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Dynamical Analysis of an Innovation Diffusion Model with Evaluation Period

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ABSTRACT: A nonlinear form of innovation diffusion model consisting of two driving equations governed by two variables for adopter and non-adopter population density is proposed to lay stress on the evaluation period. The model is analyzed qualitatively with stability theory and Hopf-bifurcation analysis by considering the evaluation period as a control parameter to see the role of evaluation period in shaping the dynamics of adopter and non-adopter populations. The threshold value of evaluation period is determined beyond which small amplitude oscillations of adopter and non-adopter population occur and goes on decreasing with the increase in carrying capacity of non-adopter class. The sensitivity analysis of the state variables w.r.t. the model parameters is performed at the positive equilibrium point. The effect of external influences to achieve maturity stage is also discussed. The analytical results so obtained are verified with the help of numerical simulations.

Key Words: Nonlinear dynamical system, Evaluation period, Stability analysis, Sensitivity analysis, Hopf-bifurcation.

Contents

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1. Introduction

Modeling and forecasting the diffusion of innovations has been a topic of practical and academic interest since the early sixties due to the work of [1,2,3,4,5,6]. Fourt and Woodlock in [1] assumed that the diffusion process is influenced solely by external factors and proposed the external influence model. This model assumed no interaction between the members of the social system. The internal influence model was proposed which was based on a contagion paradigm that diffusion occurs only through personal contacts. Mansfield in [2] illustrated the applications of internal influence model which was also known as pure imitation diffusion model. The mathematical study of Bass Model gives the deeper insight into the study of Marketing Science [6]. The first marketing innovation diffusion model was proposed by Bass, and it is still providing the foundation for developing the new hypotheses for gaining insight into the diffusion and penetration of innovations among the potential adopters.

Bass model has wide acceptance in the literature, but it works under certain set of assumptions such as constant market size, absence of repeat purchasers, effect of marketing variables etc., which limits its applicability to describe a typical product adoption behavior. Several researchers have worked to develop more generalized models applicable to diverse marketing environments and been used to understand the spread of new products in the market [8,9,10,11,13,14,15,17,20]. The study due to Bass formed basis for the development of many of these models. A dynamic diffusion model was proposed by [12], where the market size was permitted to vary over time. Other dynamic diffusion models have been developed by [5,16,18,19]. One of the first to use a heterogeneous population argument was Rogers [4]. He suggests that an adopting unit may pass through a series of five stages in the innovation decision process. These stages are awareness (individual is exposed to innovation), interest (individual seeks more information), evaluation (individual applies innovation to his or her situation), trial (individual uses innovation on a small scale) and adoption (individual makes full use of the innovation). According to Rogers, "a technological innovation is a design for instrumental action that reduces the uncertainty in the cause-effect relationships involved in achieving a desired outcome". In [21,22], the author's found that the five steps innovation diffusion processes can be simplified into two-step flow process, that is, external factor influences innovative opinion leaders to adopt a product, who in turn influence other population to adopt the product. The multistage nature of the diffusion models was also studied by [23]. To forecast the use of ethical drugs, repeat purchase models were developed in [24,25].

Another innovation diffusion model comprising of three compartment model consisting of non-adopter, adopter and frustrated classes of population is presented to discuss the influence of media coverage in spreading and controlling of adopter of a particular product in a region [26]. The diffusion process has frequently been modeled via a two-stage single differential equation approach, representing the epidemics manner in which the penetration and adoption of the innovation are influenced simultaneously by external and internal sources [32]. Mahajan *et al.* in the articles [11,15,22] discussed all the earlier contributions of the management and marketing literature to the cumulative understanding of the innovation diffusion dynamics.

The models with a time delay have been proposed by many researchers which exhibit the evaluation stage of a product [28,29,30,31,32,33,34]. Wang *et al.* in [33] proposed a nonlinear model with various stages to describe the process of awareness, evaluation and decision-making whereas Shukla *et al.* in [35] proved that diffusion process is affected by variable external influences, the change of density of non-adopters population, emigration or death rate, etc. The analysis proves that the adopter's population density increases as the parameters related to a growth rate of non-adopters population as well as the rate of external influences increase. Kumar *et al.* in [27] observed the effect of evaluation period and proved that it is responsible for Hopf bifurcation in the innovation diffusion system. The studies on the diffusion of innovations are now challenging and important issues from the marketing point of view for various companies. Very few studies have been performed in this direction and to the best of our knowledge and no attention has been paid so far to consider logistically growing non-adopters population together with evaluation period (time delay). The innovation diffusion patterns identified by the theory of stability analysis, sensitivity analysis, and Hopf bifurcation are qualitative. Most frequently, the Hopf bifurcation theory is being used to classify diffusion patterns which are used for design or forecasting. Applying various analytic techniques in the theory of dynamical systems to the real world scenario has been a thoroughly interdisciplinary attempt.

