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A New Differential Operator of Analytic Functions Involving Binomial Series

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abstract: In this paper, we introduce a new differential operator of analytic functions involving binomial series. Furthermore, we derive some subordination and superordination results for this operator. Some applications and examples are also obtained.

Key Words: Analytic functions, Differential subordinations, Differential superordinations, Dominant, Subordinant.

Contents

1. Introduction and Definitions

Let $\mathcal{H}$ be the class of functions analytic in $\mathcal{U} := \{z : |z| < 1\}$ and $\mathcal{H}(a, n)$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z), \tag{1.2}$$

then p is a solution of the differential superordination (1.2). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a *subordinant* if $q \prec p$ for all p satisfying (1.2). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant. Miller and Mocanu [7] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2 p''(z); z) \Rightarrow q(z) \prec p(z).$$

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Using the results of Miller and Mocanu [7], Bulboacă [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [3] (see also, [1,5,10]). Shanmugam et al. [9] obtained sufficient conditions for a normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z) \quad \text{and} \quad q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z).$$

where q_1 and q_2 are given univalent functions in $\mathcal{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$.

For a function f in $\mathcal{A}$, and making use of the binomial series

$$(1-\lambda)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \lambda^j \qquad (m \in \mathbb{N} = \{1, 2, \ldots\},\ j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}),$$

we now define the differential operator $D^{\zeta}_{m,\lambda} f(z)$ as follows:

$$D^0 f(z) = f(z), \tag{1.3}$$

$$D^1_{m,\lambda} f(z) = (1-\lambda)^m f(z) + (1-(1-\lambda)^m) z f'(z) \tag{1.4}$$

$$= D_{m,\lambda} f(z),\ \lambda > 0;\ m \in \mathbb{N}, \tag{1.5}$$

$$D^{\zeta}_{m,\lambda} f(z) = D_{m,\lambda}(D^{\zeta-1} f(z)) \qquad (\zeta \in \mathbb{N}). \tag{1.6}$$

If f is given by (1.1), then from (1.5) and (1.6) we see that

$$D^{\zeta}_{m,\lambda} f(z) = z + \sum_{n=2}^{\infty} \left(1 + (n-1) \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j \right)^{\zeta} a_n z^n,\ \zeta \in \mathbb{N}_0. \tag{1.7}$$

Using the relation (1.7), it is easily verified that

$$C^m_j(\lambda) z (D^{\zeta}_{m,\lambda} f(z))' = D^{\zeta+1}_{m,\lambda} f(z) - (1 - C^m_j(\lambda)) D^{\zeta}_{m,\lambda} f(z) \tag{1.8}$$

where $C^m_j(\lambda) := \sum\limits_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j$.

We observe that for $m = 1$, we obtain the differential operator $D^{\zeta}_{1,\lambda}$ defined by Al-Oboudi [2] and for $m = \lambda = 1$,we get Sălăgean differential operator D^{ζ} [8].

The main object of the present paper is to apply a method based on the differential subordination in order to derive several subordination results involving the operator $D^{\zeta}_{m,\lambda}$. Furthermore, we obtain the previous results of Srivastava and Lashin [11] as special cases of some of the results presented here.

2. Preliminaries

In order to prove our results, we shall require the following known definition and lemmas.

Definition 2.1. *[7, Definition 2, p. 817] Denote by Q, the set of all functions $f(z)$ that are analytic and injective on $\overline{\mathcal{U}} - E(f)$, where*

$$E(f) = \{\eta \in \partial\mathcal{U} : \lim_{z \to \eta} f(z) = \infty\},$$

and are such that $f'(\eta) \neq 0$ for $\eta \in \partial\mathcal{U} - E(f)$.

Lemma 2.2. *[6, Theorem 3.4h, p. 132] Let $q(z)$ be univalent in the unit disk $\mathcal{U}$ and θ and ϕ be analytic in a domain D containing $q(\mathcal{U})$ with $\phi(w) \neq 0$ when $w \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

1. *$Q(z)$ is starlike univalent in $\mathcal{U}$, and*

2. *$\Re \frac{zh'(z)}{Q(z)} > 0$ for $z \in \mathcal{U}$.*

If

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2.3. *[4] Let $q(z)$ be convex univalent in the unit disk $\mathcal{U}$ and ϑ and φ be analytic in a domain D containing $q(\mathcal{U})$. Suppose that*

1. *$\Re\left[\vartheta'(q(z))/\varphi(q(z))\right] > 0$ for $z \in \mathcal{U}$,*

2. *$zq'(z)\varphi(q(z))$ is starlike univalent in $\mathcal{U}$.*

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathcal{U}) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in $\mathcal{U}$, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{2.1}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

3. Subordination for Analytic Functions

We begin by proving the following result.

