Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 38** 5 (2020): 215–232.
ISSN-00378712 in press
doi:10.5269/bspm.v38i5.40511

C^*-algebras Generated by Isometries and True Representations

Mamoon Ahmed

Abstract: Let (G, P) be a quasi-lattice ordered group. In this paper we present a modified proof of Laca and Raeburn's theorem about the covariant isometric representations of amenable quasi-lattice ordered groups [7, Theorem 3.7], by following a two stage strategy. First, we construct a universal covariant representation for a given quasi-lattice ordered group (G, P) and show that it is unique. The construction of this object is new; we have not followed either Nica's approach in [10] or Laca and Raeburn's approach in [7], although all three objects are essentially the same. Our approach is a very natural one and avoids some of the intricacies of the other approaches. Then we show if (G, P) is amenable, true representations of (G, P) generate C^*-algebras which are canonically isomorphic to the universal object.

Key Words: Quasi-lattice ordered groups, C^*-algebra, True representation, Amenable groups.

Contents

Introduction L. A. Coburn proved interesting results in [3] about C^*-algebras generated by isometries. That paper has consequences which are relevant to this work and are summarized in Theorem 0.1 below. This theorem is not presented in the form originally given by the author, rather it is rephrased to draw attention to its similarity with later results.

Theorem 0.1. *There is an isometry U which generates a C^*-algebra $C^*(U)$ that has the following properties*

1. *Let $C^*(V)$ be the C^*-algebra generated by some isometry V. Then there is a $*$-homomorphism $\phi_V : C^*(U) \to C^*(V)$ such that $\phi_V(U) = V$.*
2. *If V is non-unitary, then ϕ_V is an isomorphism.*

2010 *Mathematics Subject Classification:* 46L55.
Submitted November18, 2017. Published February 05, 2018

Which says that all C^*-algebras generated by non-unitary isometries are the same and in some sense, universal.

J.Cuntz in [4] and [5] gave generalizations of this result. Of particular interest in this paper are two results concerning pairwise orthogonal families of isometries.

Nica introduced a class of groups termed quasi-lattice ordered groups. To each quasi-lattice ordered group (G, P) there corresponds representations of P by isometries called covariant representations. There is also a unique covariant representation with the universal property. Nica used this universal object to define amenability, which is an interesting property of some quasi-lattice ordered groups. The term 'amenability' is already used in group representation theory, but an amenable quasi-lattice ordered group (G, P) is not necessarily amenable in the usual sense. However, a quasi-lattice ordered group (G, P) is necessarily amenable in Nica's sense if G is amenable in the usual sense. In this paper we follow Nica's sense.

Nica also showed that the amenability of abelian groups can be used to establish the results of Douglas and Murphy. Laca and Raeburn introduced a subclass of covariant representations for a given quasi-lattice ordered group, which are called here true representations. Based on Nica's work, Laca and Raeburn proved the following theorem

Theorem 0.2. *Let (G, P) be a quasi-lattice ordered group. There is a covariant representation (A, U) of P by isometries with the properties*

1. *Let (B, V) be a covariant representation of P by isometries. There is a $*$-homomorphism $\phi_V : C^*(U) \to C^*(V)$ such that $\phi_V(U_p) = V_p$ for each $p \in P$.*
2. *If (G, P) is amenable and (B, V) is a true representation then ϕ_V is an isomorphism.*

Again this theorem is not presented in the form originally given by the authors, rather it is rephrased to draw attention to its similarity with later results.

In this paper we give a modified proof of Laca and Raeburn's Theorem above. We will follow Laca and Raeburn in employing a two stage strategy. First, we construct a universal covariant representation for a given quasi-lattice ordered group (G, P). Then we show that if (G, P) is amenable, then true representations of (G, P) generate C^*-algebras that are canonically isomorphic to the C^*-algebra generated by the universal covariant representation.

In section 2, quasi-lattice ordered groups and covariant representations will be discussed. In section 3, true representations and their properties will be discussed. In section 4, we construct a universal covariant representation and we give our modified proof of Laca and Raeburn's Theorem.

1. Preliminaries

Let P be a subsemigroup of a group G with identity e such that $P \bigcap P^{-1} = \{e\}$. There is a relation '$\leq$' on G with respect to P where $x \leq y$ if $x^{-1}y \in P$. This relation is a partial order on G which is left invariant in the sense that $x \leq y$

implies $zx \leq zy$ for any $x, y, z \in G$. It is the natural partial order determined by P.

Convention 1. *We now use* (G, P) *to refer to the group* G *with the natural partial order* $\leq$ *on* G *determined by* P.

Definition 1.1. *The partially ordered group* (G, P) *is* quasi-lattice ordered *if every finite subset of* G *with an upper bound in* P *has a least upper bound in* P *[2, Section 2].*

Equivalently, (G, P) is quasi-lattice ordered if and only if every element of G with an upper bound in P has a least upper bound in P, and every two elements in P with a common upper bound in P have a least upper bound in P [10, Section 2.1].

Notation 1. *The least upper bound or* sup *of the elements* x *and* y *will be denoted by* $x \vee y$.

We conclude the introduction to quasi-lattice ordered groups with the following property which was observed by Nica [10].

Lemma 1.2. *Let* (G, P) *be a quasi-lattice ordered group. If* $x, y \in G$ *have a common upper bound in* P *and* $z \in G$ *satisfies* $z(x \vee y) \in P$ *then* zx *and* zy *have a common upper bound in* P. *If, in addition,* $z \leq zx \vee zy$, *then* $zx \vee zy = z(x \vee y)$.

Definition 1.3. *Let* (G, P) *be a quasi-lattice ordered group. A representation of* (G, P) *by isometries is a pair* (A, V) *consisting of a unital* C^**-algebra* A *and a map* V *from* P *to* A *that satisfies the following three conditions:*

(i) $V_e = 1_A$;

(ii) $V_p^* V_p = 1_A$ *for all* $p \in P$;

(iii) $V_p V_q = V_{pq}$ *for all* $p, q \in P$.

If in addition V *satisfies*

$$V_p V_p^* V_q V_q^* = \begin{cases} V_{p\vee q} V_{p\vee q}^*, & \text{if } p, q \text{ have a common upper bound in } P; \\ 0, & \text{otherwise.} \end{cases}$$

then V *is a* covariant isometric representation.

