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Three Weak Solutions for a Class of Neumann Boundary Value Systems Involving the $(p_1,\ldots,p_n)$-Laplacian

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ABSTRACT: In this paper, we establish the existence of two intervals of positive real parameters λ for which a class of Neumann boundary value equations involving the $(p_1,\ldots,p_n)$-Laplacian admits three weak solutions, whose norms are uniformly bounded with respect to λ belonging to one of the two intervals. The approach is based on variational methods.

Key Words: p-Laplacian, Neumann problem, Mmultiplicity results, Variational methods.

Contents

1. Introduction

Throughout the paper, $\Omega\subset\mathbb{R}^N$ $(N\geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$, $p_i>N$ (and $p_i\geq 2$) for $1\leq i\leq n$ are natural numbers and λ is a positive parameter.

The aim of this paper is to investigate the following quasilinear elliptic system

$$\begin{cases}\Delta_{p_i}u_i+\lambda F_{u_i}(x,u_1,\ldots,u_n)=a_i(x)|u_i|^{p_i-2}u_i & \text{in } \Omega,\\ \partial u_i/\partial\nu=0 & \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

for $1\leq i\leq n$, where $\Delta_{p_i}u_i:=\operatorname{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ is the so called p_i-Laplacian operator, $a_1,\ldots,a_n\in L^\infty(\Omega)$ be n functions such that $\min_{1\leq i\leq n}\{\operatorname{ess\,inf}_\Omega a_i\}>0$, $F:\Omega\times\mathbb{R}^n\to\mathbb{R}$ is a function such that the map $x\mapsto F(x,t_1,t_2,\ldots,t_n)$ is measurable in Ω for all $(t_1,\ldots,t_n)\in\mathbb{R}^n$ and the map $(t_1,t_2,\ldots,t_n)\mapsto F(x,t_1,t_2,\ldots,t_n)$ is C^1 in $\mathbb{R}^n$ for a.e. $x\in\Omega$, F_{u_i} denotes the partial derivative of F with respect to u_i, and ν is the outer unite normal to $\partial\Omega$.

Moreover, F satisfy the following additional assumptions:

(F_1) for every $M>0$,

$$\sup_{|(t_1,\ldots,t_n)|\leq M}|F_{u_i}(x,t_1,\ldots,t_n)|\in L^1(\Omega).$$

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(F_2) $F(x, 0, \dots, 0) = 0$ for a.e. $x \in \Omega$.

In recent years, many publications (see, e.g., [2,4,5,6,7,8,9,11,13] and references therein) have appeared concerning quasilinear elliptic systems which have been used in a great variety of applications. Multiplicity results for this kind of systems have been broadly investigated in which the technical approach is based on the three critical points theorems.

Bonanno in [3] established the existence of two intervals of positive real parameters λ for which the functional $\Phi - \lambda J$ has three critical points, whose norms are uniformly bounded with respect to λ belonging to one of the two intervals. He illustrated the result for a two point boundary value problem. In this paper, by assuming that $F(x, \cdot)$ has a $(p-1)$-sublinear growth at ∞ and satisfies a certain local condition near to 0, we prove the existence of two intervals $\Lambda_1^{'}$ and $\Lambda_2^{'}$ such that, for each $\lambda \in \Lambda_1^{'} \cup \Lambda_2^{'}$, the system (1.1) admits at least three weak solutions whose norms are uniformly bounded with respect to $\lambda \in \Lambda_2^{'}$.

This paper is arranged as follows. In Section 2, we recall some basic notations and definitions and our main tool (Theorem 2.1), while Section 3 is devoted to our main result, some consequences and one example that illustrates the result. For a thorough account on the subject, we refer the reader to the very recent monographs [10,12].

2. Preliminaries

First we recall for the reader's convenience Theorem 2.1 of [3] to transfer the existence of three solutions of the system (1.1) into the existence of critical points of the Euler functional. Here, X^* denotes the dual space of X.

Theorem 2.1. *Let X be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a non-negative continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $J : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = J(x_0) = 0$ and that*

(i) $\lim_{\|x\| \to +\infty} (\Phi(x) - \lambda J(x)) = +\infty$ *for all* $\lambda \in [0, +\infty[$.