The organization of this paper is as follows: Section 1 deals with the brief introduction. In section 2, the model is formulated with the help of ordinary differential equations. The basic preliminaries including the positivity and boundedness of the system is discussed in section 3. After that in section 4, the existence of various equilibrium points has been examined. Taking evaluation period as the bifurcation parameter, the dynamical behavior of the model equations around the positive equilibrium is discussed in section 5. Sensitivity analysis is performed in section 6. In section 7, some numerical simulations are carried out to support our analytical findings. The section 8 deals with the core results of our mathematical analysis and their significance in the innovation diffusion system.

2. Formulation of the Mathematical Model

We proposed a non-linear dynamical mathematical model consisting of two populations- (i) the logistically growing non-adopter population density $N(t)$ with the evaluation period τ and (ii) the adopter population having density $A(t)$ respectively at any time t. The evaluation period is the period during which non-adopter evaluates the product, and then they finally decide, whether to adopt it or not. In other words, it is the time from the exposure (through external as well as internal influences) of the innovation to the adoption of the same. In the first stage, non-adopter population evaluate the product innovation and in the second stage, they adopt it. This is very much justifiable and realistic in practice because an observation process can never be neglected for the consumers of many products, especially the products with higher values.

Further, we make the following assumptions about the innovation diffusion system.

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- The non-adopter population grows logistically with intrinsic growth rate r_0 and carrying capacity K.
- Assume that δ be the natural death rate of the two populations, γ and ϕ be the intensity of an advertisement of a product and the conversional efficiency of word of mouth of adopters of the product for converting potential adopters (non-adopters) to adopters respectively.
- The variable external (advertisements) as well as internal (world of mouths) influences affect the individuals decision. Therefore, the rate of change of adopters is because of external as well as internal influences, their natural deaths, their rate of discontinuance to use the product.
- Here v is the discontinuance rate of adopters of the product, who may join later on, but α the rate by which the adopters leave the adopters class permanently and would never use it again. It is assumed that the population who leave the adopter class because of their disinterest in the innovation enter only in the non-adopter population.
- It is obvious that non-adopter enter into the adopter class only after going through the evaluation stage (i.e. each non-adopter of the system take average evaluation time τ to evaluate the product) so as to decide whether to adopt it or not, after having exposure through world of mouth and through advertisement at time $t-\tau$.
- The transfer rate from the potential non-adopter class to the adopter class at time t is $(\gamma+\phi A(t-\tau))N(t-\tau)$. In other words, it is the rate of transfer of those who got knowledge about the innovation at time $t-\tau$ due to external as well as internal influences and decided to be the member of adopter class at time t.

In this model, we are taking evaluation period as control parameter so as to see the impact of evaluation period in shaping the dynamics of the non-adopter and adopter population. It is also assumed that all the parameters are positive constants.

In the light of above mentioned factors, the governing eqns. of our proposed model are given by (2.1) and (2.2):

$$\frac{dN(t)}{dt} = r_0\left(N(t)-\frac{N^2(t)}{K}\right)-(\gamma+\phi A(t-\tau))N(t-\tau)+(\alpha+v)A(t)-\delta N(t), \quad (2.1)$$

$$\frac{dA(t)}{dt} = (\gamma + \phi A(t-\tau))N(t-\tau) - (\delta + \alpha + v)A(t). \tag{2.2}$$

System (2.1)-(2.2) will be studied with the initial conditions

$$N(\theta) = \phi_1(\theta), A(\theta) = \phi_2(\theta), \phi_1(0) > 0, \phi_2(0) > 0,$$

where $\theta \in [-\tau, 0]$ and $\phi_1(\theta), \phi_2(\theta) \in C([-\tau, 0], R^2_+)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^2_+, where

$$R^2_+ = \{(x_1, x_2) : x_i > 0, i = 1, 2\}$$

For a nonlinear delay system, there are two types of stability: absolute stability (independent of evaluation period) and conditional stability (depending on the evaluation period). Here, we consider the two cases separately with and without an evaluation period.

For $\tau = 0$, that is, in the absence of evaluation period, system (2.1)-(2.2) takes the following form

$$\frac{dN(t)}{dt} = r_0\left(N(t) - \frac{N^2(t)}{K}\right) - (\gamma + \phi A(t))N(t) + (\alpha + v)A(t) - \delta N(t), \tag{2.3}$$

$$\frac{dA(t)}{dt} = (\gamma + \phi A(t))N(t) - (\delta + \alpha + v)A(t). \tag{2.4}$$

In the next section, we will study basic results including of the positivity and the boundedness of the solutions.

3. Basic Preliminaries

3.1. Positivity

In the present section, we shall develop the positivity conditions for the system (2.1)-(2.2). In the following lemma, we are developing the conditions that the system (2.1)-(2.2) is positive for all times.