Notation 2. *The* C^**-algebra generated by the set* $\{V_p : p \in P\}$ *will be denoted by* $C^*(V)$. *We write* $A_V = \{V_p V_p^* : p \in P\}$ *and* $B_V = \{V_p V_q^* : p, q \in P\}$.

Remark 1.4. *A covariant isomeric representation of the quasi-lattice ordered group* (G, P) *may be defined as a pair* (A, V) *consisting of a unital* C^**-algebra* A *and a map* V *from* P *to* A *such that*

1. $V_e = 1_A$;
2. $V_p V_q = V_{pq}$ *for all* $p, q \in P$,

3. $V_p^*V_q = \begin{cases} V_{p^{-1}(p\vee q)}V^*_{q^{-1}(p\vee q)}, & \text{when } p,q \text{ have a common upper bound in } P; \\ 0, & \text{otherwise.} \end{cases}$

To see that the first definition implies the second, notice first that if $p, q \in P$ have no common upper bound in P then the covariance condition gives

$$V_P V_p^* V_q V^* q = 0$$

and hence

$$V_p^*V_q = (V_p^*V_p)V_p^*V_q(V_q^*V_q) = 0$$

However if p, q have a common upper bound in P, then

$$V_p^*V_q = (V_p^*V_p)V_p^*V_q(V_q^*V_q) = V_p^*V_{p\vee q}V^*_{p\vee q}V_q.$$

But $p \le p \vee q$, so $p^{-1}(p \vee q) \in P$. Therefore,

$$V_{p\vee q}V^*_{p\vee q} = V_pV_{p^{-1}(p\vee q)}V^*_{q^{-1}(p\vee q)}V_q^*$$

thus the result follows. The reverse implication is easily checked.

Example 1.5. *Any representation (A, V) of a totally ordered group (G, P) by isometries is covariant. To see this, let $p, q \in P$ and suppose that $p \le q$. Then $p^{-1}q \in P$ and*

$$V_p^*V_q = V_p^*V_pV_{p^{-1}q} = V_{p^{-1}q} = V_{p^{-1}(p\vee q)}V^*_{q^{-1}(p\vee q)}$$

where $p \vee q$=max$\{p, q\}$=q. Similarly if $q \le p$ then

$$V_p^*V_q = V^*_{q^{-1}p} = V_{p^{-1}(p\vee q)}V^*_{q^{-1}(p\vee q)}.$$

Thus (A, V) is covariant.

1.1. The Toeplitz representation

For any quasi-lattice ordered group (G, P), the semigroup P has an important representation known as the Toeplitz or Weiner-Hopf representation $(B(\ell^2(P)), \mathrm{T})$. Where $B(\ell^2(P))$ is the C^*-algebra of bounded linear operators on the Hilbert space

$$\ell^2(P) := \{h : P \to \mathbb{C} : \sum_{p\in P} |h(p)|^2 < \infty\}$$

with pointwise addition, scalar multiplication and inner product

$$(h, k) := \sum_{s\in P} h(s)\overline{k(s)}.$$

The map $\mathrm{T} : P \to B(\ell^2(P))$ is defined for each $p \in P$ and $h \in \ell^2(P)$ by

$$(\mathrm{T}_p h)(q) = \begin{cases} h(p^{-1}q), & \text{if } p^{-1}q \in P \text{ (ie. } p \le q) \text{ ;} \\ 0, & \text{otherwise,} \end{cases}$$

for all $q \in P$.

Now,

$$\sum_{q\in P} |(\mathrm{T}_p h)(q)|^2 = \sum_{\{q\in P: q^{-1}p\in P\}} |h(p^{-1}q)|^2 = \sum_{s\in p^{-1}P\bigcap P} |h(s)|^2 = \|h\|^2,$$

this is true since $p^{-1}P\bigcap P = p^{-1}(P\bigcap pP) = p^{-1}pP = P$. Thus for each $p \in P$ and $h \in \ell^2(P)$, $\mathrm{T}_p h \in \ell^2(P)$. In fact, T_p is an isometry, since

$$\|\mathrm{T}_p h\|^2 = \sum_{q\in P} |(\mathrm{T}_p h)(q)|^2 = \|h\|^2.$$

Notice also that T_e is the identity. Next note that for all $p, q, s \in P$ such that $(pq)^{-1}s \in P$ and $h \in \ell^2(P)$,

$$(\mathrm{T}_{pq}h)(s) = h(q^{-1}p^{-1}s) = (\mathrm{T}_q h)(p^{-1}s) = (\mathrm{T}_p\mathrm{T}_q h)(s)$$

since $p^{-1}s \in qP \subset P$. Notice also that $(pq)^{-1}s \notin P$, then $(\mathrm{T}_{pq}h)s = 0$ and either $p^{-1}s \notin P$, in which case $(\mathrm{T}_p(\mathrm{T}_q h))s = 0$, or else $q^{-1}(p^{-1}s) \notin P$, so $(\mathrm{T}_q h)(p^{-1}s) = 0$. In any case, $\mathrm{T}_p\mathrm{T}_q = \mathrm{T}_{pq}$ for all $p, q \in P$, and $(B(\ell^2(P)), \mathrm{T})$ is a representation of (G, P) by isometries.

For a quasi-lattice ordered group (G, P), the Toeplitz representation is covariant. To see this, note first that for $p \in P$ and $h, k \in \ell^2(P)$,

$$\begin{aligned}
(\mathrm{T}_p^* h, k) &= (h, \mathrm{T}_p k) \\
&= \sum_{s\in P} h(s)\overline{(\mathrm{T}_p k)(s)} \\
&= \sum_{\{s\in P: p^{-1}s\in P\}} h(s)\overline{k(p^{-1}s)} \\
&= \sum_{t\in p^{-1}P\cap P} h(pt)\overline{k(t)},
\end{aligned}$$

where again $p^{-1}P \cap P = P$. By the uniqueness of the adjoints, $(\mathrm{T}_p^* h)(t) = h(pt)$ for all $t \in P$, and so given $s \in P$

$$\begin{aligned}
(\mathrm{T}_p\mathrm{T}_p^* h)(s) &= \begin{cases} (\mathrm{T}_p^* h)(p^{-1}s), & \text{if } p \le s \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} h(s), & \text{if } p \le s \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

It follows that for all $p, q, s \in P$ and $h \in \ell^2(P)$,

$$(\mathrm{T}_p\mathrm{T}_p^*\mathrm{T}_q\mathrm{T}_q^* h)(s) = \begin{cases} h(s), & \text{if } p \le s \text{ and } q \le s \\ 0, & \text{otherwise.} \end{cases}$$

Hence $\mathrm{T}_p\mathrm{T}_p^*\mathrm{T}_q\mathrm{T}_q^* = 0$ if p and q have no common upper bound in P. Finally, if (G, P) is quasi-lattice ordered, $p \le s$ and $q \le s$ if and only if $p \vee q \le s$, and therefore $\mathrm{T}_p\mathrm{T}_p^*\mathrm{T}_q\mathrm{T}_q^* = \mathrm{T}_{p\vee q}\mathrm{T}_{p\vee q}^*$.