Further, assume that there are $r > 0$, $x_1 \in X$ such that:

(ii) $r < \Phi(x_1)$;

(iii) $\sup\limits_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x) < \frac{r}{r + \Phi(x_1)} J(x_1)$.

Here $\overline{\Phi^{-1}(]-\infty, r[)}^w$ denotes the closure of $\Phi^{-1}(]-\infty, r[)$ in the weak topology. Then, for each

$$\lambda \in \Lambda_1 := \left] \frac{\Phi(x_1)}{J(x_1) - \sup\limits_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x)}, \frac{r}{\sup\limits_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x)} \right[,$$

the equation

$$\Phi'(x) - \lambda J'(x) = 0 \tag{2.1}$$

has at least three solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2 \subseteq \left[0, \frac{hr}{r\frac{J(x_1)}{\Phi(x_1)} - \sup_{x \in \overline{\Phi^{-1}(]-\infty, r[)}^w} J(x)}\right]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2$, the equation (2.1) has at least three solutions in X whose norms are less than σ.

In the sequel, X will denote the Cartesian product of the n Sobolev spaces $W^{1,p_i}(\Omega)$ for $1 \leq i \leq n$, i.e., $X = W^{1,p_1}(\Omega) \times W^{1,p_2}(\Omega) \times \cdots \times W^{1,p_n}(\Omega)$ equipped with the norm

$$\|(u_1, u_2, \ldots, u_n)\| := \sum_{i=1}^{n} \|u_i\|_{p_i},$$

where

$$\|u_i\|_{p_i} := \left(\int_\Omega |\nabla u_i(x)|^{p_i} dx + \int_\Omega a_i(x)|u_i(x)|^{p_i} dx \right)^{1/p_i}$$

for $1 \leq i \leq n$, which is equivalent to the usual one.

Put

$$k := \max \left\{ \sup_{u_i \in W^{1,p_i}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}} : \text{ for } 1 \leq i \leq n \right\}. \tag{2.2}$$

Since $p_i > N$ for $1 \leq i \leq n$, the embedding $X \hookrightarrow (C^0(\overline{\Omega}))^n$ is compact, and so $k < +\infty$. It follows from Proposition 4.1 of [1] that

$$\sup_{u_i \in W^{1,p_i}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u_i(x)|^{p_i}}{\|u_i\|_{p_i}^{p_i}} > \frac{1}{\|a_i\|_1} \quad \text{for } \ 1 \leq i \leq n,$$

where $\|a_i\|_1 := \int_\Omega |a_i(x)| dx$ for $1 \leq i \leq n$, and so $\frac{1}{\|a_i\|_1} \leq k$ for $1 \leq i \leq n$. In addition, if Ω is convex, it is known [1] that

$$\sup_{u_i \in W^{1,p_i}(\Omega) \setminus \{0\}} \frac{\max_{x \in \overline{\Omega}} |u_i(x)|}{\|u_i\|_{p_i}} \leq 2^{\frac{p_i - 1}{p_i}} \max \left\{ \left(\frac{1}{\|a_i\|_1} \right)^{\frac{1}{p_i}}, \frac{\operatorname{diam}(\Omega)}{N^{\frac{1}{p_i}}} \left(\frac{p_i - 1}{p_i - N} m(\Omega) \right)^{\frac{p_i - 1}{p_i}} \frac{\|a_i\|_\infty}{\|a_i\|_1} \right\}$$

for $1 \leq i \leq n$, where $m(\Omega)$ is the Lebesgue measure of the set Ω, and equality occurs when Ω is a ball.

We recall that a function $u = (u_1, \ldots, u_n) \in X$ is said to be a (weak) solution of the system (1.1) if

$$\begin{aligned} \int_\Omega \sum_{i=1}^{n} |\nabla u_i(x)|^{p_i - 2} \nabla u_i(x) \nabla v_i(x) dx \quad &- \quad \lambda \int_\Omega \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \ldots, u_n(x)) v_i(x) dx \\ &+ \quad \int_\Omega \sum_{i=1}^{n} a_i(x) |u_i(x)|^{p_i - 2} u_i(x) v_i(x) dx = 0 \end{aligned}$$

for all $v = (v_1, \ldots, v_n) \in X$.

