Bol. Soc. Paran. Mat. (3s.) **v. 38** 5 (2020): 19–31.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v38i5.37053

New Notions in Ideal Topological Space

C. Santhini and M. Suganya

abstract: In this paper, we apply the notion of $I_g^{**\alpha}$- closed sets to present and study a new class of locally closed sets called $I_g^{**\alpha}$-locally closed sets in ideal topological spaces along with their several characterizations and mutual relationships between the new notion and other locally closed sets. Further we introduce $I_g^{**\alpha}$-submaximal spaces and some properties of such notion are investigated.

Key Words: $I_g^{**\alpha}$-closed sets, $I_g^{**\alpha}$-open sets, $I_g^{**\alpha}$-locally closed sets, $I_g^{**\alpha}$-submaximal spaces.

Contents

1. Introduction

In topological spaces locally closed sets were studied by Bourbaki [4]. Kuratowski [13], introduced the local function in ideal spaces. Balachandran, Sundaram and Maki [2], introduced and investigated the concept of generalized locally closed sets. In 1999, Dontchev [5], introduced I-locally closed subsets in an ideal topological spaces. Navaneethakrishnan and Sivaraj [17], introduced the concept of I_g-locally $*$-closed sets in ideal topological spaces in 2009. Recently Santhini et al [20] introduced $I_g^{**\alpha}$-closed sets by using α-local functions. In this paper, we define and study a new class of generalized locally closed sets called $I_g^{**\alpha}$-locally closed sets in ideal topological spaces. Also several properties of $I_g^{**\alpha}$-locally closed sets are discussed. Submaximality plays a very significant role in topology. The concept of submaximality of general topological spaces was introduced by Hewitt [9] in 1943. In 2005, Acikgoz et al [1], introduced and studied the properties of

2010 *Mathematics Subject Classification:* 54A05,54A10.
Submitted May 08, 2017. Published January 17, 2018

I-submaximal space in ideal topological spaces. Bhavani and Sivaraj [3] introduced I_g-submaximal spaces and studied their characterizations and properties in 2015. In this paper, we introduce $I_g^{**\alpha}$-submaximal space via $I_g^{**\alpha}$-closed sets and some properties of such spaces are investigated.

2. Preliminaries

First we recall some basic definitions and some properties which will be useful in the sequel.
A collection $I \subseteq P(X)$ is called an ideal on X if it satisfies the following two conditions: (a): $A \in I$ and $B \subseteq A \Rightarrow B \in I$. (b): $A \in I$ and $B \in I \Rightarrow A \cup B \in I$. An ideal topological space is a topological space (X, τ) with an ideal I on X and it is denoted by (X, τ, I). For $A \subseteq X$, $A^*(I, \tau) = \{x \in X : U \cap A \notin I$ for every $U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$ is called the local function [13] of A with respect to τ and I. We simply write A^* instead of $A^*(I, \tau)$. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(I, \tau)$ called the $*$-topology finer than τ is defined by $cl^*(A)$=$A \cup A^*$. A subset A of an ideal space (X, τ, I) is $*$-closed [10] ($*$-perfect) if $A^* \subseteq A (A = A^*)$. By a space, we always mean a topological space (X, τ, I) with no separation properties assumed. A subset A of a topological space (X, τ) is α-open [19] if $A \subseteq int(cl(int(A)))$. Let A be a subset of an ideal topological space (X, τ, I). Then $A^{*\alpha} = \{x \in X : U \cap A \notin I$ for every $U \in \tau^{\alpha}(x)\}$ is called α-local function of A [11] with respect to I and τ^{α}. We simply write $A^{*\alpha}$ instead of $A^{*\alpha}(I, \tau)$ in case there is no ambiguity. A Kuratowski α-closure operator $cl^{*\alpha}(.)$ for a topology $\tau^{*\alpha}(I, \tau)$ called the $*^{\alpha}$ - topology, finer than τ^* and τ is defined by $cl^{*\alpha}(A) = A \cup A^{*\alpha}$. In an ideal topological space (X, τ, I) if $A \subseteq X$, $cl^*(A)$, $int^*(A)$ will denote the closure and interior of A in (X, τ^*) and $cl^{*\alpha}(A)$, $int^{*\alpha}(A)$ will denote the closure and interior of A in $(X, \tau^{*\alpha})$.

Definition 2.1. *A subset A of an ideal topological space (X, τ, I) is said to be*

(i) $*^{\alpha}$-closed [15] if $A^{*\alpha} \subseteq A$.

(ii) $*$-dense [6] if $cl^*(A) = X$.

(iii) $*^{\alpha}$-dense [18] if $cl^{*\alpha}(A) = X$.

(iv) $*$-codense [7] if $X - A$ is $*$-dense.

Definition 2.2. *[14] A subset A of a topological space (X, τ) is said to be g-closed, if $cl(A) \subseteq U$ whenever $A \subseteq U$ where U is open. The complement of g-closed is called g-open.*

Definition 2.3. *A subset A of an ideal topological space (X, τ, I) is said to be an*

(i) I_g-closed [6] if $A^* \subseteq U$ whenever $A \subseteq U$, where U is open in X.

(ii) $I_g^{**\alpha}$-closed [20] if $A^{*\alpha} \subseteq U$ whenever $A \subseteq U$, where U is $*$-open in X.