2. True Representations

Definition 2.1. *A covariant representation (A, V) of a quasi-lattice ordered group (G, P) is called a true representation if $\prod_{p\in F}(1 - V_pV_p^*) \neq 0$ for all finite subsets F of $P\backslash\{e\}$.*

Remark 2.2. *The name 'true' reflects that V_p is a true isometry (that is, $V_pV_p^* \neq 1$) for all $p \in P$.*

We consider the following examples to illustrate the importance of true representations.

2.1. Totally ordered groups

A representation (A, V) of a totally ordered group (G, P) by isometries is true if and only if V_p is non-unitary for all $p \in P$. To see this, recall that all such representations are covariant. Notice that any $p, q \in P$ are comparable, so with out loss of generality, suppose that $p \leq q$. Then

$$(1 - V_pV_p^*)(1 - V_qV_q^*) = 1 - V_pV_p^* - V_qV_q^* + V_{p\vee q}V_{p\vee q}^* = 1 - V_pV_p^*$$

since $p \vee q = \max\{p, q\} = q$. Similarly for a finite subset $F \subset P$ we have

$$\prod_{p\in F}(1 - V_pV_p^*) = 1 - V_tV_t^*$$

where $t = \min\{p \in F\}$. Hence $\prod_{p\in F} = 0$ if and only if $V_tV_t^* = 1$, and the result follows.

2.2. The Toeplitz representation

The Toeplitz representation of a quasi-lattice ordered group (G, P) is a true representation. To see this, consider $\delta_e \in \ell^2(P)$, defined by

$$\delta_e(s) = \begin{cases} 1 & \text{if } s = e, \\ 0 & \text{otherwise.} \end{cases}$$

Then for any $p \in P \setminus \{e\}$, $(\mathrm{T}_p^*\delta_e)(s) = \delta_e(ps) = 0$ for all $s \in P$, and hence $(1 - \mathrm{T}_p\mathrm{T}_p^*)\delta_e = \delta_e$. This gives $\big(\prod_{p\in F}(1 - \mathrm{T}_p\mathrm{T}_p^*)\big)\delta_e = \delta_e$ for any $F \subset P$, and the result follows.

2.3. Free Products

Let (G, P) be the free products of the family $\{(G_i, P_i) : i \in I\}$ where I is a finite set and $(G_i, P_i) = (\mathbb{Z}, \mathbb{N})$ for every $i \in I$. We claim that a covariant representation (A, V) of (G, P) is true if and only if

$$\sum_{i\in I} V_{a_i}V_{a_i}^* < 1(\text{ that is } \neq 1)$$

in which a_i denote the element $\rho_i(1) \in G$, where ρ_i is the canonical group homomorphism from G_i into G.
It is easily verified that for $i, j \in I$, $V_{a_i}V_{a_j}^* = 0$ unless $i = j$. Hence if (A, V) is covariant,

$$0 \neq \prod_{i \in I}(1 - V_{a_i}V_{a_i}^*) = 1 - \sum_{i \in I} V_{a_i}V_{a_i}^*$$

and so $\sum_{i \in I} V_{a_i}V_{a_i}^* \neq 1$. Now suppose that $\sum_{i \in I} V_{a_i}V_{a_i}^* \neq 1$ and consider a finite set $F \subset P$. For each $p \in F$, there is $i \in I$ such that the first element in the reduced word p is some power of a_i. Following Cuntz' results in [4,5] one can see that

$$1 - V_{a_i}V_{a_i}^* \leq \prod_{i \in I}(1 - V_pV_p^*)$$

and thus $\prod_{i \in I}(1 - V_{a_i}V_{a_i}^*) \neq 0$ for all finite $F \subset P$ if and only if $\prod_{i \in I}(1 - V_{a_i}V_{a_i}^*) \neq 0$.

Lemma 2.3. *Let (G, P) be a quasi-lattice ordered group. If F is a finite subset of P and $x \in P$ satisfies $p \nleq x$ for all $p \in F$, then the representation (A, V) is true implies*

$$Q = V_xV_x^* \prod_{p \in F}(1 - V_pV_p^*)$$

is a non-zero projection.

Proof: Notice that $Q^* = Q$ since $(V_pV_p^*)^* = V_pV_p^*$ for all $p \in P$. Also $QQ = Q$ since $V_pV_p^*V_pV_p^* = V_pV_p^*$ and $(1 - V_pV_p^*)(1 - V_pV_p^*) = (1 - V_pV_p^*)$ for all $p \in P$ which is true by the covariance condition. Hence Q is a projection.

Now, for every $p \in F$ we have,

$$V_pV_p^*V_x = V_pV_{p^{-1}(p\vee x)}V^*_{x^{-1}(p\vee x)} = V_xV_{x^{-1}(p\vee x)}V^*_{x^{-1}(p\vee x)}$$

and this is true by the covariance condition. Thus

$$V_{x^*}[V_xV_x^* \prod_{p \in F}(1 - V_pV_p^*)]V_x = \prod_{p \in F}(1 - V_{x^{-1}(p\vee x)}V^*_{x^{-1}(p\vee x)})$$

since $V_x^*V_x = 1$. The last expression does not equal zero since (A, V) is true and $x^{-1}(p \vee x) \neq e$ (otherwise $x = p \vee x$ and $p \leq x$). Hence

$$V_xV_x^* \prod_{p \in F}(1 - V_pV_p^*) \neq 0$$

as required. □

Proposition 2.4. *A true representation (A, V) of a quasi-lattice ordered group (G, P) has the following properties*

1. *Let* $a = \sum_{p\in F} \gamma_p V_p V_p^* \in \operatorname{span}(A_V)$ *where* F *is a finite subset of* P *and* $\gamma_p \in \mathbb{C}$ *for all* $p \in F$. *Then* $\|a\| = \max\left\{|\sum_{p\in S}\gamma_p| : S \in i(F)\right\}$, *where* $i(F)$ *is the set of initial segments of* F.