For all $c > 0$ we denote by $K(c)$ the set

$$\left\{(t_1, \ldots, t_n) \in \mathbb{R}^n \ : \ \sum_{i=1}^{n} \frac{|t_i|^{p_i}}{p_i} \le c\right\}.$$

This set will be used in some of our hypotheses with appropriate choices of c.

3. Main results

Our main result is the following theorem.

Theorem 3.1. *Assume that there exist $n+1$ positive constants r and s_i for $1 \le i \le n$, with $s_i < p_i$ for $1 \le i \le n$, and two functions $\alpha \in L^1(\Omega)$ and $w = (w_1, \ldots, w_n) \in X$ such that*

(j) $\sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i} > r$;

(jj) $$\int_\Omega \sup_{(t_1,\ldots,t_n)\in K(kr)} F(x, t_1, \ldots, t_n)dx < \frac{r \displaystyle\int_\Omega F(x, w_1(x), \ldots, w_n(x))dx}{2\displaystyle\sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i}};$$

(jjj) $F(x, t_1, \ldots, t_n) \le \alpha(x)\left(1 + \sum_{i=1}^{n} |t_i|^{s_i}\right)$ *for a.e.* $x \in \Omega$ *and all* $(t_1, \ldots, t_n) \in \mathbb{R}^n$.

Then, for each

$$\lambda \in \Lambda_1' := \left] \frac{\displaystyle\sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i}}{\displaystyle\int_\Omega F(x, w_1(x), \ldots, w_n(x))dx - \int_\Omega \sup_{(t_1,\ldots,t_n)\in K(kr)} F(x, t_1, \ldots, t_n)dx}, \right.$$
$$\left. \frac{r}{\displaystyle\int_\Omega \sup_{(t_1,\ldots,t_n)\in K(kr)} F(x, t_1, \ldots, t_n)dx} \right[,$$

the system (1.1) admits at least three weak solutions in X and, moreover, for each $h > 1$, there exist an open interval

$$\Lambda_2' \subseteq \left[0, \frac{hr}{r\frac{\int_\Omega F(x, w_1(x), \ldots, w_n(x))dx}{\sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i}} - \displaystyle\int_\Omega \sup_{(t_1,\ldots,t_n)\in K(kr)} F(x, t_1, \ldots, t_n)dx}\right]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2'$, the system (1.1) admits at least three weak solutions in X whose norms are less than σ.

Proof: In order to apply Theorem 2.1, we begin by setting

$$\Phi(u) := \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} \tag{3.1}$$

and

$$J(u) := \int_\Omega F(x, u_1(x), \dots, u_n(x))dx \tag{3.2}$$

for all $u = (u_1, \dots, u_n) \in X$. It is known that Φ and J are well defined and continuously Gâteaux differentiable functionals with

$$\begin{aligned}\Phi'(u)(v) \quad &= \quad \int_\Omega \sum_{i=1}^{n} |\nabla u_i(x)|^{p_i-2} \nabla u_i(x) \nabla v_i(x) dx \\ &\quad + \int_\Omega \sum_{i=1}^{n} a_i(x)|u_i(x)|^{p_i-2} u_i(x) v_i(x) dx\end{aligned}$$

and

$$J'(u)(v) = \int_\Omega \sum_{i=1}^{n} F_{u_i}(x, u_1(x), \dots, u_n(x)) v_i(x) dx$$

for every $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in X$, as well as Φ is sequentially weakly lower semicontinuous (see Proposition 25.20 of [15]). Also, $\Phi' : X \to X^*$ is a uniformly monotone operator in X (for more details, see (2.2) of [14]), and since Φ' is coercive and hemicontinuous in X, by applying Theorem 26.A of [15], Φ' admits a continuous inverse on X^*.