Lemma 3.1. *Let the functions $p(z)$ and $q(z)$ be analytic in $\mathcal{U}$ and suppose that $q(z) \neq 0$ $(z \in \mathcal{U})$ is also univalent in $\mathcal{U}$ and that*

$$\frac{zq'(z)}{q(z)} \text{ is starlike univalent in } \mathcal{U}. \tag{3.1}$$

If $q(z)$ satisfies

$$\Re\left(1 + \frac{c_1}{\beta}q(z) + \frac{2c_2}{\beta}(q(z))^2 + \cdots + \frac{nc_n}{\beta}(q(z))^n - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0 \tag{3.2}$$

$$(z \in \mathfrak{U}; c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0)$$

and

$$\begin{aligned} & c_0 + c_1 p(z) + c_2 (p(z))^2 + \cdots + c_n (p(z))^n + \beta \frac{zp'(z)}{p(z)} \\ \prec \quad & c_0 + c_1 q(z) + c_2 (q(z))^2 + \cdots + c_n (q(z))^n + \beta \frac{zq'(z)}{q(z)} \end{aligned}$$

$$(z \in \mathfrak{U}; c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C}; \beta \neq 0)$$

then $p(z) \prec q(z) \quad (z \in \mathfrak{U})$ *and* q *is the best dominant.*

Proof: Let

$$\theta(\omega) := c_0 + c_1\omega + c_2\omega^2 + \cdots + c_n\omega^n \text{ and } \phi(\omega) := \frac{\beta}{\omega}.$$

Then, we observe that $\theta(\omega)$ is analytic in $\mathbb{C}$, $\phi(\omega)$ is analytic in $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$ and that $\phi(\omega) \neq 0$ $(\omega \in \mathbb{C}^*)$.

Also, by letting

$$Q(z) = zq'(z)\phi(q(z)) = \beta \frac{zq'(z)}{q(z)}$$

and

$$\begin{aligned} h(z) &= \theta(q(z)) + Q(z) \\ &= c_0 + c_1 q(z) + c_2 (q(z))^2 + \cdots + c_n (q(z))^n + \beta \frac{zq'(z)}{q(z)}, \end{aligned}$$

we find from (3.1) and (3.2), $Q(z)$ is starlike univalent in $\mathfrak{U}$ and that

$$\begin{aligned} & \Re\left(\frac{zh'(z)}{Q(z)}\right) \\ = \quad & \Re\left(1 + \frac{a_1}{\beta} q(z) + \frac{2a_2}{\beta}(q(z))^2 + \cdots + \frac{na_n}{\beta}(q(z))^n - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right) > 0 \end{aligned}$$

$$(z \in \mathfrak{U};\ c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0).$$

Our result now follows by an application of Lemma 2.2. □

We first prove the following subordination theorem involving the operator $D^{\zeta}_{m,\lambda}$
.

Theorem 3.2. *Let the function* $q(z)$ *be analytic and univalent in* $\mathfrak{U}$ *such that* $q(z) \neq 0$ $(z \in \mathfrak{U})$. *Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* $\mathfrak{U}$ *and the inequality (3.2) holds true. Let*

$$\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f) := c_0 + c_1\left(\frac{D^{\zeta}_{m,\lambda} f(z)}{z}\right) + c_2\left(\frac{D^{\zeta}_{m,\lambda} f(z)}{z}\right)^2$$

$$+\cdots+c_n\left(\frac{D^{\zeta}_{m,\lambda}f(z)}{z}\right)^n+\frac{\beta}{C^m_j(\lambda)}\left(\frac{D^{\zeta+1}_{m,\lambda}f(z)}{D^{\zeta}_{m,\lambda}f(z)}-(1-C^m_j(\lambda))\right). \quad (3.3)$$

If $q(z)$ satisfies

$$\Omega^m_j(c_0,c_1,c_2,\ldots,c_n,\beta,\zeta,\lambda,f)\prec c_0+c_1q(z)+c_2(q(z))^2+\cdots+c_n(q(z))^n+\beta\frac{zq'(z)}{q(z)} \quad (3.4)$$

$$(z\in\mathcal{U};\ c_0,c_1,c_2,\ldots,c_n,\beta\in\mathbb{C};\ \beta\neq 0)$$

then

$$\frac{D^{\zeta}_{m,\lambda}f(z)}{z}\prec q(z)\quad (z\in\mathcal{U}\backslash\{0\})$$

and q is the best dominant.