2. *The set* B_V *is linearly independent with* $\operatorname{span}(B_V)$ *dense in* $C^*(V)$.

3. *For each finite subset* $F \subset P$ *and* $S \in i(F)$ *there is a nonzero projection* $Q_{F,S} \in \operatorname{span}(A_V)$ *such that for all* $p, q \in F$

$$Q_{F,S} V_p V_q^* Q_{F,S} = \begin{cases} Q_{F,S} & \textit{if } p = q \in S \\ 0 & \textit{otherwise.} \end{cases}$$

4. *There is a continuous linear map* Φ_V *of* $C^*(V)$ *onto* $\overline{\operatorname{span}}(A_V)$ *such that*

$$\Phi_V(V_p V_q^*) = \begin{cases} V_p V_p^*, & \textit{if } p = q \\ 0, & \textit{otherwise.} \end{cases}$$

Recall that:

1. The C^*-algebra generated by A_V is commutative and hence any product in $\operatorname{span}(A_V)$ may be rearranged as necessary.

2. For a quasi-lattice ordered group (G, P) and a finite subset F of P. A subset I of F is an initial segment if and only if whenever $x, y \in F$, $x \le y$ and $y \in I$ imply that $x \in I$.

Proof:

1. For each initial segment $S \subset F$, define

$$X_S = \{p \in F \setminus S : p \text{ and } \vee S \text{ have c.u.b. in } P\}$$

and

$$P_S = \begin{cases} V_{\vee S} V_{\vee S}^* & \text{if } X_S = \phi; \\ V_{\vee S} V_{\vee S}^* \prod_{p\in X_S}(1 - V_p V_p^*) & \text{othewise.} \end{cases}$$

Now if $X_S = \phi$ then $V_{\vee S}^* P_S V_{\vee S} = 1$ and hence $P_S \neq 0$. Also if $X_S \neq \phi$ then $P_S \neq 0$ by Lemma 2.3 since $S \in i(F)$ and hence $p \nleq \vee S$ for all $p \in F \setminus S$. We claim that the set $\{P_S : S \in i(F)\}$ has the properties:

(a) For each $p \in F$ and $S \in i(F)$

$$V_p V_p^* P_S = \begin{cases} P_S & \text{if } p \in S \\ 0 & \text{otherwise,} \end{cases}$$

(b) For all $S, B \in i(F)$,

$$P_B P_S = \begin{cases} P_S & \text{if } S = B \\ 0, & \text{otherwise,} \end{cases}$$

(c) $\sum_{S\in i(F)} P_S = 1$.

To prove the above claims, notice that if $p \in S$, then $p \vee S = \vee S$ so by the covariance condition

$$V_pV_p^*P_S = V_{p\vee S}V_{p\vee S}^* \prod_{p\in X_S}(1 - V_pV_p^*) = P_S.$$

Moreover, $V_pV_p^*P_S \neq 0$ implies $p \in S$. To see this, notice that for

$$V_pV_p^*V_{\vee S}V_{\vee S}$$

to be nonzero, p must share a common upper bound with S. In which case, $V_pV_p^*P_S$ contains the factor $V_pV_p^*(1 - V_pV_p^*) = 0$ unless $p \leq S$, so $p \in S$.

Next consider $S, B \in i(F)$. If $S = B$ then $P_SP_B = P_S$ by Lemma 2.3. However, if $S \neq B$ then by the covariance condition there is some $z \in (S \setminus B)\bigcup(B \setminus S)$ and so P_SP_B contains a factor

$$V_{\vee S}V_{\vee S}^*(1 - V_zV_z^*)V_{\vee B}V_{\vee B}^*V_{S\vee B}V_{S\vee B}^* - V_{S\vee z\vee B}V_{S\vee z\vee B}^*.$$

But $S \vee z \vee B = S \vee B$, so $P_SP_B = 0$.

Also,

$$\begin{aligned}
1 &= \prod_{p\in F}(V_pV_p^* + (1 - V_pV_p^*)) \\
&= \sum_{S\subset F}\prod_{p\in S}V_pV_p^* \prod_{p\in F\setminus S}(1 - V_pV_p^*) \\
&= \sum_{S\in i(F)} V_{\vee S}V_{\vee S}^* \prod_{p\in F\setminus S}(1 - V_pV_p^*) \\
&= \sum_{S\in i(F)} P_S.
\end{aligned}$$

The second last equality follows by the covariance condition, since

$$\prod_{p\in S}V_pV_p^* = \begin{cases} V_{\vee S}V_{\vee S}^* & \text{if } S \text{ has an upper bound in } P \\ 0 & \text{otherwise} \end{cases}$$

and moreover if S has an upper bound but $S \notin i(F)$ then there is some $p \in F \setminus S$ such that $p \leq \vee S$, and hence $V_{\vee S}V_{\vee S}^*(1 - V_pV_p^*) = 0$, again by the covariance condition. The last equality follows since if $p \in F \setminus S$ has no common upper bound with S then $V_{\vee S}V_{\vee S}^*(1 - V_pV_p^*) = 1$, by the covariance condition. Notice that if $S = \phi$ the $\prod_{p\in S}V_pV_p^* = 1$ and if $S = F$ then $\prod_{p\in F\setminus S}(1 - V_pV_p^*) = 1$.

Thus by claims three and one,

$$\begin{aligned} a &= \sum_{p\in F} \gamma_p V_p V_p^* (\sum_{S\in i(F)} P_S) \\ &= \sum_{S\in i(F)} \sum_{p\in F} \gamma_p V_p V_p^* P_S \\ &= \sum_{S\in i(F)} (\sum_{p\in S} \gamma_p) P_S. \end{aligned}$$

Now, for each $B \in i(F)$,

$$(a - (\sum_{p\in S} \gamma_p)1)P_B = aP_B - (\sum_{p\in S} \gamma_p)P_B = 0$$

by claim two. Hence $\beta_B \in \sigma(a)$ since $P_B \neq 0$.