The complement of I_g-closed set and $I_g^{**\alpha}$-closed set are called I_g-open set and $I_g^{**\alpha}$-open set. Also the class of $I_g^{**\alpha}$-open sets and $I_g^{**\alpha}$-closed sets are denoted by $I_g^{**\alpha}$-$O(X)$ and $I_g^{**\alpha}$-C(X) respectively.

Definition 2.4. *[20] A subset A of an ideal topological space (X, τ, I) is said to be a $*^{\alpha}$-perfect if $A = A^{*\alpha}$.*

Definition 2.5. *[20] A subset A of an ideal topological space (X, τ, I) is said to be an A_I^{α}-set if $A = U \cap V$ where U is open and V is $*^{\alpha}$- perfect.*

Definition 2.6. *A subset A of (X, τ) is said to be*

(i) locally closed [4] if $A = U \cap V$ where U is open and V is closed.

(ii) generalized locally closed set (briefly GLC-set) [12] if $A = U \cap V$ where U is g-open and V is g-closed.

(iii) GLC-set [12] if $A = U \cap V$ where U is g-open and V is closed.*

*(iii) GLC**-set [12] if $A = U \cap V$ where U is open and V is g-closed.*

Definition 2.7. *A subset A of (X, τ, I) is said to be*

(i) I-locally closed [5] if $A = U \cap V$ where U is open and V is $*$-perfect.

(ii) I-locally $*$-closed [17] if $A = U \cap V$ where U is open and V is $*$-closed.

(iii) I_g-locally $*$-closed [17] if $A = U \cap V$ where U is I_g-open and V is $*$-closed.

(iv) $I^{*\alpha}$-locally closed [16] if $A = U \cap V$ where U is open and V is $*^{\alpha}$-closed.

Remark 2.8.

(i) Every I-locally closed set is I-locally $*$-closed.

(ii) Every I-locally closed set is $I^{*\alpha}$-locally closed.

Definition 2.9. *A topological space (X, τ) is said to be a*

(i) submaximal space [9] if every dense subset of X is open.

(ii) g-submaximal space [2] if every dense subset of X is g-open.

Definition 2.10. *An ideal topological space (X, τ, I) is said to be an*

(i) I-submaximal space [1] if every $*$-dense subset of X is open.

(ii) I_g-submaximal space [3] if every $*$-dense subset of X is I_g-open.

Definition 2.11. *[6] A space X is said to be a T_I-space, if every I_g-closed set is $*$- closed in X.*

Definition 2.12. *[8] Nonempty subsets A, B of an ideal space (X, τ, I) are called $*$-separated if $Cl^*(A) \cap B = A \cap Cl(B) = \phi$.*

3. $I_g^{**\alpha}$- Locally closed sets

In this section, a new class of generalized locally closed sets namely $I_g^{**\alpha}$-locally closed sets in an ideal topological space is defined and several characterizations of this notion are derived.

Definition 3.1. *A subset A of an ideal topological space (X, τ, I) is said to be $I_g^{**\alpha}$-locally closed if there exist an $I_g^{**\alpha}$-open set U and a $*^\alpha$-closed set V such that $A = U \cap V$.*

The family of all $I_g^{**\alpha}$- locally closed sets is denoted by $I_g^{**\alpha}$- LC(X).

Example 3.2. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Then $\{a, b, d\}$ is an $I_g^{**\alpha}$-locally closed set.*

Remark 3.3.

(i) *Every $*^\alpha$-closed set is $I_g^{**\alpha}$-locally closed.*

(ii) *Every $I_g^{**\alpha}$-open set is $I_g^{**\alpha}$-locally closed.*

Remark 3.4. *The converses of the above remark are not true as seen from the following example.*

Example 3.5. *Let X= $\{$a,b,c$\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\}$ and I=$\{\phi\}$. Then*

(i) *$\{$a,b$\}$ is $I_g^{**\alpha}$-locally closed but not $*^\alpha$-closed.*

(ii) *$\{$b,c$\}$ is $I_g^{**\alpha}$-locally closed but not $I_g^{**\alpha}$-open.*

Theorem 3.6. *Every locally closed set is $I_g^{**\alpha}$- locally closed.*

Proof: *Let A be a locally closed set such that $A = U \cap V$ where U is an open set and V is a closed set. By theorem 4.14 [20], U is an $I_g^{**\alpha}$-open set and since every closed is $*^\alpha$-closed, V is a $*^\alpha$-closed set, then A is $I_g^{**\alpha}$- locally closed.* □

Remark 3.7. *The converse of the above theorem is not true as seen from the following example.*

Example 3.8. *Let $X = \{a, b, c\}$,$\tau = \{\phi, X, \{c\}\}$ and I=$\{\phi\}$. Then $\{b\}$ is an $I_g^{**\alpha}$ - locally closed set but not a locally closed set.*

Theorem 3.9. *Every I-locally closed set is $I_g^{**\alpha}$-locally closed.*

Proof: *Let A be an I-locally closed set such that $A = U \cap V$ where U is an open set and V is $*$-perfect. By theorem 4.14 [20], U is an $I_g^{**\alpha}$-open set. Since every $*$-perfect is $*$-closed, V is $*$-closed and hence V is a $*^\alpha$-closed set. Consequently, A is $I_g^{**\alpha}$-locally closed.* □