We claim that $J' : X \to X^*$ is a compact operator. To this end, it is enough to show that J' is strongly continuous on X. For this, for fixed $(u_1, \dots, u_n) \in X$, let $(u_{1m}, \dots, u_{nm}) \to (u_1, \dots, u_n)$ weakly in X as $m \to +\infty$. Then we have $(u_{1m}, \dots, u_{nm})$ converges uniformly to $(u_1, \dots, u_n)$ on Ω as $m \to +\infty$ (see [15]). Since $F(x, \cdot, \dots, \cdot)$ is C^1 in $\mathbb{R}^n$ for every $x \in \Omega$, the derivatives of F are continuous in $\mathbb{R}^n$ for every $x \in \Omega$, so for $1 \le i \le n$, $F_{u_i}(x, u_{1m}, \dots, u_{nm}) \to F_{u_i}(x, u_1, \dots, u_n)$ strongly as $m \to +\infty$. By the Lebesgue control convergence theorem, $J'(u_{1m}, \dots, u_{nm}) \to J'(u_1, \dots, u_n)$ strongly as $m \to +\infty$. Thus we proved that J' is strongly continuous on X, which implies that J' is a compact operator by [15, Proposition 26.2]. Hence the claim is true.

Thanks to the assumption (jjj), for each $\lambda > 0$ one has

$$\lim_{\|u\| \to +\infty} (\Phi(u) - \lambda J(u)) = +\infty.$$

Also, from (j) and (3.1) we get $\Phi(w) > r$. Due to (2.2), for each $u_i \in W^{1,p_i}(\Omega)$

$$\sup_{x \in \Omega} |u_i(x)|^{p_i} \le k \|u_i\|_{p_i}^{p_i}$$

for $1 \le i \le n$, so we have

$$\sup_{x \in \Omega} \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \le k \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} = k\Phi(u) \tag{3.3}$$

for every $u = (u_1, \dots, u_n) \in X$. From (3.3), for each $r > 0$ we obtain

$$\begin{aligned}
\Phi^{-1}(]-\infty, r]) &= \left\{ u = (u_1, \dots, u_n) \in X : \Phi(u) \le r \right\} \\
&= \left\{ u = (u_1, \dots, u_n) \in X : \sum_{i=1}^{n} \frac{\|u_i\|_{p_i}^{p_i}}{p_i} \le r \right\} \\
&\subseteq \left\{ u = (u_1, \dots, u_n) \in X : \sum_{i=1}^{n} \frac{|u_i(x)|^{p_i}}{p_i} \le kr \ \text{ for all } x \in \Omega \right\},
\end{aligned}$$

and, since $\overline{\Phi^{-1}(]-\infty, r[)}^{w} = \Phi^{-1}(]-\infty, r])$, owing to our assumptions, we have

$$\begin{aligned}
\sup_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} J(u) &\le \int_\Omega \sup_{(t_1,\dots,t_n) \in K(kr)} F(x, t_1, \dots, t_n)\,dx \\
&< \frac{r \displaystyle\int_\Omega F(x, w_1(x), \dots, w_n(x))\,dx}{2 \displaystyle\sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i}} \\
&< r \frac{\displaystyle\int_\Omega F(x, w_1(x), \dots, w_n(x))\,dx}{r + \displaystyle\sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i}} \\
&= \frac{r}{r + \Phi(w)} J(w).
\end{aligned}$$

We can apply Theorem 2.1 at this point and obtain two intervals Λ_1 and Λ_2 such that if $\lambda \in \Lambda_1 \cup \Lambda_2$, then system (1.1) has at least three weak solutions. Next, we derive the upper and lower bounds of Λ_1 and Λ_2. For each $x \in \Omega$ we have

$$\frac{r}{\displaystyle\sup_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} J(u)} \ge \frac{r}{\displaystyle\int_\Omega \sup_{(t_1,\dots,t_n) \in K(kr)} F(x, t_1, \dots, t_n)\,dx}$$

and

$$\begin{aligned}
&\frac{\Phi(w)}{J(w) - \displaystyle\sup_{u \in \overline{\Phi^{-1}(]-\infty, r[)}^{w}} J(u)} \\
&\le \frac{\displaystyle\sum_{i=1}^{n} \frac{\|w_i\|_{p_i}^{p_i}}{p_i}}{\displaystyle\int_\Omega F(x, w_1(x), \dots, w_n(x))\,dx - \int_\Omega \sup_{(t_1,\dots,t_n) \in K(kr)} F(x, t_1, \dots, t_n)\,dx}.
\end{aligned}$$