In fact $\sigma(a) = \{\sum_{p\in S} \gamma_p : S \in i(F)\}$. To see this consider $\lambda \in \mathbb{C}$ such that $\lambda \notin \sum_{p\in S} \gamma_p$ for all $S \in i(F)$. Then by claim three,

$$a - \lambda 1 = \sum_{S\in i(F)} (\sum_{p\in S} \gamma_p - \lambda) P_S$$

But this has inverse

$$\sum_{S\in i(F)} (\sum_{p\in S} \gamma_p - \lambda)^{-1} P_S$$

by claims two and three. Thus $\lambda \notin \sigma(a)$, and hence $\sigma(a) = \{\sum_{p\in S} \gamma_p : S \in i(F)\}$. Since $\operatorname{span}(A_V)$ is commutative, then by [8, Theorem 1.3.6], we have

$$\|a\| = r(a) = \max\{|\sum_{p\in S} \gamma_p| : S \in i(F)\}.$$

2. Notice that $\operatorname{span}(B_V)$ is dense in $C^*(V)$ since products of the form

$$V_{p_1} V_{q_1}^* V_{p_2} V_{q_2}^* ... V_{p_n} V_{q_n}^*$$

can be reduced to the form $V_p V_q^*$ for some $p, q \in P$ using the covariance condition.

The proof of linear independence is by contradiction. Suppose that

$$a = \sum_{p,q\in F} \gamma_{p,q} V_p V_q^* = 0$$

where F is a finite subset of P and $\{\gamma_{p,q}\} \subset \mathbb{C}$ are not all zero. Then the set

$$\{q \in F : \gamma_{p,q} \neq 0 \text{ for some } p \in F\}$$

is finite and nonempty, and hence has a minimal element s. Moreover, the set

$$\{p \in F : \gamma_{p,s} \neq 0\}$$

is finite and nonempty and hence has a minimal element r.

Put

$$P_1 = \prod_{\{p \in F : p \nleq r\}} (1 - V_p V_p^*) \text{ and } P_2 = \prod_{\{q \in F : q \nleq s\}} (1 - V_q V_q^*).$$

Then for each $p \in F$ such that $p \nleq r$, $P_1 V_p$ contains a factor

$$(1 - V_p V_p^*) V_p = 0$$

and hence $P_1 V_p = 0$. Similarly for any $q \in F$ such that $q \leq s$, $V_q^* P_2 = 0$. So now

$$0 = P_1 a P_2 = \sum_{\{p \in F : p \leq r, q \leq s\}} \gamma_{p,q} P_1 V_p V_q^* P_2 = \gamma_{r,s} P_1 V_r V_s^* P_2.$$

We claim that $P_1 V_r V_s^* P_2 \neq 0$ and thus $\gamma_{r,s} = 0$ which contradicts the choice of r and s.

To establish the claim, notice that

$$V_r^* P_1 V_r V_s^* P_2 V_s = \prod_{p \in S} (1 - V_{r^{-1}(p \vee r)} V_{r^{-1}(p \vee r)}^*) \prod_{q \in B} (1 - V_{s^{-1}(q \vee s)} V_{s^{-1}(q \vee s)}^*)$$

where

$$S = \{p \in F : p, r \text{ have c.u.b. in } P, p \nleq r\}$$

and

$$B = \{q \in F : q, s \text{ have c.u.b. in } P, q \nleq s\}.$$

Now notice that $r^{-1}(p \vee r) \neq e$ for all $p \in S$, otherwise $p \vee r = r$ and $p \leq r$. Similarly, $q^{-1}(q \vee s) \neq e$ for all $q \in B$. Thus

$$V_r^* P_1 V_r V_q^* P_2 V_q \neq 0$$

since (A, V) is true and the claim follows.

3. Let F be a finite subset of P and $S \in i(F)$. Put

$$X_S = \{p \in F \setminus S : p \text{ and } \vee S \text{ have c.u.b. in } P\}$$

and

$$Z_S = \{z(\vee S) \vee S : z \in SS^{-1}, z(\vee S), \vee S \text{ have c.u.b. in } P\} \setminus \{\vee S\}.$$

Define

$$Q_{F,S} = \begin{cases} V_{\vee S} V_{\vee S}^* & \text{if } X_S \cup Z_S = \phi \\ V_{\vee S} V_{\vee S}^* \prod_{p \in X_S \cup Z_S} (1 - V_p V_p^*) & \text{otherwise.} \end{cases}$$

Clearly $Q_{F,S} \in \operatorname{span}(A_V)$.

Now if $X_S \cup Z_S = \phi$ then $V_{\vee S}^* Q_{F,S} V_{\vee S} = 1$ and hence $Q_{F,S} \neq 0$. Also note that $p \nleq \vee S$ for all $p \in F \setminus S$ since S is an initial segment of F. Moreover, for $p \in Z_S$, $p \vee S = p \nleq \vee S$. Hence $Q_{F,S} \neq 0$ by Lemma 2.3.

Now notice that if $p = q \in S$, $V_p V_p^* V_{\vee S} V_{\vee S}^* = V_{\vee S} V_{\vee S}^*$ by the covariance condition and hence $Q_{F,S} V_p V_q^* Q_{F,S} = Q_{F,S} Q_{F,S} = Q_{F,S}$. We claim that for $p, q \in F$, $Q_{F,S} V_p V_q^* Q_{F,S} \neq 0$ implies that $p = q \in S$. To see this, notice first that $p \in S$, otherwise $Q_{F,S} V_p$ contains a factor $(1 - V_p V_p^*) V_p = 0$. Hence,

$$V_{\vee S}^* V_p = V_{(\vee S)^{-1}(p \vee S)} V_{p^{-1}(p \vee S)}^* = V_{p^{-1}(\vee S)}^*$$

by the covariance condition and since $p \leq \vee S$ for all $p \in S$. Similarly $q \in S$ and $V_q^* V_{\vee S} = V_{q^{-1}(\vee S)}$. But then $Q_{F,S} V_p V_q^* Q_{F,S}$ contains the factor