Remark 3.10. *The converse of the above theorem is not true as seen from the following example.*

Example 3.11. *Let* $X = \{a, b, c\}$,$\tau = \{\phi, X, \{a\}, \{b, c\}\}$ *and* I=$\{\phi, \{a\}\}$. *Then* $\{b\}$ *is an* $I_g^{**\alpha}$*-locally closed set but not an I-locally closed set.*

Theorem 3.12. *Every I-locally* $*$*-closed set is* $I_g^{**\alpha}$*-locally closed.*

Proof: *Let* A *be an* I*-locally* $*$*-closed set such that* $A = U \cap V$ *where* U *is an open set and* V *is a* $*$*-closed set. By theorem 4.14* [20]*,* U *is an* $I_g^{**\alpha}$*-open set. Since every* $*$*-closed is* $*^{\alpha}$*-closed,* V *is a* $*^{\alpha}$*-closed set. Hence* A *is* $I_g^{**\alpha}$*-locally closed.* □

Remark 3.13. *The converse of the above theorem is not true as seen from the following example.*

Example 3.14. *Let* $X = \{a, b, c\}$*,* $\tau = \{\phi, X, \{a\}, \{a, c\}\}$ *and* I=$\{\phi\}$. *Then* $\{a, b\}$ *is an* $I_g^{**\alpha}$*-locally closed set but not an I-locally* $*$*-closed set.*

Theorem 3.15. *Every* $I^{*\alpha}$*-locally closed set is* $I_g^{**\alpha}$*-locally closed.*

Proof: *Let* A *be an* $I^{*\alpha}$*-locally closed set such that* $A = U \cap V$ *where* U *is an open set and* V *is a* $*^{\alpha}$*-closed set. By theorem 4.14* [20]*,* U *is an* $I_g^{**\alpha}$*-open set. Hence* A *is* $I_g^{**\alpha}$*-locally closed.* □

Remark 3.16. *The converse of the above theorem is not true as seen from the following example.*

Example 3.17. *Let* X= $\{a,b,c,d\}$*,* $\tau = \{\phi, X, \{a\}, \{b, c, d\}\}$ *and* I=$\{\phi\}$. *Then* $\{a, b, c\}$ *is an* $I_g^{**\alpha}$*-locally closed set but not an* $I^{*\alpha}$*-locally closed set.*

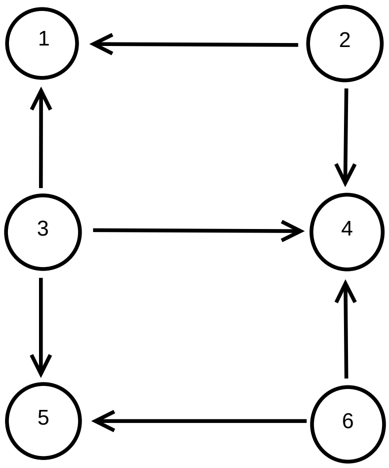
Theorem 3.18. *Every* A_I^{α}*-set is* $I_g^{**\alpha}$*- locally closed.*

Proof: *Let* A *be an* A_I^{α}*-set such that* $A = U \cap V$ *where* U *is an open set and* V *is* $*^{\alpha}$*-perfect. By theorem 4.14* [20]*,* U *is an* $I_g^{**\alpha}$*-open set. Since every* $*^{\alpha}$*-perfect is* $*^{\alpha}$*-closed,* V *is* $*^{\alpha}$*-closed. Hence* A *is* $I_g^{**\alpha}$*-locally closed.* □

Remark 3.19. *The converse of the above theorem is not true as seen from the following example.*

Example 3.20. *Let* $X = \{a, b, c, d\}$*,* $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ *and* I=$\{\phi\}$. *Then* $\{a,b,d\}$ *is an* $I_g^{**\alpha}$*-locally closed set but not an* A_I^{α}*-set.*

Remark 3.21. *From the above relations between several locally closed sets and* $I_g^{**\alpha}$*- locally closed set we have the following diagram where (1)* $\rightarrow$ *(2) represents (1) implies (2) but not conversely.*



(1): I_g-locally $*$-closed (2): I-locally closed (3): I-locally $*$- closed

(4): $I^{*\alpha}$-locally closed (5): $I_g^{**\alpha}$-locally closed (6): A_I^{α}-set

4. Characterizations of $I_g^{**\alpha}$- Locally closed sets

In this section, several characterizations of $I_g^{**\alpha}$-locally closed sets are derived.

Theorem 4.1. *For a subset A of (X, τ, I) the following are equivalent.*

(i) A is $I_g^{**\alpha}$-locally closed.

(ii) $A = U \cap cl^{*\alpha}(A)$ for some $I_g^{**\alpha}$-open set U.

(iii) $cl^{*\alpha}(A) - A$ is $I_g^{**\alpha}$-closed.

(iv) $A \cup (X - cl^{*\alpha}(A))$ is $I_g^{**\alpha}$-open.