Note also that (jj) immediately implies

$$\begin{aligned}
&\frac{\displaystyle\sum_{i=1}^{n}\frac{\|w_i\|_{p_i}^{p_i}}{p_i}}{\displaystyle\int_\Omega F(x,w_1(x),\dots,w_n(x))dx-\int_\Omega \sup_{(t_1,\dots,t_n)\in K(kr)} F(x,t_1,\dots,t_n)dx}\\
&<\frac{\displaystyle\sum_{i=1}^{n}\frac{\|w_i\|_{p_i}^{p_i}}{p_i}}{\left(\dfrac{2\sum_{i=1}^{n}\frac{\|w_i\|_{p_i}^{p_i}}{p_i}}{r}-1\right)\displaystyle\int_\Omega \sup_{(t_1,\dots,t_n)\in K(kr)} F(x,t_1,\dots,t_n)dx}\\
&<\frac{\displaystyle\sum_{i=1}^{n}\frac{\|w_i\|_{p_i}^{p_i}}{p_i}}{\left(\dfrac{r+\sum_{i=1}^{n}\frac{\|w_i\|_{p_i}^{p_i}}{p_i}}{r}-1\right)\displaystyle\int_\Omega \sup_{(t_1,\dots,t_n)\in K(kr)} F(x,t_1,\dots,t_n)dx}\\
&=\frac{r}{\displaystyle\int_\Omega \sup_{(t_1,\dots,t_n)\in K(kr)} F(x,t_1,\dots,t_n)dx}.
\end{aligned}$$

Also

$$\begin{aligned}
&\frac{hr}{r\frac{J(w)}{\Phi(w)}-\displaystyle\sup_{u\in\overline{\Phi^{-1}(]-\infty,r[)}^{w}} J(u)}\\
&\leq \frac{hr}{r\dfrac{\displaystyle\int_\Omega F(x,w_1(x),\dots,w_n(x))dx}{\displaystyle\sum_{i=1}^{n}\frac{\|w_i\|_{p_i}^{p_i}}{p_i}}-\displaystyle\int_\Omega \sup_{(t_1,\dots,t_n)\in K(kr)} F(x,t_1,\dots,t_n)dx}\\
&=\rho.
\end{aligned}$$

So

$$\int_\Omega \sup_{(t_1,\dots,t_n)\in K(kr)} F(x,t_1,\dots,t_n)dx<r\frac{\displaystyle\int_\Omega F(x,w_1(x),\dots,w_n(x))dx}{\sum_{i=1}^{n}\frac{\|w_i\|_{p_i}^{p_i}}{p_i}},$$

and now apply (jj). Thus, by choosing $x_0=0$, $x_1=w$, from Theorem 2.1 it follows that, for each $\lambda\in\Lambda_1'$ the system (1.1) admits at least three weak solutions and there exist an open interval $\Lambda_2'\subseteq[0,\rho]$ and a real positive number σ such that, for each $\lambda \in \Lambda_2^{'}$, the system (1.1) admits at least three weak solutions whose norms in X are less than σ. $\square$

Now, we give a particular consequence of Theorem 3.1 for a fixed test function w. Moreover, F dose not depend on $x \in \Omega$.