$$V_{\vee S}^* V_p V_q^* V_{\vee S} = V_{p^{-1}(\vee S)}^* V_{q^{-1}(\vee S)}.$$

Again by the covariance condition, $p^{-1}(\vee S)$ and $q^{-1}(\vee S)$ must have a common upper bound in P. it follows that

$$V_{\vee S} V_{\vee S}^* V_p V_q^* V_{\vee S} V_{\vee S}^* = V_{p[p^{-1}(\vee S) \vee q^{-1}(\vee S)]} V_{q[p^{-1}(\vee S) \vee q^{-1}(\vee S)]}^*$$

by the covariance condition. Now by Lemma 1.2

$$p[p^{-1}(\vee S) \vee q^{-1}(\vee S)] = pq^{-1}(\vee S) \vee S$$

and similarly

$$q[p^{-1}(\vee S) \vee q^{-1}(\vee S)] = qp^{-1}(\vee S) \vee S.$$

Hence

$$V_{\vee S} V_{\vee S}^* V_p V_q^* V_{\vee S} V_{\vee S}^* = V_{z(\vee S) \vee S} V_{z^{-1}(\vee S) \vee S}^*$$

where $z = pq^{-1}$. So $z(\vee S) \vee S \in Z_S$ otherwise $Q_{F,S} V_p V_q^* Q_{F,S}$ contains the factor

$$(1 - V_{z(\vee S) \vee S} V_{z(\vee S) \vee S}^*) V_{z(\vee S) \vee S} = 0.$$

Hence $z(\vee S) \vee S = \vee S$ and $z(\vee S) \leq \vee S$. Similarly, $z^{-1}(\vee S) \leq \vee S$. But then $\vee S \leq z(\vee S)$, so $z(\vee S) = \vee S$ and $z = e$, giving $p = q \in S$, as required.

4. By part (2), B_V is linearly independent. Define $\tilde{\Phi}_V$ to be the linear extension to $\operatorname{span}(B_V)$ of the map

$$V_p V_q^* \mapsto \begin{cases} V_p V_p^*, & \text{if } p = q \\ 0, & \text{otherwise.} \end{cases}$$

Then $\tilde{\Phi}_V$ is norm reducing. To see this, let $a = \sum_{p,q \in F} \gamma_{p,q} V_p V_q^* \in \operatorname{span}(B_V)$, for some finite subset $F \subset P$ and complex numbers $\gamma_{p,q}$. Since $i(F)$ is finite, there is $S \in i(F)$ such that $|\sum_{p \in S} \gamma_{p,p}|$ is a maximum. By (1),

$$\|\tilde{\Phi}_V(a)\| = \Big|\sum_{p \in S} \gamma_{p,p}\Big|$$

By (3) there is a non-zero projection Q such that

$$QaQ = \big|\sum_{p\in F} \gamma_{p,p}\big| Q = \|\tilde{\Phi}_V(a)\| Q.$$

Now $\|Q\| = 1$ since

$$\|Q\|^2 = \|Q * Q\| = \|QQ\| = \|Q\|$$

and $Q \neq 0$. Then

$$\|\tilde{\Phi}_V(a)\| = \|\tilde{\Phi}_V(a)\| \, \|Q\| = \|QaQ\| \leq \|a\| \, \|Q\|^2 = \|a\|.$$

Thus $\tilde{\Phi}_V$ is contractive and has a continuous linear extension $\Phi_V : C^*(V) \to \overline{\operatorname{span}}(A_V)$ with the required property.

□

Remark 2.5.

1. *Notice that the items (1) and (2) in Proposition 2.4 each imply that (A, V) is a true representation.*
2. *Notice also that (1) of Proposition 2.4 demonstrates that the norm of an element of the algebra $\operatorname{span}(A_V)$ depends only on the coefficients of the set A_V, and not on the choice of true representation. In fact, this is also true on $\overline{\operatorname{span}}(A_V)$ by the continuity of the norm.*

The next lemma shows that there is a strong similarity between any two true representations of a quasi-lattice ordered group.

Lemma 2.6. *Let (A, U) and (B, V) be covariant representations of the quasi-lattice ordered group (G, P). If (A, U) is true then there is a $*$-homomorphism $\tilde{\Phi} : \operatorname{span}(B_U) \to \operatorname{span}(B_V)$ such that $\tilde{\Phi}(U_p U_q^*) = V_p V_q^*$ for all $p, q \in P$. Moreover, if (B, V) is true then $\tilde{\Phi}$ is a $*$-isomorphism.*

Proof: By Proposition 2.4 (2), B_U is linearly independent, so the linear extension $\tilde{\Phi}$ of the map

$$U_p U_q^* \mapsto V_p V_q^*$$

for each $p, q \in P$ is well defined. Then $\tilde{\Phi}$ is a $*$-homomorphism, since for all $p, q, r, s \in P$ such that q, r have a common upper bound,

$$\begin{aligned}
\tilde{\Phi}(U_p U_q^* U_r U_s^*) &= \tilde{\Phi}(U_p U_{q^{-1}(q\vee r)} U_{r^{-1}(q\vee r)}^* U_s^*) \\
&= V_{pq^{-1}(q\vee r)} V_{sr^{-1}(q\vee r)}^* \\
&= V_p V_q^* V_r V_s^* \\
&= \tilde{\Phi}(U_p U_q^*) \tilde{\Phi}(U_r U_s^*)
\end{aligned}$$

by the covariance condition. However if q, r have no common upper bound then

$$\tilde{\Phi}(U_p U_q^* U_r U_s^*) = \tilde{\Phi}(0) = 0 = V_p V_q^* V_r V_s^*$$

by the covariance condition. Also,

$$[\tilde{\Phi}(U_p U_q^*)]^* = [V_p V_q^*]^* = V_q V_p^* = \tilde{\Phi}([U_p U_q^*]^*).$$

These properties extend to $\operatorname{span}(B_U)$ by the linearity of the map.

