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A Study on a Class of Modified Bessel-Type Integrals in a Fréchet Space of Boehmians

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ABSTRACT: In this paper, an attempt is being made to discuss a class of modified Bessel- type integrals on a set of generalized functions known as Boehmians. We show that the modified Bessel-type integral, with appropriately defined convolution products, obeys a fundamental convolution theorem which consequently justifies pursuing analysis in the Boehmian spaces. We describe two Fréchet spaces of Boehmians and extend the modified Bessel-type integral between the different spaces. Furthermore, a convolution theorem and a class of basic properties of the extended integral such as linearity, continuity and compatibility with the classical integral, which provide a convenient extension to the classical results, have been derived..

Key Words: Bessel-type integral transform, Boehmians, Convolutions.

Contents

1. Introduction

Bessel type functions have received an importance due to their frequent use in mathematics and physical applications. In physical applications, the modified Bessel type functions of the third kind appear as solutions of certain radial Schrödinger equations and as Dirichlet problems with boundary conditions on a wedge. Besides, they play an important role in diffraction and hydrodynamics problems and are the approximant in certain uniform asymptotic expansions as well. More and above, Bessel type functions serve as kernels of various integral transforms as in the case of Kantorovich-Lebedev transform [1], Hankel transform [20, 21], extended Hankel transform [4], Y_η-transform [22], Struve transform [8],

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Mejer K_η-transform [23], Bessel-type K_η^ρ and $L_\eta^{(m)}$ [1] transforms, modified Bessel-type transform $L_{\gamma,\sigma}^{\beta}$ [3], Lommel-Maitland transform [14], Hardy-Titchmarsh transform [17], generalized Hardy-Titchmarsh transform [15] and few others.

Generalized functions which are continuous linear functionals over a space of infinitely differentiable functions are useful in making discontinuous functions more likely smooth and describe physical phenomena as point charges that consequently led to an extensive use in applied physics and engineering problems. The recent space of generalized functions known as the Boehmian space is defined by an algebraic construction which is alike to that of field of quotients $[2, 5, 7, 10-13, 20, 21]$. When the construction is applied to various function spaces and the multiplication is interpreted as a convolution, the construction yields various spaces of generalized functions. Delta sequences whose supports shrink to the origin are used in the construction of the Boehmian spaces. This indeed led, among other things, to a uniqueness theorem which can be interpreted as an uncertainty principle for Boehmian spaces. However, as Boehmians have an abstract algebraic definition they allow different interpretations of those extended transforms to be isomorphisms between the different spaces of Boehmians.

The purpose of this article, as part of long term research, is to show that the modified Bessel-type integral has a real extension into appropriately defined spaces of Boehmians and to define an isomorphism between the different spaces. We spread the article into six sections. In Section 2 we recall some definitions and notations from the distributional context. In Section 3 we introduce convolution products and delta sequences which provide settings for the construction of Boehmian spaces where the modified integral can be defined. In Section 4 we prove some further necessary results. Sections 5 and 6 are devoted for discussions and conclusions.

2. Definitions and auxiliary results

Let $\beta > 0; \mathrm{Re}\,(\gamma) > \frac{1}{\beta} - 1; \sigma \in \mathbb{R}; \mathrm{Re}\,(z) > 0$. Then a refined generalization of the modified Bessel type function of the third kind or Macdonald function is given by [25]

$$\lambda_{\gamma,\sigma}^{(\beta)}(z) = \frac{\beta}{\Gamma\left(\gamma + 1 - \frac{1}{\beta}\right)} \int_1^\infty \left(t^\beta - 1\right)^{\gamma - \frac{1}{\beta}} t^\sigma e^{-zt} dt. \tag{1}$$

The asymptotic behaviour of the modified function $\lambda_{\gamma,\sigma}^{(\beta)}$ near zero and infinity is given as follows $[25, (2.8), (2.9)]$

If $\beta \in \mathbb{R}^+, \sigma \in \mathbb{R}$ and $\gamma \in \mathbb{C}$, $\mathrm{Re}\,(\gamma) > \frac{1}{\beta} - 1$. Then $\lambda_{\gamma,\sigma}^{(\beta)}(z) \sim A\,(z \to 0)$ where $A = \frac{\Gamma\left(-\gamma - \frac{\sigma}{\beta}\right)}{\Gamma\left(1 - \frac{(\sigma+1)}{\beta}\right)}$ and that

$$\lambda_{\gamma,\sigma}^{(\beta)}(z) \sim B e^{-z} z^{-1-\gamma+\frac{1}{\beta}}\,(z \to \infty)$$

where $\mathrm{Re}\,(\gamma) < \dfrac{-\sigma}{\beta}$ and $B = \beta^{1+\gamma-\frac{1}{\beta}}$.