Lemma 3.1. *The interior equilibrium of the given system (2.1)-(2.2) is invariant in positive quadrant.*

Proof: To prove that for all $t \in [0, M)$, $(M > 0)$, $N(t) > 0$ and $A(t) > 0$ under the initial conditions $N(0) > 0$, $A(0) > 0$, we suppose otherwise i.e., there exists a $0 < T < M$ such that for all $t \in [0, T)$, $N(t) > 0$ and $A(t) > 0$ and either $N(T) = 0$ or $A(T) = 0$. For any $t \in [-\tau, T)$, integration of system (2.1)-(2.2) yields,

$$N(T) = N(0)exp\int_0^t \left(r_0 - \frac{r_0 N}{K} - \phi A(s-\tau) + (\alpha + v)\frac{A}{N} - \gamma - \delta\right)ds,$$

$$A(T) = A(0)exp\int_0^t \left(\phi N(s-\tau) - \delta - \alpha - v\right)ds.$$

Since $N(t)$ and $A(t)$ are both continuous functions in $[-\tau, T]$, there exists an $S > 0$ such that for all $t \in [\tau, T)$

$$N(T) = N(0)exp\int_0^t \Big(r_0 - \frac{r_0 N}{K} - \phi A(s-\tau) + (\alpha + v)\frac{A}{N} - \gamma - \delta\Big)ds \geq N(0)e^{-TS},$$

$$A(T) = A(0)exp\int_0^t \Big(\phi N(s-\tau) - \delta - \alpha - v\Big)ds \geq A(0)e^{-TS}.$$

Taking $t \to T$, we get $N(T) > 0$ and $A(T) > 0$, a contradiction. Thus $N(t) > 0$, $A(t) > 0$ for any $t \in [0, M)$. □

3.2. Boundedness

Here, we shall prove that the system (2.1)-(2.2) will have all the solutions bounded in some region. For this, we state the following lemma:

Lemma 3.2. *All the solutions of the given system (2.1)-(2.2) which start in R^2_+ are uniformly bounded.*

Proof: Let $(N(t), A(t))$ be any solution of the system with positive initial conditions.

Let us consider a time function Z(t) = N(t) + A(t), then we have

$$\frac{dZ}{dt} = r_0 N(t)\Big(1 - \frac{N(t)}{K}\Big) - \delta A - \delta N,$$

and

$$\frac{dZ}{dt} + \delta Z = r_0 N(t)\Big(1 - \frac{N(t)}{K}\Big).$$

After simplification, we get

$$\frac{dZ}{dt} + \delta Z \leq \frac{r_0 K}{4}, \tag{3.1}$$

for any $\delta > 0$. It is clear that the right hand side of the above expression is bounded. Applying a theorem on differential inequalities [36], we obtain $0 \leq Z(N, A) \leq \frac{W}{\delta} + \frac{Z\big(N(0), A(0)\big)}{e^{\delta t}}$, where $W = \frac{r_0 K}{4}$. Which, upon letting $t \to \infty$, yield $0 \leq Z(N, A) \leq \frac{W}{\delta}$. Therefore, all the solutions of the system (2.1)-(2.2) that starts in R^2_+ enter into the region $U = \{(N(t), A(t)) : 0 \leq Z(t) \leq \frac{W}{\delta} + \epsilon\}$, for any $\epsilon > 0$, that is, all the solutions of the system (2.1)-(2.2) are confined in the region U. □

4. Equilibrium Points

There are three feasible equilibria for the system, (i) $E_0(0,0)$ is the trivial steady state; (ii) $E_1(N,0)$ is the adopter free equilibrium point; (iii) $E_2(N^*,A^*)$ is the positive equilibrium point. At $E_0(0,0)$, the system is asymptotically stable provided $\delta > r_0$, that is, death rate of the population is more than the intrinsic growth rate of the non-adopter population and this condition is obvious.

For $E_1(N,0)$, where $N = \frac{K(r_0-\gamma-\delta)}{r_0}$ exists if $r_0 > \gamma+\delta$ and the necessary and sufficient conditions for adopter free equilibrium point $E_1(N,0)$ to be stable are

$$r_0 > Max\{\alpha+v, 2\gamma+\delta\}. \tag{4.1}$$

From the above expression, we find that the instability condition of trivial equilibrium, that is, $\delta < r_0$ facilitates the existence as well as the stability of adopter free equilibrium $E_1(N,0)$.

Now the equilibrium point $E_2(N^*,A^*)$ is the positive steady state, where $N^* = \frac{(\delta+v+\alpha)A^*}{\gamma+\phi A^*}$ and A^* are the roots of the equation

$$K\phi^2\delta A^2 + (r_0\mu^2 - K\phi\mu(r_0-\delta) + 2K\gamma\phi\delta)A + [K\gamma^2\delta - K\gamma\mu(r_0-\delta)] = 0. \tag{4.2}$$

Here $A^* = \frac{-\beta\pm\sqrt{\beta^2+K_1}}{2\gamma_1}$ is positive provided $r_0 > \frac{(\mu+\gamma)\delta}{\mu}$;

where $\beta = r_0\mu^2 - K\phi\mu(r_0-\delta) + 2K\gamma\delta\phi$, $K_1 = 4K^2\phi^2\delta\gamma(\mu r_0 - \mu\delta - \gamma\delta)$, $\gamma_1 = K\delta\phi^2$ and $\delta+v+\alpha = \mu$.