Proof: Define the function $p(z)$ by

$$p(z):=\frac{D^{\zeta}_{m,\lambda}f(z)}{z}\qquad (z\in\mathcal{U}\backslash\{0\};\ f\in\mathcal{A}).$$

Then a computation shows that

$$\frac{zp'(z)}{p(z)}=\frac{z(D^{\zeta}_{m,\lambda}f(z))'}{D^{\zeta}_{m,\lambda}f(z)}-1.$$

By using the identity (1.8), we obtain

$$\frac{zp'(z)}{p(z)}=\frac{1}{C^m_j(\lambda)}\left(\frac{D^{\zeta+1}_{m,\lambda}f(z)}{D^{\zeta}_{m,\lambda}f(z)}-(1-C^m_j(\lambda))\right)$$

which, in light the hypothesis (3.4), yields the following subordination

$$\begin{aligned}&c_0+c_1p(z)+c_2(p(z))^2+\cdots+c_n(p(z))^n+\beta\frac{zp'(z)}{p(z)}\\ \prec\ &c_0+c_1q(z)+c_2(q(z))^2+\cdots+c_n(q(z))^n+\beta\frac{zq'(z)}{q(z)}\end{aligned}$$

and Theorem 3.2 follows by an application of Lemma 3.1. □

For the choices $q(z)=\frac{1+Az}{1+Bz}, -1\leq B<A\leq 1$ and $q(z)=\left(\frac{1+z}{1-z}\right)^{\mu}, 0<\mu\leq 1$ in Theorem 3.2, we get Corollaries 3.3 and 3.4 below.

Corollary 3.3. *Assume that (3.2) holds true. If $f \in \mathcal{A}$ and*

$$\begin{aligned} &\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f) \\ \prec\ & c_0 + c_1\left(\frac{1+Az}{1+Bz}\right) + c_2\left(\frac{1+Az}{1+Bz}\right)^2 + \cdots + c_n\left(\frac{1+Az}{1+Bz}\right)^n \\ &+\frac{\beta(A-B)z}{(1+Az)(1+Bz)} \end{aligned}$$

$$(z \in \mathfrak{U};\ c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0),$$

where $\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f)$ is as defined in equation (3.3), then

$$\frac{D^{\zeta}_{m,\lambda} f(z)}{z} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathfrak{U}\backslash\{0\})$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Corollary 3.4. *Assume that (3.2) holds true. If $f \in \mathcal{A}$ and*

$$\begin{aligned} &\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f) \\ \prec\ & c_0 + c_1\left(\frac{1+z}{1-z}\right)^{\mu} + c_2\left(\frac{1+z}{1-z}\right)^{2\mu} + \cdots + c_n\left(\frac{1+z}{1-z}\right)^{2n\mu} + \frac{2\beta\mu z}{1-z^2} \end{aligned}$$

$$(z \in \mathfrak{U};\ c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0),$$

where $\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f)$ is as defined in equation (3.3), then

$$\frac{D^{\zeta}_{m,\lambda} f(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{\mu} \quad (z \in \mathfrak{U}\backslash\{0\})$$

and $\left(\frac{1+z}{1-z}\right)^{\mu}$ is the best dominant.

For $q(z) = e^{\epsilon Az}$, $(|\epsilon A| < \pi)$, in Theorem 3.2, we get the following result.

Corollary 3.5. *Assume that (3.2) holds true. If $f \in \mathcal{A}$ and*

$$\begin{aligned} &\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f) \\ \prec\ & c_0 + c_1 e^{\epsilon Az} + c_2 e^{2\epsilon Az} + \cdots + c_n e^{n\epsilon Az} + \beta\epsilon Az \end{aligned}$$

$$(z \in \mathfrak{U};\ c_0, c_1, c_2, \ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0),$$

where $\Omega_j^m(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f)$ is as defined in equation (3.3), then

$$\frac{D^{\zeta}_{m,\lambda} f(z)}{z} \prec e^{\epsilon Az} \quad (z \in \mathfrak{U}\backslash\{0\})$$

and $e^{\epsilon A z}$ *is the best dominant.*

For $q(z) = \frac{1}{(1-z)^{2b}}, (b \in \mathbb{C}^*)$, $c_0 = \zeta = \lambda = m = 1, c_1 = c_2 = \ldots = c_n = 0$ and $\beta = \frac{1}{b}$ in Theorem 3.2, we get the following result obtained by Srivastava and Lashin [11].

Corollary 3.6. *Let b be a non zero complex number. If $f \in \mathcal{A}$, and*

$$1 + \frac{1}{b}\left[\frac{zf'(z)}{f(z)} - 1\right] \prec \frac{1+z}{1-z},$$

then

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{2b}}$$

and $\frac{1}{(1-z)^{2b}}$ *is the best dominant.*

4. Superordination for Analytic Functions

Next, applying Lemma 2.3, we obtain the following two theorems.