Proof:

(i)⇒(ii): Let A be an $I_g^{**\alpha}$-locally closed set such that $A = U \cap V$ where U is $I_g^{**\alpha}$-open and V is $*^{\alpha}$- closed. Clearly, $A \subseteq U \cap cl^{*\alpha}(A)$. Also $A \subseteq V$. By [11] $cl^{*\alpha}(A) \subseteq cl^{*\alpha}(V)$. Since V is $*^{\alpha}$-closed, $cl^{*\alpha}(A) \subseteq cl^{*\alpha}(V) = V$ which implies that $U \cap cl^{*\alpha}(A) \subseteq U \cap V = A$. Hence $A = U \cap cl^{*\alpha}(A)$ for some $I_g^{**\alpha}$-open set U.

(ii) ⇒ (i): Let $A = U \cap cl^{*\alpha}(A)$ for some $I_g^{**\alpha}$-open set U. Since U is $I_g^{**\alpha}$-open and $cl^{*\alpha}(A)$ is $*^{\alpha}$- closed, A is $I_g^{**\alpha}$-locally closed.

(iii) ⇒ (iv): Let $U = cl^{*\alpha}(A) - A$. By (iii), U is $I_g^{**\alpha}$-closed and so $X - U$ is $I_g^{**\alpha}$-open. Therefore $X - U = X - (cl^{*\alpha}(A) - A) = A \cup (X - cl^{*\alpha}(A))$ which is $I_g^{**\alpha}$-open.

(iv) ⇒ (iii): Let $U = A \cup (X - cl^{*\alpha}(A))$. *By (iv), U is* $I_g^{**\alpha}$*-open and so* $X - U$ *is* $I_g^{**\alpha}$*-closed. Therefore* $X - U = X - (A \cup (X - cl^{*\alpha}(A))) = cl^{*\alpha}(A) - A$. *Hence* $cl^{*\alpha}(A) - A$ *is* $I_g^{**\alpha}$*-closed.*

(iv) ⇒ (ii): Let $U = A \cup (X - cl^{*\alpha}(A))$. *By (iv), U is* $I_g^{**\alpha}$*-open. Now* $U \cap cl^{*\alpha}(A) = (A \cup (X - cl^{*\alpha}(A))) \cap cl^{*\alpha}(A) = A$.

(ii) ⇒ (iv): Let $A = U \cap cl^{*\alpha}(A)$ *for some* $I_g^{**\alpha}$*-open set U. Now* $A \cup (X - cl^{*\alpha}(A)) = (U \cap cl^{*\alpha}(A)) \cup (X - cl^{*\alpha}(A)) = U$. *Since U is* $I_g^{**\alpha}$*-open,* $A \cup (X - cl^{*\alpha}(A))$ *is* $I_g^{**\alpha}$*-open.*

□

Theorem 4.2. *Let* (X, τ, I) *be an ideal space which is a* T_I*-space. Then the following hold.*

(i) Every I_g*-locally ∗-closed set is* $I_g^{**\alpha}$*- locally closed.*

(ii) Every GLC-set is $I_g^{**\alpha}$*-locally closed.*

(iii) Every GLC-set is* $I_g^{**\alpha}$*-locally closed.*

*(iv) Every GLC**-set is* $I_g^{**\alpha}$*-locally closed.*

Proof:

(i) : Let A be an I_g*-locally ∗-closed set such that* $A = U \cap V$ *where U is a* I_g*-open set and V is ∗-closed. Now X is a* T_I*-space implies U is ∗-open. Since every ∗-open is* $I_g^{**\alpha}$*-open, U is an* $I_g^{**\alpha}$*-open set and every ∗-closed is* $*^\alpha$*-closed implies V is a* $*^\alpha$*-closed set. Consequently, A is* $I_g^{**\alpha}$*-locally closed.*

(ii) : Let A be a GLC-set such that $A = U \cap V$ *where U is a g-open set and V is a g-closed set. By theorem 2.1 [6], U is* I_g*-open, V is* I_g*-closed. Now X is a* T_I*-space implies U is ∗-open and V is ∗-closed. Since every ∗-open is* $I_g^{**\alpha}$*-open, U is an* $I_g^{**\alpha}$*-open set and V is a* $*^\alpha$*-closed set. Consequently, A is* $I_g^{**\alpha}$*- locally closed.*

(iii) : Let A be a GLC-set such that* $A = U \cap V$ *where U is a g-open set and V is a closed set. By theorem 2.1 [6], U is* I_g*-open. Now X is a* T_I*-space implies U is ∗-open. Since every ∗-open is* $I_g^{**\alpha}$*-open, U is an* $I_g^{**\alpha}$*-open set and since every closed set is* $*^\alpha$*-closed, V is a* $*^\alpha$*-closed set and so A is* $I_g^{**\alpha}$*-locally closed.*

*(iv) : Let A be a GLC**-set such that* $A = U \cap V$ *where U is an open set and V is a g-closed set. By theorem 2.1 [6], V is* I_g*-closed. Now X is a* T_I*-space implies V is ∗-closed. Since every ∗-closed set is* $*^\alpha$*-closed, V is a* $*^\alpha$*-closed set and by theorem 4.14 [20], and U is an* $I_g^{**\alpha}$*-open set. Hence A is* $I_g^{**\alpha}$*-locally closed.*

□

Theorem 4.3. *Let (X, τ, I) be an ideal space and A be a subset of X. If A is $I_g^{**\alpha}$-locally closed and $*^\alpha$-dense, then A is $I_g^{**\alpha}$-open.*

Proof: *Let A be $I_g^{**\alpha}$-locally closed. Then by theorem 4.1(ii), $A = U \cap cl^{*\alpha}(A)$ for some $I_g^{**\alpha}$-open set U. Since A is $*^\alpha$-dense, $cl^{*\alpha}(A) = X$ which implies that $A = U \cap X = U$. Hence A is $I_g^{**\alpha}$-open.*

□

Corollary 4.4. *Let (X, τ, I) be an ideal topological space. Then the following are equivalent.*

(i) Every subset of X is $I_g^{**\alpha}$-locally closed.