The stability and the Hopf-bifurcation analysis of the delayed innovation diffusion model about the positive equilibrium point $E_2(N^*,A^*)$ will be discussed in the next section.

5. Dynamical Behavior of the System

The characteristic equation obtained from the variational matrix of the delayed innovation diffusion model system (2.1)-(2.2) takes the form

$$\Delta(\lambda,\tau) = (\lambda^2 + A\lambda + B) + (C\lambda + D)e^{-\lambda\tau} = 0, \tag{5.1}$$

where

$$\begin{cases} A = \{2\delta+v+\alpha\} - r_0\{1-\frac{2N^*}{K}\}, \\ B = \{\delta - r_0(1-\frac{2N^*}{K})\}\{\mu\}, \\ C = \gamma+\phi A^* - \phi N^*, \\ D = \phi N^*\{r_0(1-\frac{2N^*}{K}) - \delta\} + \{\gamma+\phi A^*\}\{\delta+v+\alpha\}. \end{cases}$$

In the absence of evaluation period ($\tau = 0$), the transcendental equation (5.1) becomes

$$\lambda^2 + (A+C)\lambda + (B+D) = 0. \tag{5.2}$$

Therefore, all the roots of the characteristic equation will have negative real parts if

$H_1 : A + C > 0$ and $H_2 : B + D > 0$

Therefore, the positive equilibrium point $E_2(N^*, A^*)$ is locally asymptotically stable if H_1 and H_2 hold good.

Lemma 5.1. *The necessary and sufficient conditions for the stable equilibrium $E_2(N^*, A^*)$ without any evaluation period are*

$$\delta > r_0(1 - \frac{2N^*}{K})$$

and

$$\mu > \phi N^*.$$

we shall now investigate the dynamics of the delayed system, that is, we want to determine if the real part of some root of Eqn.(5.1) increases to reach zero and eventually becomes positive as τ varies. This will prove that the evaluation period results in Hopf-bifurcation.

Theorem 5.2. *[37] (i) The positive equilibrium $E_2(N^*, A^*)$ of the system (2.1)-(2.2) is absolutely stable if and only if the equilibrium $E_2(N^*, A^*)$ of the corresponding ODE system is asymptotically stable and the characteristic Eqn.(5.1) has no purely imaginary roots for any $\tau > 0$. (ii) The positive equilibrium $E_2(N^*, A^*)$, of the system (2.1)-(2.2) is conditionally stable if and only if all the roots of the characteristic Eqn.(5.1) have negative real parts at $\tau = 0$ and there exists some positive value of τ such that the characteristic Eqn.(5.1) has a pair of purely imaginary roots $\pm i\omega$.*

Theorem 5.3. *The necessary and sufficient conditions for $E_2(N^*, A^*)$ to be asymptotically stable in the presence of an evaluation period are*

1. *the real parts of all the roots of $\Delta(\lambda, \tau) = 0$ are negative,*
2. *for all real ω and for $\tau > 0$, $\Delta(\lambda, \tau) \neq 0$.*

Proof: Assume that for some $\tau > 0$, $\lambda = i\omega$ ($\omega > 0$ and $i = \sqrt{-1}$) is a root of characteristic equation (5.1), where ω is a positive real number. If we substitute $\lambda = i\omega$ into (5.1), then we have

$$RealPart : Dcos\omega\tau + C\omega sin\omega\tau = \omega^2 - B, \tag{5.3}$$

$$ImaginaryPart : C\omega cos\omega\tau - Dsin\omega\tau = -A\omega. \tag{5.4}$$

Squaring and adding the real and imaginary parts from equations (5.3) and (5.4), we have the following fourth order equation in ω:

$$\omega^4 - (C^2 - A^2 + 2B)\omega^2 + (B^2 - D^2) = 0. \tag{5.5}$$

From above equation if $H_3 : A^2 - C^2 - 2B > 0$ and $B^2 - D^2 > 0$. Now $A^2 - C^2 - 2B > 0$ if $\mu + \phi N^* > \gamma + \phi A^*$ and $\mu + \gamma + \phi A^* > \phi N^*$ or if $\mu + \phi N^* < \gamma + \phi A^*$ and $\mu + \gamma + \phi A^* < \phi N^*$. The sufficient conditions for $B^2 - D^2 > 0$ are $\delta > r_0(1 - \frac{2N^*}{K})$ and $\mu > \frac{\phi N^* \delta}{2(\gamma + \phi A^*)}$. If $B^2 - D^2 > 0$, then the equation (5.5) does not have positive roots. Therefore characteristic equation (5.1) does not have purely imaginary roots. Since H_1 and H_2 ensure that all roots of (5.2) have negative real parts. By Rouche's Theorem, it follows that all roots of (5.5) will have negative real parts too. □

