned} \tag{4.2}$$

On the other hand the second term of the equation is

$$\frac{f}{(1-S(r))^2} = \frac{1}{2\ r^{3/2}}. \tag{4.3}$$

In view of (4.2) and (4.3),we have

$$-\operatorname{div}(w(r)|\nabla u(r)|\nabla u(r)) = \frac{f}{(1-S(r))^2}.$$

Which proves the result of theorem 4.1.

□

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