(ii) Every $*^\alpha$-dense set is $I_g^{**\alpha}$-open.

Proof:

(i) ⇒ (ii): Let $A \subseteq X$ be $*^\alpha$-dense. By (i), A is $I_g^{**\alpha}$-locally closed. Then by theorem 4.1(iv), $A \cup (X - cl^{*\alpha}(A))$ is $I_g^{**\alpha}$-open. Since $cl^{*\alpha}(A) = X$, A is $I_g^{**\alpha}$-open.

(ii) ⇒ (i): For any subset A of X, consider $U = A \cup (X - cl^{*\alpha}(A))$. Then $cl^{*\alpha}(U) = cl^{*\alpha}(A \cup (X - cl^{*\alpha}(A))) = X$ and so U is $*^\alpha$-dense. By (ii), U is $I_g^{**\alpha}$-open. By theorem 4.1(ii), A is $I_g^{**\alpha}$-locally closed.

□

Theorem 4.5. *Intersection of two $I_g^{**\alpha}$-locally closed sets is $I_g^{**\alpha}$-locally closed.*

Proof: *Let A and B be $I_g^{**\alpha}$-locally closed sets. Then $A = U_1 \cap V_1$ and $B = U_2 \cap V_2$ where U_1, U_2 are $I_g^{**\alpha}$-open and V_1, V_2 are $*^\alpha$-closed. Then $A \cap B = (U_1 \cap V_1) \cap (U_2 \cap V_2) = (U_1 \cap U_2) \cap (V_1 \cap V_2)$ where $U_1 \cap U_2$ is $I_g^{**\alpha}$-open and $V_1 \cap V_2$ is $*^\alpha$-closed. Consequently, $A \cap B$ is $I_g^{**\alpha}$-locally closed.*

□

Theorem 4.6. *Let (X, τ, I) be an ideal topological space. Then the following hold*

(i) If A is $I_g^{**\alpha}$-locally closed, B is locally closed then $A \cap B$ is $I_g^{**\alpha}$-locally closed.

(ii) If A is $I_g^{**\alpha}$-locally closed, B is I-locally closed then $A \cap B$ is $I_g^{**\alpha}$-locally closed.

(iii) If A is $I_g^{**\alpha}$-locally closed, B is I-locally $*$-closed then $A \cap B$ is $I_g^{**\alpha}$-locally closed.

(iv) If A is $I_g^{**\alpha}$-locally closed, B is $I^{*\alpha}$-locally closed then $A \cap B$ is $I_g^{**\alpha}$-locally closed.

Proof:

(i): Let A be an $I_g^{**\alpha}$-locally closed set then $A = U_1 \cap V_1$ where U_1 is $I_g^{**\alpha}$-open and V_1 is $*^{\alpha}$- closed. Since B is a locally closed set, $B = U_2 \cap V_2$ where U_2 is open and V_2 is closed. Then $A \cap B = (U_1 \cap V_1) \cap (U_2 \cap V_2) = (U_1 \cap U_2) \cap (V_1 \cap V_2)$. Since every open is $I_g^{**\alpha}$-open, $U_1 \cap U_2$ is $I_g^{**\alpha}$-open and $V_1 \cap V_2$ is $*^{\alpha}$- closed. Hence $A \cap B$ is $I_g^{**\alpha}$- locally closed.

(ii): Let A be an $I_g^{**\alpha}$-locally closed set then $A = U_1 \cap V_1$ where U_1 is $I_g^{**\alpha}$-open and V_1 is $*^{\alpha}$-closed. Since B is I-locally closed, $B = U_2 \cap V_2$ where U_2 is open and V_2 is $*$-perfect. Then $A \cap B = (U_1 \cap V_1) \cap (U_2 \cap V_2) = (U_1 \cap U_2) \cap (V_1 \cap V_2)$. Since every open is $I_g^{**\alpha}$-open, $U_1 \cap U_2$ is $I_g^{**\alpha}$-open and $V_1 \cap V_2$ is $*^{\alpha}$-closed. Hence $A \cap B$ is $I_g^{**\alpha}$-locally closed.

(iii): By remark 2.8, proof follows from (ii).

(iv): By remark 2.8, proof follows from (ii).