Further, if $H_4 : B^2 - D^2 < 0$, then from Routh-Hurwitz criterion, Eqn.(5.5) has a unique positive root ω_0^2. Under this condition, the characteristic equation (5.1) will have a pair of purely imaginary roots of the form $\pm i\omega_0$. Put ω_0^2 in (5.3) and (5.4) and solving for τ, we shall have

$$\tau_n^* = \frac{1}{\omega_0} Cos^{-1} \frac{(\omega_0^2 - B)D - AC\omega_0^2}{C^2\omega_0^2 + D^2} + \frac{2n\pi}{\omega_0}; n = 0, 1, 2, 3.... \tag{5.6}$$

Theorem 5.4. *[38] (a) If $H_1 - H_3$ hold, then all roots of Eqn.(5.1) have negative real parts for $\tau \geq 0$. (b) If H_1, H_2 and H_4 hold, then the equilibrium point $E_2(N^*, A^*)$ is asymptotically stable for $\tau < \tau_0^*$ and unstable for $\tau > \tau_0^*$ and as τ increases through τ_0^* , $(E_2(N^*, A^*)$ bifurcates into small periodic solutions, where $\tau_0^* = \tau_n^*$ for $n = 0$ is given by Eqn.(5.6).*

For $\tau = 0$, $E_2(N^*, A^*)$ is asymptotically stable if H_1 and H_2 holds. Hence, by Butler's Lemma, $E_2(N^*, A^*)$ remains stable for $\tau < \tau_0$, where $\tau_0 = \tau_n^*$ for $n = 0$.

Let us now investigate whether model system undergoes a Hopf-bifurcation phenomenon at $E_2(N^*, A^*)$ when τ increases through τ_0^*. We now investigate how the real part of characteristic Eqn.(5.1) varies as τ increases in a small neighborhood of τ_0^*. For this purpose, let us now compute the transversality condition for hopf-bifurcation, and we turn to showing that $\{\frac{d}{d\tau}(Re\lambda)\}_{\tau=\tau_0^*,\omega=\omega_0} > 0$. This will signify that there exists at least one eigenvalue with positive real part for $\tau > \tau_0^*$. Moreover, the conditions of Hopf-bifurcation are then satisfied yielding the required periodic solution [7,28]. So, differentiate the transcendental Eqn.(5.1) w.r.t. τ, we have

$$\begin{aligned} sign\left\{Re\frac{d\lambda}{d\tau}\right\}_{\tau=\tau_0{}^*,\lambda=i\omega_0} &= sign\left\{Re\left\{\frac{d\lambda}{d\tau}\right\}^{-1}\right\}_{\lambda=i\omega_0} \\ &= sign\left\{\frac{\sqrt{(C^2 - A^2 + 2B)^2 - 4(B^2 - D^2)}}{((-\omega_0^2 + B)^2 + A^2\omega_0^2)(D^2 + C^2\omega_0^2)}\right\}. \end{aligned} \tag{5.7}$$

Therefore, by virtue of H_4, $\left\{\frac{d}{d\tau}(Re\lambda)\right\}_{\tau=\tau_0^*,\omega=\omega_0} > 0$. Thus, the transversality condition is true if H_4 hold good, and hence, Hopf-bifurcation occurs at $\omega = \omega_0$, and at the critical value of evaluation period $\tau = \tau_0{}^*$.

Remark 5.5. *[39] As τ passes through the threshold value τ_0^*, the equilibrium E_2 losses its stability and Hopf bifurcation occurs with emergence of a small amplitude periodic oscillations.*

6. Sensitivity Analysis

In this section, normalized sensitivity analysis of state variables of the system without any evaluation period is done at non-zero equilibrium point with respect to model parameters and is shown in Table 1.

Definition 6.1. *[40,41] The normalized forward sensitivity index of a variable, u, that depends on a parameter, p, is defined as:*

$$\Upsilon_p^u = \frac{\partial u}{\partial p} \times \frac{p}{u}$$

Estimation of highly sensitive parameter should be done very carefully, because a small variation in the parameter will lead to relatively large quantitative change. On the other hand, a less sensitive parameter does not require as much effort to estimate, since a small variation in that parameter will not produce large change to the quantity of interest. Sensitivity indices can be positive or negative which indicates that the nature of the relationship, and it is the magnitude that ranks the strength of the relationship as compared to the other parameters. Since we don't have an explicit formula for the quantity we are interested in (positive steady state), we estimate $\frac{\partial u}{\partial p}$ using the central difference approximation:

$$\frac{\partial u}{\partial p} = \frac{u(p + \Delta p) - u(p - \Delta p)}{2\Delta p} + O(\Delta p^2).$$

We perform sensitivity analysis of state variables at positive steady state E_2 with respect to model parameters. Sensitivity indices of state variables at positive equilibrium are shown in Table.(1).

Table 1: The sensitivity indices $\Upsilon_{y_j}^{x_i} = \frac{\partial x_i}{\partial y_j} \times \frac{y_j}{x_i}$ of the state variables of the system (2.1)-(2.2) to the parameters y_j for the parameter values.