Corollary 3.2. *Let $F : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function and assume that there exist $n+3$ positive constants γ, δ, α and s_i for $1 \le i \le n$, with $s_i < p_i$ for $1 \le i \le n$, such that*

(k) $\sum_{i=1}^n \frac{\delta^{p_i}}{p_i} > \frac{\gamma}{\prod_{i=1}^n p_i}$;

(kk) $\max_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(t_1,\ldots,t_n) < \frac{\gamma}{2k\prod_{i=1}^n p_i} \frac{F(\delta,\ldots,\delta)}{\sum_{i=1}^n \frac{\delta^{p_i}}{p_i}\|a_i\|_1}$;

(kkk) $F(t_1,\ldots,t_n) \le \alpha(1+\sum_{i=1}^n |t_i|^{s_i})$ *for all* $(t_1,\ldots,t_n) \in \mathbb{R}^n$.

Then, for each

$$\lambda \in \Lambda_1^{'} := \left] \frac{\sum_{i=1}^n \frac{\delta^{p_i}}{p_i}\|a_i\|_1}{m(\Omega)\Big(F(\delta,\ldots,\delta) - \max_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(t_1,\ldots,t_n)\Big)}, \right.$$
$$\left. \frac{\frac{\gamma}{k\prod_{i=1}^n p_i}}{m(\Omega)\max_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(t_1,\ldots,t_n)} \right[,$$

the system

$$\begin{cases} \Delta_{p_i} u_i + \lambda F_{u_i}(u_1,\ldots,u_n) = a_i(x)|u_i|^{p_i-2}u_i & \textit{in } \Omega, \\ \partial u_i/\partial \nu = 0 & \textit{on } \partial\Omega, \end{cases} \tag{3.4}$$

for $1 \le i \le n$, admits at least three weak solutions in X and, moreover, for each $h > 1$, there exist an open interval

$$\Lambda_2^{'} \subseteq \left[0, \frac{\frac{h\gamma}{k\prod_{i=1}^n p_i}}{m(\Omega)\Big(\frac{\gamma}{k\prod_{i=1}^n p_i}\frac{F(\delta,\ldots,\delta)}{\sum_{i=1}^n \frac{\delta^{p_i}}{p_i}\|a_i\|_1} - \max_{(t_1,\ldots,t_n)\in K(\frac{\gamma}{\prod_{i=1}^n p_i})} F(t_1,\ldots,t_n)\Big)}\right]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2^{'}$, the system (3.4) admits at least three weak solutions in X whose norms are less than σ.

Proof: We prove that all assumptions of Theorem 3.1 are fulfilled with $w(x) := (\delta,\ldots,\delta)$ and $r := \frac{\gamma}{k\prod_{i=1}^n p_i}$. If we put $w(x) := (\delta,\ldots,\delta)$ for each $x \in \Omega$, then we have $\|w_i\|_{p_i} = \|a_i\|_1^{\frac{1}{p_i}}\delta$ for $1 \le i \le n$. By (k) and the fact that $\frac{1}{\|a_i\|_1} \le k$ for $1 \le i \le n$, we get $\Phi(w) = \sum_{i=1}^n \frac{\delta^{p_i}}{p_i}\|a_i\|_1 > r$. The other assumptions of Theorem 3.1 are clearly satisfied. $\square$

Corollary 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(t) = \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. Assume that there exist four positive constants γ, δ, α and s with $\delta^p > \gamma$ and $s < p$, such that*

(l) $\max_{t\in[-\sqrt[p]{\gamma},\, \sqrt[p]{\gamma}]} F(t) < \frac{F(\delta)}{2k\|a\|_1}$;

(ll) $F(t) \leq \alpha(1 + |t|^s)$ *for all* $t \in \mathbb{R}$.

Then, for each

$$\lambda \in \Lambda_1^{'} := \left] \frac{\delta^p \|a\|_1}{p\, m(\Omega)\Big(F(\delta) - \max_{t\in[-\sqrt[p]{\gamma},\, \sqrt[p]{\gamma}]} F(t)\Big)}, \frac{\gamma}{(kp)m(\Omega) \max_{t\in[-\sqrt[p]{\gamma},\, \sqrt[p]{\gamma}]} F(t)} \right[,$$

the problem

$$\begin{cases} \Delta_p u + \lambda f(u) = a(x)|u|^{p-2}u & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{3.5}$$

admits at least three weak solutions in $W^{1,p}(\Omega)$ and, moreover, for each $h > 1$, there exist an open interval

$$\Lambda_2^{'} \subseteq \left[0, \frac{h\gamma}{(kp)m(\Omega)\Big(\frac{\gamma F(\delta)}{k\delta^p\|a\|_1} - \max_{t\in[-\sqrt[p]{\gamma},\, \sqrt[p]{\gamma}]} F(t)\Big)}\right]$$

and a positive real number σ such that, for each $\lambda \in \Lambda_2^{'}$, the problem (3.5) admits at least three weak solutions in $W^{1,p}(\Omega)$ whose norms are less than σ.