□

Theorem 4.7. *If A is $I_g^{**\alpha}$-locally closed, B is closed ($*$-closed, $*^{\alpha}$-closed, $*$-perfect, $*^{\alpha}$-perfect), then $A \cap B$ is $I_g^{**\alpha}$-locally closed.*

Proof:

Let A be an $I_g^{**\alpha}$-locally closed set such that $A = U \cap V$ where U is $I_g^{**\alpha}$-open and V is $*^{\alpha}$-closed. Then $A \cap B = (U \cap V) \cap B$ where B is closed and so $A = U \cap (V \cap B)$. Hence $A \cap B$ is $I_g^{**\alpha}$-locally closed.

□

5. $I_g^{**\alpha}$- Submaximal space

Now $I_g^{**\alpha}$- submaximal space is defined and characterized.

Definition 5.1. *An ideal space (X, τ, I) is called $I_g^{**\alpha}$- submaximal space if every $*^{\alpha}$-dense subset of X is $I_g^{**\alpha}$-open.*

Example 5.2. *Let $X = \{a, b, c, d\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\phi, \{b\}, \{c\}, \{b, c\}\}$. Then X is an $I_g^{**\alpha}$-submaximal space.*

Remark 5.3.

(i) Every $*^{\alpha}$-dense is dense.

(ii) Every $*^{\alpha}$-dense is $*$-dense.

Proof: *Since $cl^{*\alpha}(A) \subseteq cl^{*}(A) \subseteq cl(A)$ [11], the proof follows.* □

Theorem 5.4. *Every submaximal space is an $I_g^{**\alpha}$-submaximal space.*

Proof: *Let A be $*^{\alpha}$-dense in X. Since every $*^{\alpha}$-dense is dense, A is dense. Also, since X is a submaximal space, A is open. By theorem 4.14 [20], A is $I_g^{**\alpha}$-open. Hence X is an $I_g^{**\alpha}$-submaximal space.* □

Remark 5.5. *The converse of the above theorem is not true as seen from the following example.*

Example 5.6. *Let* $X = \{a, b, c\}$ *,*$\tau = \{\phi, X, \{a\}, \{b, c\}\}$ *and* I=$\{\phi, \{a\}\}$*. Then* X *is an* $I_g^{**\alpha}$*-submaximal space. But* X *is not a submaximal space since* $\{a,b\}$ *is a dense subset of* X *which is not open.*

Theorem 5.7. *Every I-submaximal space is an* $I_g^{**\alpha}$*-submaximal space.*

Proof: *Let* A *be* $*^\alpha$*-dense in X. Since every* $*^\alpha$*-dense is* $*$*-dense,* A *is* $*$*-dense. Also* X *is an I-submaximal space,* A *is open. By theorem 4.14 [20],* A *is* $I_g^{**\alpha}$*-open. Hence* X *is an* $I_g^{**\alpha}$*-submaximal space.* □

Remark 5.8. *The converse of the above theorem is not true as seen from the following example.*

Example 5.9. *Let* $X = \{a, b, c, d\}$ *,*$\tau = \{\phi, X, \{a\}, \{a, b\}\}$ *and* I=$\{\phi\}$*. Then* X *is an* $I_g^{**\alpha}$*-submaximal space. But* X *is not an I-submaximal space since* $\{a,b,d\}$ *is a* $*$*-dense subset of* X *which is not open.*

Theorem 5.10. *A* T_I*-space which is a g-submaximal space is an* $I_g^{**\alpha}$*-submaximal space.*

Proof: *Let X be a* T_I*-space which is a g-submaximal space and A be* $*^\alpha$*-dense in X. Since every* $*^\alpha$*-dense is dense,* A *is dense. Also, since* X *is a g-submaximal space,* A *is g-open and hence* A *is* I_g*-open. Now* X *is a* T_I*-space implies* A *is* $*$*-open and so* A *is* $I_g^{**\alpha}$*-open. Hence* X *is an* $I_g^{**\alpha}$*-submaximal space.* □

Theorem 5.11. *A* T_I*-space which is an* I_g*-submaximal space is an* $I_g^{**\alpha}$*-submaximal space.*

Proof: *By theorem 2.4 [3], and by theorem 5.10, the proof follows.* □

Theorem 5.12. *Let* (X, τ, I) *be an ideal topological space. Then the following are equivalent.*

(i) X *is an* $I_g^{**\alpha}$*-submaximal space.*

(ii) Every subset of X *is* $I_g^{**\alpha}$*-locally closed.*

(iii) Every $*^\alpha$*-dense subset of* X *is an intersection of a* $*^\alpha$*-closed set and an* $I_g^{**\alpha}$*-open subset of* X*.*

Proof: *(i)⇒(ii): Let* X *be an* $I_g^{**\alpha}$*-submaximal space. Since every* $*^\alpha$*-dense is* $I_g^{**\alpha}$*-open and by corollary 4.4, every subset of* X *is* $I_g^{**\alpha}$*-locally closed.*
(ii)⇒(iii): Let A *be* $*^\alpha$*-dense in X. By (ii)* A *is an* $I_g^{**\alpha}$*-locally closed. Then by theorem 4.1, there exists an* $I_g^{**\alpha}$*-open set* U *such that* $A = U \cap cl^{*\alpha}(A)$ *where* $cl^{*\alpha}(A)$ *is* $*^\alpha$*-closed.*
(iii)⇒ (i): Let A *be* $*^\alpha$*-dense in X. By (iii)* $A = U \cap V$*, where* U *is* $I_g^{**\alpha}$*-open and* V *is* $*^\alpha$*-closed. Since* $A \subseteq V$*,* V *is* $*^\alpha$*-dense and so* $V = X$*. Hence* $A = U$ *which is* $I_g^{**\alpha}$*-open. Thus* X *is an* $I_g^{**\alpha}$*-submaximal space.* □

Theorem 5.13. *Let (X, τ, I) be an ideal space. Then the following are equivalent.*

(i) X is an $I_g^{**\alpha}$-submaximal space.