The mth derivative of the function $\lambda_{\gamma,\sigma}^{(\beta)}$ is given as follows [25, (2.13)]

If $\beta \in \mathbb{R}^+, \gamma \in \mathbb{C}, \sigma \in \mathbb{R}$ and $\operatorname{Re}(\gamma) > \frac{1}{\beta} - 1$, then for every $m \in \mathbb{N}$, we have

$$\left(\frac{d}{dx}\right)^m \lambda_{\gamma,\sigma}^{(\beta)}(x) = (-1)^m \lambda_{\gamma,\sigma+m}^{(\beta)}(x).$$

A generalization of the Bessel-type integral

$$\left(L_{\gamma,f}^{(n)}\right)(x) = \int_0^\infty \lambda_\gamma^{(n)}(xt) f(t)\, dt \;\; (x > 0),$$

with the kernel function

$$\lambda_\gamma^{(n)}(z) = \frac{(2\pi)^{\frac{(n-1)}{2}}\sqrt{n}}{\Gamma\left(\gamma + 1 - \frac{1}{n}\right)}\left(\frac{z}{n}\right)^{\gamma_n}\int_1^\infty (t^n - 1)^{\gamma - \frac{1}{n}} e^{-zt} dt \;\left(\operatorname{Re}(\gamma) > \frac{1}{n} - 1\right)$$

is known as the modified Bessel-type integral associated with $\lambda_{\gamma,\sigma}^{(\beta)}$ is defined by

$$\left(L_{\gamma,\sigma}^{(\beta)} f\right)(x) = \int_0^\infty \lambda_{\gamma,\sigma}^{(\beta)}(xt) f(t)\, dt \;\; (x > 0). \tag{2}$$

Under suitable conditions of fractional differentiation, compositions of $L_{\gamma,\sigma}^{(\beta)}$ with the left-sided $\left(I_{0+}^\alpha \text{ and } D_{0+}^\alpha\right)$ and the right-sided $\left(I_-^\alpha \text{ and } D_-^\alpha\right)$ Liouville fractional integrals and derivatives;

$$\left(I_{0+}^\alpha \varphi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x \frac{\varphi(t)\, dt}{(x-t)^{1-\alpha}}, \left(D_{0+}^\alpha \varphi\right)(x) = \left(\frac{d}{dx}\right)^{[\alpha]+1}\left(I_{0+}^{1-[\alpha]}\varphi\right)(x),$$
$$\left(I_-^\alpha \varphi\right)(x) = \frac{1}{\Gamma(\alpha)}\int_x^\infty \frac{\varphi(t)\, dt}{(t-x)^{1-\alpha}}, \left(D_-^\alpha \varphi\right)(x) = \left(-\frac{d}{dx}\right)^{[\alpha]+1}\left(I_-^{1-[\alpha]}\varphi\right)(x),$$

are given as [25, (1.7) - (1.10)]

$$\begin{aligned} L_{\gamma,\sigma}^{(\beta)} I_{0+}^\alpha \varphi &= x^{-\alpha} L_{\gamma,\sigma-\alpha}^{(\beta)}\varphi,\; L_{\gamma,\sigma}^{(\beta)} D_{0+}^\alpha \varphi = x^{-\alpha} L_{\gamma,\sigma+\alpha}^{(\beta)}\varphi, \\ I_-^\alpha L_{\gamma,\sigma}^{(\beta)}\varphi &= L_{\gamma,\sigma-\alpha}^{(\beta)} x^{-\alpha}\varphi,\; D_-^\alpha L_{\gamma,\sigma}^{(\beta)}\varphi = L_{\gamma,\sigma+\alpha}^{(\beta)} x^\alpha \varphi, \end{aligned}$$

where $\alpha, x > 0$, $[\alpha]$ and $\{\alpha\}$ are the integral and fractional parts of α, respectively. For $\mu \in \mathbb{C}, k \in \mathbb{N}_0$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $1 \le p < \infty$, the space $F_{p,\mu}$ is defined by

$$F_{p,\mu} = \left\{\varphi \in C_0^\infty\left(\mathbb{R}^+\right) : x^k \frac{d^k}{dx^k}\left(x^{-\mu}\varphi(x)\right) \in L^p\left(\mathbb{R}^+\right) \;\; (k \in \mathbb{N}_0)\right\} \tag{3}$$

where the space $F_{\infty,\mu}$ is defined by

$$F_{\infty,\mu} = \left\{\varphi \in C_0^\infty\left(\mathbb{R}^+\right) : x^k \frac{d^k}{dx^k}\left(x^{-\mu}\varphi(x)\right) \to 0 \text{ as } x \to 0 \text{ and } x \to \infty\right\}. \tag{4}$$

With the topology generated by the family $\{\gamma_k^{p,\mu}\}$ of seminorms

$$\gamma_k^{p,\mu}(\varphi) = \left\| x^k \frac{d^k}{dx^k}\left(x^{-\mu}\varphi\right)\right\|_p \qquad (k \in \mathbb{N}_0),$$

$F_{p,\mu}$ defines a Fréchet space with the usual L_p-norm $\|.\|_p$.
Now we deem it proper to recall some results we request for the next discussion [25].