Parameter (y_j)	Values	N^*	A^*
r_0	0.5501	1.19991	2.79304
K	10	0.16401	0.381769
ϕ	0.11	-0.83599	-0.618231
γ	0.11	-0.629645	-0.465635
α	0.0016	0.0101122	0.00747816
δ	0.23	0.253719	-1.71806
v	0.0003	0.00189604	0.00140215

Here, we see that r_0, K, α, v have a positive impact on the N^* and A^* (the value of N^* and A^* increase with a positive unit change in the value of r_0, K, α, v) and the rest of the parameters have a negative impact.

Moreover, the intrinsic growth rate (r_0) is the most sensitive parameter for the non-adopter and adopter population and it has a big role to play in shaping the dynamics of these populations.

7. Numerical Simulations

In this section, we present numerical results of the system (2.1)-(2.2). For this purpose, we consider a set of parametric values as $K = 10$, $r_0 = 0.5501$, $\phi = 0.11$, $\gamma = 0.11$, $\alpha = 0.0016$, $\delta = 0.23$, $v = 0.0003$. The system (2.1)-(2.2) transformed into the following form:

$$\begin{cases} \frac{dN(t)}{dt} = 0.5501\left(N(t) - \frac{N^2(t)}{10}\right) - (0.11 + 0.11A(t-\tau))N(t-\tau) \\ \qquad\quad +(0.0016 + 0.0003)A(t) - 0.23N(t), \\ \frac{dA(t)}{dt} = (0.11 + 0.11A(t-\tau))N(t-\tau)) - (0.23 + 0.0016 + 0.0003)A(t). \end{cases} \tag{7.1}$$

In the absence of evaluation period i.e. $\tau = 0$, together with the above set of parametric values, the system converges to asymptotically stable interior equilibrium point $E_2(1.202, 1.328)$ with initial values as $N(t) = 0.1$ and $A(t) = 0.1$, as shown in Fig. 1. By using DDE23 pack of Matlab, the system is integrated with evaluation period τ and attains stability for $\tau = 1.351$. Fig. 2 shows stable dynamic of innovation diffusion system around E_2 for $\tau = 1.351$. But if we gradually increase the value of τ by keeping other parameters fixed, it is seen that the system possesses small periodic orbits and a Hopf bifurcation incurred into the system. Numerically, by using first condition of H_3, we have $A^2 - C^2 - 2B = 0.0737 > 0$ and $H_4 = B^2 - D^2 = -0.0052 < 0$ are satisfied, that is, by Routh-Hurwitz criterion, there exists a unique positive root of Eqn.(5.5), and a purely imaginary root $i\omega_0$ with $\omega_0 = 0.2100$ from equation (5.5) is calculated. By using this value in (5.6), we have found the threshold value of evaluation period $\tau = \tau_0^*$ for the model system (7.1) such that $E_2(N^*, A^*)$ loses its stability as τ passes through τ_0^* (taking other parameters fixed) and this threshold value is $\tau_0^* \simeq 1.8067$. Moreover at $\tau = \tau_0^*$, the transversality condition $\left\{\frac{d}{d\tau}(Re\lambda)\right\}_{\tau=\tau_0^*,\omega=\omega_0} = 2.7047e + 03 > 0$ for the existence of Hopf-bifurcation is also satisfied. This shows that the positive equilibrium $E_2(N^*, A^*)$ remains stable for $0 \leq \tau < 1.8067$ and becomes unstable for $\tau \geq 1.8067$. A Hopf-bifurcation in the form of a limit cycle is shown in Fig. 3 for $\tau_0^* = 1.8067$. A more stable limit cycle for both Non-adopter and adopter classes is also shown at $\tau = 1.9467$ (Fig. 4). It indicates that there is a threshold limit of evaluation period below which the system attains stability and beyond it, system becomes unstable. Moreover, if we gradually increase the carrying capacity K of the dynamic system, we see that the evaluation period always decreases (Fig. 5-Fig. 7). Using $K = 100, 1000$ and 10000, we have been able to calculate the evaluation period $\tau = 0.7448$, 0.6578 and 0.6477 respectively from Eqn.(5.6), which gives purely imaginary eigenvalues $i\omega_0$ (the system sets into oscillations) and it is shown in Table.2. The possible justification lies in the fact that more the number of non-adopters (more the value of K), more will be the number of interactions between adopters and non-adopters and hence lesser will be the evaluation period for attaining Hopf-bifurcation. Thus, it can be concluded that the system around the positive equilibrium E_2 enters into a Hopf bifurcation and exhibits the periodic oscillations for a certain period of evaluation.