Now, suppose (B, V) is true and $\tilde{\Phi}(a) = 0$ for some

$$a = \sum_{p,q \in F} \gamma_{p,q} U_p U_q^* \in \operatorname{span}(B_U)$$

where F is a finite subset of P and $\gamma_{p,q} \in \mathbb{C}$ for each $p, q \in F$. Then

$$\sum_{p,q \in F} \gamma_{p,q} V_p V_q^* = \tilde{\Phi}(a) = 0,$$

so by Proposition 2.4(2) $\gamma_{p,q} = 0$ for all $p, q \in F$. Hence $a = 0$ and $\tilde{\Phi}$ is injective. It is clear that $\tilde{\Phi}$ is surjective and the result follows. □

Thus the algebraic structure of $\operatorname{span}(B_V)$ is essentially the same irrespective of the choice of a true representation (A, V). It may be that for certain quasi-lattice ordered groups there is only one way to complete this algebra as a C^*-algebra and then all true representations must be the same. It turns out that this happens if and only if (G, P) is amenable. Amenability will be discussed in the next section. However, let us point out here that amenability is in some sense a topological restriction on the algebra $\operatorname{span}(B_V)$ generated by a true representation (A, V). Amenability can be regarded as the requirement that all norms on this algebra with the essential C^*-algebra properties are equivalent.

3. The universal covariant representation and amenability

In this section we give two of the main results in this paper. We will follow the two stage strategy outlined in the introduction to show Laca and Raeburn's Theorem. First, we construct a universal covariant representation for a given quasi-lattice ordered group (G, P) and show that it is unique. Then we discuss amenability and its relationship to true representations.

Definition 3.1. *A universal covariant representation (A, U) of the quasi-lattice ordered group (G, P) is a covariant representation such that if (B, V) is any other covariant representation of (G, P), there is a unique $*$-homomorphism $\phi : C^*(U) \to C^*(V)$ such that $\phi(U_p) = V_p$ for all $p \in P$.*

Theorem 3.2. *Let (G, P) be a quasi-lattice ordered group. Then there is a universal covariant representation (A, U) of (G, P).*

Proof: Let $\{(B(H_V), V) : V \in \Lambda\}$ be the set of covariant representations of (G, P) on closed subspaces $\{H_V\}_{V\in\Lambda}$ of the Hilbert space $\ell^2(P)$. Define $A = B(\bigoplus_{V\in\Lambda} H_V)$ and $U : P \to A$ by $U_p = \bigoplus_{V\in\Lambda} V_p$ for each $p \in P$. We claim that $(B(\bigoplus_{V\in\Lambda} H_V), U)$ is a covariant representation of (G, P). To see this, first notice that,

$$U_e = \bigoplus_{V\in\Lambda} V_e = \bigoplus_{V\in\Lambda} 1_{H_V} = 1.$$

Also,

$$U_pU_q = \bigoplus_{V\in\Lambda} V_p \bigoplus_{V\in\Lambda} V_q = \bigoplus_{V\in\Lambda} V_pV_q = \bigoplus_{V\in\Lambda} V_{pq} = U_{pq}$$

for all $p, q \in P$. Finally, if $p, q \in P$ have a common upper bound in P, then by the covariance condition

$$\begin{aligned} U_p^*U_q &= \bigoplus_{V\in\Lambda} V_p^* \bigoplus_{V\in\Lambda} V_q = \bigoplus_{V\in\Lambda} V_p^*V_q = \bigoplus_{V\in\Lambda} V_{p^{-1}(p\vee q)}V_{q^{-1}(p\vee q)}^* \\ &= U_{p^{-1}(p\vee q)}U_{q^{-1}(p\vee q)}. \end{aligned}$$

However if p, q have no common upper bound, then by the covariance condition we have $U_p^*U_q = \bigoplus_{V\in\Lambda} V_p^*V_q = 0$. Thus (A, U) is a covariant representation as required.

In fact, (A, U) is true. To see this observe that $T \in \Lambda$, where $(B(\ell^2(P)), T)$ is the Toeplitz representation of (G, P). Hence if F is a finite subset of $P \setminus \{e\}$ then $\prod_{p\in F}(1 - U_pU_p^*) \neq 0$, since otherwise $\prod_{p\in F}(1 - T_pT_p^*) = 0$ for the true representation $(B(\ell^2(P)), T)$. By Proposition 2.4, $B_U = \{U_pU_q^* : p, q \in P\}$ is linearly independent with span dense in $C^*(G, P)$.

Let (B, W) be a covariant representation of (G, P). Recall from Lemma 2.6 that there is a $*$-homomorphism $\tilde{\phi} : \mathrm{span}(B_U) \to C^*(W)$ such that $\tilde{\phi}(U_pU_q^*) = W_pW_q^*$ for all $p, q \in P$. It remains to show that $\tilde{\phi}$ extends to a $*$-homomorphism on $C^*(U)$. It will be sufficient to show that the map is norm-reducing on $\mathrm{span}(B_U)$.

Let $a = \sum_{p,q\in F} \gamma_{p,q}U_pU_q^* \in \mathrm{span}(B_U)$ for some finite $F \subset P$ and $\gamma_{p,q} \in \mathbb{C}$. Observe that $\mathrm{span}(B_U)$ is a $*$-algebra with dimension equal to that of the Hilbert space $\ell^2(P)$ and hence

$$\|\tilde{\phi}(a)\| \leq \sup\{\|\rho(a)\| : \rho \in \Omega\}$$

where Ω is the set of $*$-homomorphisms taking $\mathrm{span}(B_V)$ into the bounded operators on a closed subspace of $\ell^2(P)$. Now, since each $\rho \in \Omega$ is a $*$-homomorphism, $\rho \circ U$ is a covariant representation of (G, P) on a closed subspace of the Hilbert space $\ell^2(P)$. That is $\rho \circ U \in \Lambda$ for all $\rho \in \Omega$. So now,

$$\begin{aligned}
\|\tilde{\phi}(a)\| &\leq \sup\{\|\rho(\sum_{p,q\in F} \gamma_{p,q} U_p U_q^*)\| : \rho \in \Omega\} \\
&= \sup\{\| \sum_{p,q\in F} \gamma_{p,q} (\rho \circ U)_p (\rho \circ U)_q^* \| : \rho \in \Omega\} \\
&\leq \sup\{\| \sum_{p,q\in F} \gamma_{p,q} V_p V_q^* \| : V \in \Lambda\} \\
&= \| \bigoplus_{V\in\Lambda} (\sum_{p,q\in F} \gamma_{p,q} V_p V_q^*)\| \\
&= \| \sum_{p,q\in F} \gamma_{p,q} U_p U_q^* \| \\
&= \|a\|.
\end{aligned}$$