Finally, we present the application of Theorem 3.1 in the ordinary case with $p = 2$, that Example 3.5 illustrates the result. For simplicity, we put $\Omega = (0, 1)$. Note that in this situation we have

$$k = 2\max\{\|a\|_1^{-1}, \|a\|_\infty^2 \|a\|_1^{-2}\}.$$

Corollary 3.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $F(t) = \int_0^t f(\xi)d\xi$ for each $t \in \mathbb{R}$. Assume that there exist four positive constants γ, δ, α and s with $\delta^2 > \gamma$ and $s < 2$, such that assumption (ll) in Corollary 3.3 holds, and*

(m) $\max_{t\in[-\sqrt{\gamma},\sqrt{\gamma}]} F(t) < \frac{F(\delta)}{2k\|a\|_1}$.

Then, for each

$$\lambda \in \Lambda_1^{'} := \left] \frac{\delta^2 \|a\|_1}{2\Big(F(\delta) - \max_{t\in[-\sqrt{\gamma},\sqrt{\gamma}]} F(t)\Big)}, \frac{\gamma}{2k \max_{t\in[-\sqrt{\gamma},\sqrt{\gamma}]} F(t)} \right[,$$

the problem

$$\begin{cases} u'' + \lambda f(u) = a(x)u & \text{in } (0, 1), \\ u'(0) = u'(1) = 0 \end{cases} \tag{3.6}$$

admits at least three classical solutions in $C^2([0,1])$ *and, moreover, for each* $h > 1$*, there exist an open interval*

$$\Lambda_2' \subseteq \left[0, \frac{h\gamma}{2k\left(\frac{\gamma F(\delta)}{k\delta^2\|a\|_1} - \max_{t\in[-\sqrt{\gamma},\sqrt{\gamma}]} F(t)\right)}\right]$$

and a positive real number σ *such that, for each* $\lambda \in \Lambda_2'$*, the problem (3.6) admits at least three classical solutions in* $C^2([0,1])$ *whose norms are less than* σ*.*

Example 3.5. *Consider the problem*

$$\begin{cases} u'' + \lambda e^{-u}u^{11}(12-u) = xu & \text{in } (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{3.7}$$

Set $f(t) = e^{-t}t^{11}(12-t)$ *for all* $t \in \mathbb{R}$*. A direct calculation yields* $F(t) = e^{-t}t^{12}$ *for all* $t \in \mathbb{R}$*. Note that by choosing* $\delta = 2$*,* $\gamma = 1$ *and* $a(x) = x$*, we have* $k = 8$*. A simple computation shows*

$$\frac{F(\delta)}{2k\|a\|_1} - \max_{t\in[-\sqrt{\gamma},\sqrt{\gamma}]} F(t) = \frac{2^9}{e^2} - e > 0.$$

Moreover, with $s = 1$ *and* α *sufficiently large, the assumption (ll) is satisfied. So, by Corollary 3.4, for each* $\lambda \in \Lambda_1' :=]\frac{1}{2^{12}e^{-2}-e}, \frac{1}{16\,e}[$*, the problem (3.7) admits at least three classical solutions in* $C^2([0,1])$ *and, moreover, for each* $h > 1$*, there exist an open interval* $\Lambda_2' \subseteq \left[0, \frac{h}{16(2^8e^{-2}-e)}\right]$ *and a positive real number* σ *such that, for each* $\lambda \in \Lambda_2'$*, the problem (3.7) admits at least three classical solutions in* $C^2([0,1])$ *whose norms are less than* σ*.*

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