Table 2: Impact of Carrying Capacity K on Evaluation Period τ

K	Eigenvalues	Evaluation Period
100	$\pm$ 0.2305, $\pm$ 0.4159i	0.7448
1000	$\pm$ 0.2323, $\pm$ 0.4249i	0.6578
10000	$\pm$ 0.2325, $\pm$ 0.4259i	0.6477

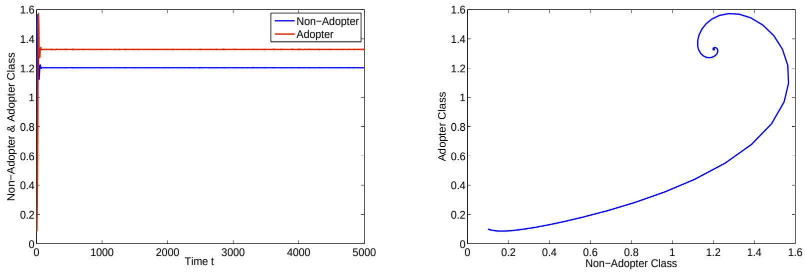

Figure 1: Solution trajectories predicting the local stability for Non-Adopter and Adopter Class without any evaluation period.

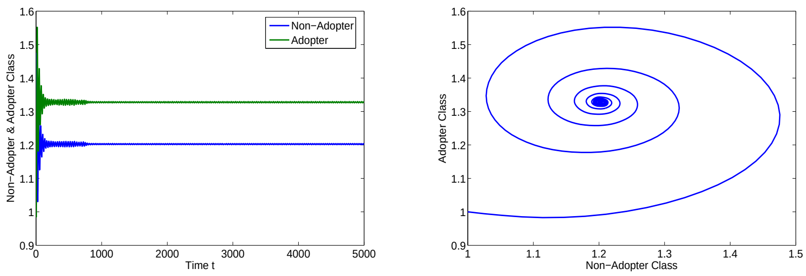

Figure 2: Convergence of solution trajectories to E_2, of system (15)-(16) at $\tau = 1.351 < \tau_0^*$.

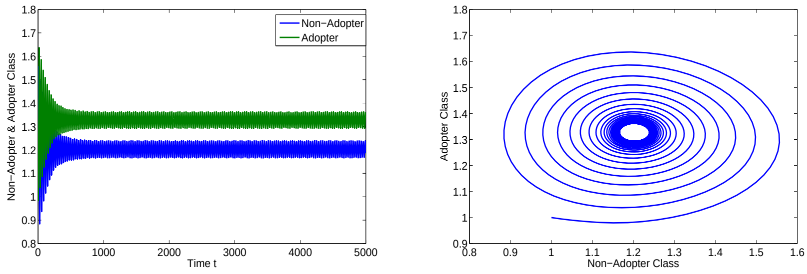

Figure 3: Hopf-Bifurcation of given system around E_2 for Non-Adopter and Adopter Class at threshold value $\tau_0^* = 1.8067$ with increasing time.

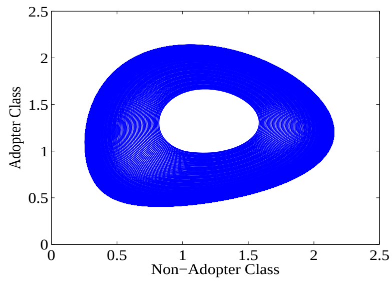

Figure 4: A more stable limit cycle is shown at $\tau = 1.9467$ with increasing time.

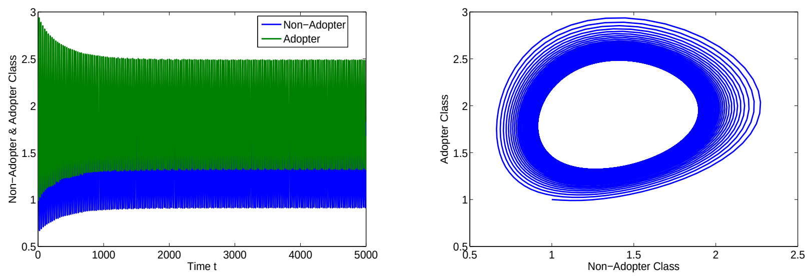

Figure 5: Oscillatory character for Non-Adopter and Adopter Class at $K = 100$, $\tau = 0.7448$.

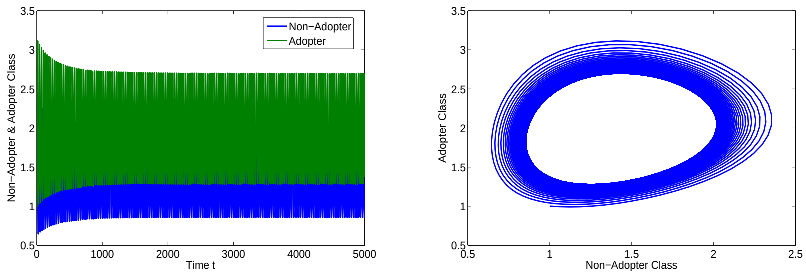

Figure 6: Time series and Phase plane gives periodic solutions for Non-Adopter and Adopter Class at $K = 1000$, $\tau = 0.6578$.