Theorem 4.1. *Let q be analytic and convex univalent in $\mathcal{U}$ such that $q(z) \neq 0$ and $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathcal{U}$. Suppose also that*

$$\Re\left(\frac{c_1}{\beta}q(z) + \frac{2c_2}{\beta}(q(z))^2 + \cdots + \frac{nc_n}{\beta}(q(z))^n\right) > 0 \tag{4.1}$$

$$(z \in \mathcal{U};\ c_1, c_2, \ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0).$$

If $f \in A$,

$$\frac{D^{\zeta}_{m,\lambda}f(z)}{z} \in \mathcal{H}[q(0), 1] \cap Q$$

and $\Omega^m_j(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f)$ defined in (3.3) is univalent in $\mathcal{U}$, then the following superordination:

$$c_0 + c_1q(z) + c_2(q(z))^2 + \cdots + c_n(q(z))^n + \beta\frac{zq'(z)}{q(z)} \prec \Omega^m_j(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f)$$

$$(z \in \mathcal{U};\ c_1, c_2, \ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0).$$

implies that

$$q(z) \prec \frac{D^{\zeta}_{m,\lambda}f(z)}{z} \quad (z \in \mathcal{U}\backslash\{0\})$$

and $q(z)$ is the best subordinant.

Proof: Let

$$\vartheta(\omega) := c_0 + c_1\omega + c_2\omega^2 + \cdots + c_n\omega^n \text{ and } \varphi(\omega) := \beta\frac{\omega'}{\omega}.$$

Then, we observe that $\vartheta(\omega)$ is analytic in $\mathbb{C}$, $\varphi(\omega)$ is analytic in $\mathbb{C}^*= \mathbb{C}\backslash\{0\}$ and that $\varphi(\omega) \neq 0$ $(\omega \in \mathbb{C}^*)$.

Since q is a convex univalent in $\mathcal{U}$, it follows that

$$\Re\left(\frac{\vartheta'(q(z))}{\varphi(q(z))}\right) = \Re\left(\frac{c_1}{\beta}q(z) + \frac{2c_2}{\beta}(q(z))^2 + \cdots + \frac{nc_n}{\beta}(q(z))^n\right) > 0$$

$$(z \in \mathcal{U};\ c_1, c_2, \ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0).$$

Theorem 4.1 follows as an application of Lemma 2.3. □

Combining the results of differential subordination and superordination, we state the following "sandwich results":

Theorem 4.2. *Let q_1 be convex univalent and q_2 be univalent in $\mathcal{U}$ such that $q_1(z) \neq 0$ and $q_2(z) \neq 0$ $(z \in \mathcal{U})$. Suppose also that q_2 satisfies (4.1) and q_1 satisfies (3.2). If $f \in A$,*

$$\frac{D^{\zeta}_{m,\lambda}f(z)}{z} \in \mathcal{H}[q(0), 1] \cap Q$$

and

$$c_0 + c_1\left(\frac{D^{\zeta}_{m,\lambda}f(z)}{z}\right) + c_2\left(\frac{D^{\zeta}_{m,\lambda}f(z)}{z}\right)^2$$
$$+ \cdots + c_n\left(\frac{D^{\zeta}_{m,\lambda}f(z)}{z}\right)^n + \frac{\beta}{C^m_j(\lambda)}\left(\frac{D^{\zeta+1}_{m,\lambda}f(z)}{D^{\zeta}_{m,\lambda}f(z)} - (1 - C^m_j(\lambda))\right)$$
$$(z \in \mathcal{U};\ c_1, c_2, \ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0).$$

is univalent in $\mathcal{U}$, then the subordination given by

$$c_0+c_1q_1(z)+c_2(q_1(z))^2+\cdots+c_n(q_1(z))^n+\beta\frac{zq_1'(z)}{q_1(z)} \prec \Omega^m_j(c_0, c_1, c_2, \ldots, c_n, \beta, \zeta, \lambda, f)$$

$$\prec c_0 + c_1q_2(z) + c_2(q_2(z))^2 + \cdots + c_n(q_2(z))^n + \beta\frac{zq_2'(z)}{q_2(z)}$$
$$(z \in \mathcal{U};\ c_1, c_2, \ldots, c_n, \beta \in \mathbb{C};\ \beta \neq 0).$$

implies that

$$q_1(z) \prec \frac{D^{\zeta}_{m,\lambda}f(z)}{z} \prec q_2(z).$$

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