(ii) Every $*$-codense subset of X is $I_g^{**\alpha}$-closed.

Proof: *(i)⇒(ii): Let X be an $I_g^{**\alpha}$-submaximal space and A be a $*$-codense subset of X. Then $X - A$ is $*$-dense. By (i), $X - A$ is $I_g^{**\alpha}$-open. Hence A is $I_g^{**\alpha}$-closed. (ii)⇒(i): Let A be a $*^{\alpha}$-dense subset of X. Then A is $*$-dense in X and so $X - A$ is $*$-codense. By (ii) $X - A$ is $I_g^{**\alpha}$-closed implies that A is $I_g^{**\alpha}$-open. Hence X is an $I_g^{**\alpha}$-submaximal space.* □

6. $I_g^{**\alpha}$- Separated sets

In this section we define $I_g^{**\alpha}$-separated sets and basic properties of this notion are derived.

Definition 6.1. *Two non-empty subsets A and B of an ideal space (X, τ, I) are called $I_g^{**\alpha}$-separated if $A \cap cl^{*\alpha}(B) = cl^{*\alpha}(A) \cap B = \phi$.*

Example 6.2. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$ and $I = \{\phi\}$. Let $A = \{a\}$ and $B = \{b\}$. Then $cl^{*\alpha}(A) \cap B = \{a, c\} \cap \{b\} = \phi$ and $A \cap cl^{*\alpha}(B) = \{a\} \cap \{b, c\} = \phi$ and so the sets A and B are $I_g^{**\alpha}$-separated.*

Theorem 6.3. *Let A and B be subsets of an ideal topological space X. Then the following hold.*

(i) If A and B are separated sets in X, then A and B are $I_g^{**\alpha}$-separated.

(ii) If A and B are $*$-separated sets in X, then A and B are $I_g^{**\alpha}$-separated.

Proof: *(i): Let A and B be two separated sets in X. Then $A \cap cl(B) = \phi$ and $cl(A) \cap B = \phi$. Now $cl^{*\alpha}(A) \subseteq cl(A)$ implies $cl^{*\alpha}(A) \cap B \subseteq cl(A) \cap B = \phi$. Similarly $A \cap cl^{*\alpha}(B) = \phi$. Hence A and B are $I_g^{**\alpha}$-separated sets. (ii): Let A and B be two $*$-separated sets. Then $A \cap cl(B) = \phi$ and $cl^*(A) \cap B = \phi$. Since $cl^{*\alpha}(A) \subseteq cl^*(A)$, $cl^{*\alpha}(A) \cap B \subseteq cl^*(A) \cap B = \phi$. Also $cl^{*\alpha}(B) \subseteq cl(B)$ implies $A \cap cl^{*\alpha}(B) \subseteq A \cap cl(B) = \phi$. Hence A and B are $I_g^{**\alpha}$-separated sets.* □

Remark 6.4. *The converses of the above theorem are not true as seen from the following examples.*

Example 6.5. *Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a, b\}\}$ and $I = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Let A=$\{a\}$ and B=$\{c\}$. Then A and B are $I_g^{**\alpha}$-separated sets but not separated sets.*

Example 6.6. *Let X= $\{a,b,c\}$, $\tau = \{\phi, X, \{a\}, \{a, c\}\}$ and I=$\{\phi\}$. Let A=$\{a\}$ and B=$\{c\}$. Then A and B are $I_g^{**\alpha}$-separated sets but not $*$-separated sets.*

Remark 6.7. *The following table shows the relationships between $I_g^{**\alpha}$-separated sets with other known existing separated sets.The symbol "1" in a cell means that a set implies the other set and the symbol "0" means that a set does not imply the other set.*

sets	separated	$*$-separated	$I_g^{**\alpha}$-separated
separated	1	1	1
$*$-separated	0	1	1
$I_g^{**\alpha}$-separated	0	0	1

Theorem 6.8. *Let (X, τ, I) be an ideal topological space. If A and B are $I_g^{**\alpha}$-separated sets of X and $A \cup B \in \tau^\alpha$, then A and B are $*^\alpha$-open.*

Proof: *Since A and B are $I_g^{**\alpha}$-separated in X, then $B = (A \cup B) \cap (X \setminus Cl^{*\alpha}(A))$. Since $A \cup B \in \tau^\alpha$ and $Cl^{*\alpha}(A)$ is $*^\alpha$-closed in X, B is $*^\alpha$-open. By a similar way, we obtain that A is $*^\alpha$-open.* □

Theorem 6.9. *Assume that $I_g^{**\alpha}$-O(X) forms a topology. For an ideal topological space (X, τ, I), let $A, B \in I_g^{**\alpha} - LC(X)$. If A and B are $I_g^{**\alpha}$-separated, then $A \cup B \in I_g^{**\alpha} - LC(X)$.*