Hence $\tilde{\phi}$ has a unique continuous extension

$$\phi : C^*(U) \to C^*(W). \tag{3.1}$$

That ϕ is a $*$-homomorphism follows from its continuity. Any other $*$-homomorphism taking U_p to W_p for all $p \in P$ must also be continuous, and hence agree with ϕ on all of $C^*(G, P)$. $\square$

Remark 3.3. *In fact, any other universal covariant representation $(\hat{A}, \hat{U})$ of the quasi-lattice ordered group (G, P) is essentially the same as $(C^*(U), U)$ in the following sense. There are $*$-homomorphisms $\phi : C^*(U) \to C^*(\hat{U})$ and $\psi : C^*(\hat{U}) \to C^*(U)$ such that*

$$\phi(U_p U_q^*) = \hat{U}_p \hat{U}_q^* \text{ and } \psi(\hat{U}_P \hat{U}_q^*) = U_p U_q^*$$

for all $p, q \in P$. Then

$$\psi \circ \phi(U_p U_q^*) = \psi(\hat{U}_p \hat{U}_q^*) = U_p U_q^*.$$

By the linearity and continuity of ψ and ϕ, $\psi \circ \phi(a) = a$ for all $a \in C^*(U)$. Similarly, $\phi \circ \psi(a') = a'$ for all $a' \in C^*(\hat{U})$, so ϕ and ψ are inverses. Moreover, since ϕ and ψ are contractive, $C^*(U)$ and $C^*(\hat{U})$ are isometrically isomorphic. Therefore, $(C^*(U), U)$ will be referred to as the universal covariant representation and $C^*(U)$ will be given the symbol $C^*(G, P)$.

Definition 3.4. *A quasi-lattice ordered group (G, P) is a amenable if*

$$\Phi_U : C^*(G, P) \to C^*(V)$$

is faithful on positive elements, in the sense that if $a \in C^*(G, P)$ then $\Phi_U(a^*a) = 0$ implies $a = 0$.

We now give a technical Lemma that will be required for our next main Theorem in this section.

Lemma 3.5. *Let (G, P) be a quasi-lattice ordered group. If (A, V) is a covariant representation of (G, P) and $\phi : C^*(G, P) \to C^*(V)$ is the $*$-homomorphism supplied by Theorem 3.2 in Equation 3.1, then*

$$\Phi_V \circ \phi = \phi \circ \Phi_U$$

where Φ is the homomorphism in Proposition 2.4.

Proof: The maps $\Phi_V \circ \phi$ and $\phi \circ \Phi_U$ operate on $U_p U_q^*$ in the following manner:

$$\begin{aligned}(\Phi_V \circ \phi)(U_p U_q^*) &= \Phi_V(V_p V_q^*)\\ &= \begin{cases} V_p V_p^*, & \text{if } p = q \\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

and

$$\begin{aligned}(\phi \circ \Phi_U)(U_p U_q^*) &= \begin{cases} \phi(U_p U_p^*), & \text{if } p = q \\ \phi(0), & \text{otherwise}\end{cases}\\ &= \begin{cases} V_p V_p^*, & \text{if } p = q \\ 0, & \text{otherwise}\end{cases}\end{aligned}$$

and thus $(\Phi_V \circ \phi)(U_p U_q^*) = (\phi \circ \Phi_U)(U_p U_q^*)$. These maps are linear and continuous, by Theorem 3.2 and Proposition 2.4(4), this equality extends to $C^*(G, P)$. □

Theorem 3.6. *The C^*-algebra generated by a true representation of an amenable quasi-lattice ordered group (G, P) is canonically isomorphic to the C^*-algebra generated by the universal covariant representation.*

Proof: Let V be a true representation of (G, P), and $\phi : C^*(G, P) \to C^*(V)$ be the $*$-homomorphism in Equation 3.1. Suppose that $\phi(a) = 0$ for some $a \in C^*(V)$. By Remark 2.5(2),

$$\|\Phi_U(a^* a)\| = \|(\phi \circ \Phi_U)(a^* a)\|$$

since $\Phi_U(a^* a) \in \overline{\mathrm{span}}(A_U)$. Thus by Lemma 3.5,

$$\|\Phi_U(a^* a)\| = \|(\Phi_V \circ \phi)(a^* a)\| = 0.$$

Hence $a = 0$ by hypothesis. By [8, Theorem 3.1.5], ϕ is an isometric $*$-isomorphism. □

References

1. S. Adji, M. Laca, M. Nilsen and I. Raeburn, *Crossed poducts by semigroups of endomorphisms and the Toeplitz algebras of ordered groups*, Proc.Amer. Math. Soc. 122 (1994), 1133–1141.

2. M. Ahmed and A. Pryde, *Semigroup crossed products and the induced algebras of lattice-ordered groups*, J. Math. Anal. Appl. 364 (2010) 498–507.

3. L. A. Coburn, *The C^*-algebra generated by an isometry*, Bull. Amer. Math. Soc. 73 (1967), 722–726.

4. J. Cuntz, *K-theory for certain C^*-algebras*, Annals of Mathematics 113 (1981), 181–197.

5. J. Cuntz, *Simple C^*-algebras generated by isometries*, Comm. Math. Phys. 57 (1977) 173–185.

6. R.G. Douglas, *On the C^*-algebra of a one parameter semigroup of isometries*, Acta. Math. 128 (1972), 143–152.

7. M. Laca and I. Raeburn, *Semigroup crossed products and the Toeplitz algebras of nonabelian groups*, J. Funct. Anal. 139 (1996), 415–440.

8. G. J. Murphy, *C*-algebras and operator theory*, Academic Press, Inc., 1990.

9. G. J. Murphy, *Ordered groups and Toeplitz algebras*, J. Operator Theory 18 (1987), no. 2, 303–326.

10. A. Nica, *C^*-algebras generated by isometries and Wiener-Hopf operators*, J. Operator Theory 27 (1992), 17–52.

Mamoon Ahmed,
Department of Basic Sciences,
Princess Sumaya University for Technology,
Jordan.
E-mail address: m.ahmed@psut.edu.jo