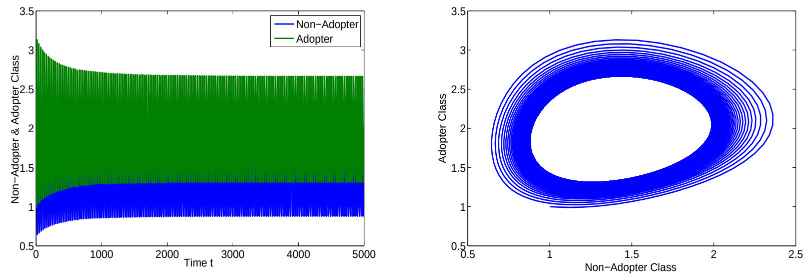

Figure 7: Time series and Phase plane shows periodic solutions for Non-Adopter and Adopter Class at $K = 10000$, $\tau = 0.6477$.

7.1. Impact of external influences

For a given value of external influences (advertisements), the innovation diffusion system (7.1) produces bifurcating periodic solutions. As and when the magnitude of external influences increases (γ changes from 0.11 to 0.18 and keeping other parameters unchanged), the system enters into stable equilibrium state. This means that the zero growth of adopter and non adopter populations would happen in the long run. In other words, the system enters into maturity stage. We observe that the given innovation diffusion system undergoes Hopf bifurcation in Fig. 3-Fig. 4 for the coefficient of cumulative density of variable external influences $\gamma = 0.11$. But in Fig. 8, the system converges to equilibrium position $E_2(0.7567, 0.9162)$ for $\gamma = 0.18$, which shows that after an increase in the magnitude of variable external influences (advertisements), the final level of adoption of the innovation can be achieved.

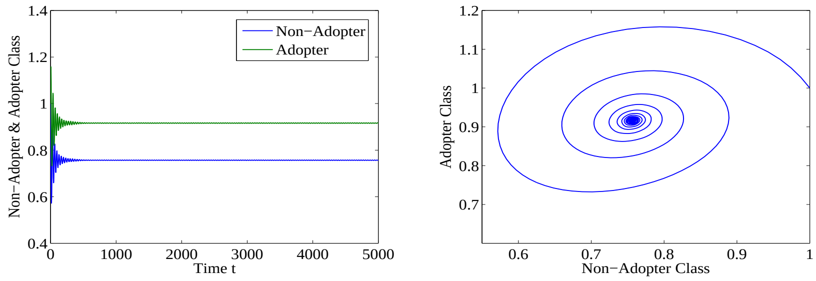

Figure 8: Convergence of periodic solutions to the equilibrium point $E_2(0.7567, 0.9162)$ for coefficient of external influences $\gamma = 0.18$ and $\tau = 1.8067$.

8. Results

In this paper, we have investigated the innovation diffusion system with an assumption that the non-adopter population is growing logistically with intrinsic growth rate, r_0 and the evaluation period, τ. As the factor of an evaluation period is always present in the innovation diffusion process. We have incorporated τ in the innovation diffusion model and nonlinear model has been analyzed qualitatively and quantitatively. The normalized sensitivity analysis of the system showed that the intrinsic growth rate of the non-adopter population, r_0, has a big role to play in shaping the dynamics of the innovation diffusion system. Our model system (2.3)-(2.4), which is without any time-delay incorporates the knowledge and the persuasion stage through justifiable parametric values of internal (ϕ) and external (γ) influences and we have been able to establish the conditions in parametric form for the stability of adopter and non-adopter population.

Oscillatory behavior in the product innovation diffusion process is quite common. In this paper, our aim was to study the effect of evaluation period(time delay) in the innovation diffusion model. So, it was assumed that shifting of non-adopter population to adopter population is not instantaneous rather it takes some time, that is, evaluation period τ. In the present paper, it is observed that the system was establishing local asymptotic stability in the absence of evaluation period. However, incorporation of evaluation period in the innovation diffusion model drives the otherwise stable system into Hopf bifurcation and periodic solutions occur around the non-zero equilibrium point. Moreover, as and when the value of evaluation period crossed over the threshold value $\tau = \tau_0^* = 1.8067$, the system showed the excitable change in the form of limit cycle (Fig. 3). Thus, we have found that evaluation period has a vital role to play in establishing the periodic orbits in the innovation diffusion system.

In Fig. 5-Fig. 7, it has been observed that the threshold value of evaluation period, for Hopf-bifurcation to take place, decreases with the increase in carrying capacity of non-adopter population. This means that if the adopters are allowed to spread word of mouths by means of interactions with non-adopters, there is a high possibility of a positive outbreak in the non-adopter population to become the members of adopter population. In other words, if the number of potential consumers are more in the system, the more will be the number of adopters, consequently they will take less average time for shifting over to the adopter class (Table.2).

Finally, we conclude that the incorporation of evaluation period together with logistically growing non-adopter population has proved to be useful in capturing the realistic scenarios of the real world applications. Here, we have been able to prove that some justifiable value of evaluation period is responsible for changing the stable innovation diffusion system to system with periodic cycles, that is, the adoption process starts and enter into the bifurcating periodic solutions. Also, we have proved that the the system enters into maturity stage by an increase in the magnitude of external influences.

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