Proof: *Given $A, B \in I_g^{**\alpha}$-LC(X) and $I_g^{**\alpha}$-separated. By theorem 4.1, there exist an $I_g^{**\alpha}$-open sets U and V of (X, τ, I) such that $A = U \cap cl^{*\alpha}(A)$ and $B = V \cap cl^{*\alpha}(B)$. Now $G = U \cap (X - cl^{*\alpha}(B))$ and $H = V \cap (X - cl^{*\alpha}(A))$ are $I_g^{**\alpha}$-open subsets of (X, τ, I). Since $A \cap cl^{*\alpha}(B) = \phi$, $A \subseteq (cl^{*\alpha}(B))^c$. Now $A = U \cap cl^{*\alpha}(A)$ becomes $A \cap (cl^{*\alpha}(B))^c = G \cap cl^{*\alpha}(A)$. Then $A = G \cap cl^{*\alpha}(A)$. Similarly $B = H \cap cl^{*\alpha}(B)$. Moreover $G \cap cl^{*\alpha}(B) = \phi$ and $H \cap cl^{*\alpha}(A) = \phi$. Since G and H are $I_g^{**\alpha}$-open sets of (X, τ, I), $G \cup H$ is $I_g^{**\alpha}$-open. Therefore $A \cup B = (G \cup H) \cap cl^{*\alpha}(A \cup B)$ and hence $A \cup B \in I_g^{**\alpha} - LC(X)$.*

□

References

1. Acikgoz.A, Yuksel.S and Noiri.T, *On α-I-preirresolute functions and β-I-preirresolute functions*, Bull. Malays. Math. Sci. Soc. 2,28(1),1-8,(2005).
2. Balachandran.K, Sundaram.P and Maki.H, *Generalized locally closed sets and GLC-Continuous functions*, Indian J. Pure and Appl. Math.,27(3),235-244,(1996).
3. Bhavani.K, Sivaraj.D, *I_g-Submaximal space*, Bol. Soc.Paran.Mat.Volume 33(1),105-110,(2015).
4. Bourbaki.N *General topology*, part I. Addison Wesley Reading Mass,(1996).
5. Dontchev.J, *Idealization of Ganster Reilly Decomposition Theorems*, Math.GN/990-1017,(5),Jan.1999.
6. Dontchev.J, Ganster.M, Noiri.T, *Unified operation approach of generalized closed sets via topological ideal*, Math.Japanica 49,395-401,(1999) .
7. Erdal Ekici and Takashi Noiri, *Properties of I-Submaximal ideal topological spaces*, Faculty of sciences and mathematics, Filomat 24:4,87-94,(2010).

8. Erdal Ekici and Takashi Noiri, *Connectedness in ideal topological spaces*, Novi Sad J.Math.Vol.38,No.2,65-70,(2008) .

9. E. Hewitt, *A problem of set-theoretic topology*, Duke Math.J.,10,309-333,(1943).

10. Jankovic.D, Hamlett.T.R, *New topologies from old via ideals*, Amer.Math.Monthly 97,295-310,(1990).

11. Jeong Gi Kang and Chang Su Gim, *On P-I open sets*, Honam Mathematical J.31, No.3,pp.293-314,(2009).

12. Krishnan Balachandran, Palaniappan Sundaram and Haruo Maki, *Generalized locally closed sets and GLC-continuous functions*,Indian J.Pure Appl. Math. 27(3). 235-244(1996).

13. Kuratowski.K, *Topology*,Vol.I.Academic press, New York, (1966).

14. Levine.N, *Generalized closed sets in topology*, Rend. Circ.Mat.Palermo 19,89-96,(1970).

15. Rajamani.M, Indhumathi.V and Krishnaprakash.S, *Some stronger local function via ideals*,J.Adv.Res.Pure Math.,2,48-52,(2010).

16. Rajamani.M, Indhumathi.V and Krishnaprakash.S, *On $I_g^{*\alpha}$-closed sets and $I_g^{*\alpha}$-continuity*, Jordan Journal of Mathematics and Statistics,5(3),201-208,(2012).

17. Navaneethakrishnan.M and Sivaraj.D, *Generalized locally closed sets in ideal topological spaces*, Bull. Allahabad Math.Soc,Vol.24, Part1,13-19,(2009).

18. Nasef A.A, Radwan A.E, Esmaeel.R.B, *Some properties of α-open sets with respect to an ideal*, International Journal of Pure and Applied Mathematics Volume 102 No.4,613-630,(2015).

19. Njastad.O, *On some classes of nearly open sets*, Pascific J. Math.,15,961-970,(1965).

20. Santhini.C, Suganya.M, *New form of generalized closed sets via ideal topology*, Proceedings of International Conference on Recent Trends in Applied mathematics,S.R.N.M College,Sattur,(2017).

C. Santhini,
Associate Professor of Mathematics,
V. V. Vanniaperumal College for Women,
Virudhunagar-626 001,
Tamil Nadu, INDIA.
E-mail address: santhinimdu@gmail.com

and

M. Suganya,
Department of Mathematics,
V. V. Vanniaperumal College for Women,
Virudhunagar-626 001,
Tamil Nadu, INDIA.
E-mail address: suganyaksa97@gmail.com