Lemma 1 The space $C_0^\infty(\mathbb{R}^+)$ of infinitely differentiable functions with compact supports over $\mathbb{R}^+$ is dense in $F_{p,\mu}$ for any $1 \le p \le \infty$ and $\mu \in \mathbb{C}$.

Lemma 2 [25,Theorem 4.1]**.** If $1 \le p \le \infty, \beta \in \mathbb{R}^+, \gamma, \mu \in \mathbb{C}, \sigma \in \mathbb{R}, (\sigma < \beta - 1)$ and $\frac{1}{\beta} - 1 < \operatorname{Re}(\gamma) - \frac{\sigma}{\beta}$, $\operatorname{Re}(\mu) > -\frac{1}{p}$, then $L_{\gamma,\sigma}^{(\beta)}$ is a continuous linear mapping from $F_{p,\mu}$ into $F_{p,\frac{2}{p}+\mu-1}$.

More on this theory, the Parseval's formula for $L_{\gamma,\sigma}^{(\beta)}$ is given as follows.

Lemma 3 [25,Theorem 4.2]**.** Let $1 \le p \le \infty$, $1 \le q \le \infty, \frac{1}{p} + \frac{1}{q} = 1, \beta \in \mathbb{R}^+$, $\gamma, \mu \in \mathbb{C}$, $\sigma \in \mathbb{R}$, $(\sigma < \beta - 1)$ and let $\frac{1}{\beta} - 1 < \operatorname{Re}(\gamma) < -\frac{\sigma}{\beta}$, $\operatorname{Re}(\mu) > -\frac{1}{q}$. Then, for $\varphi \in F_{p,\mu}$ and $\psi \in F_{q,\frac{2}{q}+\mu-1}$, there holds a formula of integration as

$$\int_0^\infty \left(L_{\gamma,\sigma}^{(\beta)}\psi\right)(x)\,\varphi(x)\,dx = \int_0^\infty \psi(x) \int_0^\infty \left(L_{\gamma,\sigma}^{(\beta)}\varphi\right)(x)\,dx.$$

By aid of Lemma 3, the operator $L_{\gamma,\sigma}^{(\beta)}$ of a distribution $f \in \acute{F}_{p,\mu}$ (the dual of $F_{p,\mu}$) can be defined as

$$\left\langle L_{\gamma,\sigma}^{(\beta)} f, \varphi\right\rangle = \left\langle f, L_{\gamma,\sigma}^{(\beta)}\varphi\right\rangle, \ \forall \varphi \in F_{p,\frac{2}{p}-\mu-1} \ (1 \le p \le \infty;\ \mu \in \mathbb{C}).$$

In a distributional sense, the right hand side of the above equation shows that $L_{\gamma,\sigma}^{(\beta)}\varphi \in F_{p,\mu}$, for every choice of $\varphi \in F_{p,\frac{2}{p}-\mu-1}$ $(1 \le p \le \infty;\ \mu \in \mathbb{C})$ which justify an existence and correctness of the inner product. Moreover, it can be easily inferred from the left hand side of the inner product that $L_{\gamma,\sigma}^{(\beta)} f \in \acute{F}_{p,\frac{2}{p}-\mu-1}$ for every $f \in \acute{F}_{p,\mu}$.

3. Extended spaces and integrals

In this section we aim to generate the Boehmian spaces β_1 and β_2. The space β_1 will be generated from the Fréchet space $F_{\rho,\mu}$, the dense subspace C_0^∞ and the shrinking set Δ of delta sequences $\{\delta_n\}$ such that:
$(i)\,\delta_n \in C_0^\infty$, $\forall n \in \mathbb{N}$ $(ii)\int_0^\infty \delta_n = 1$, $\forall n \in \mathbb{N}$ $(iii)\,|\delta_n| < \infty$, $\forall n \in \mathbb{N}$ and $(iv)\operatorname{supp}\delta_n(x) \subseteq (a_n, b_n)$, $\lim\{a_n\} = \lim\{b_n\} = 0$.
Now, we divine convolution products that we claim their appropriateness for our results .

Definition 4. (i) Let f and g be integrable functions on $\mathbb{R}^+$. Then between f and g we define a convolution product $\diamond$ as follows

$$(f \diamond g)(y) = \int_0^\infty f(xy)\, g(x)\, dx. \tag{5}$$

The complementary convolution product should be defined as follows [9]

$$(f \circ g)(y) = \int_0^\infty f\left(yx^{-1}\right) g(x) \frac{dx}{x}. \tag{6}$$

By virtue of the above convolution products, a convolution theorem for $L_{\gamma,\sigma}^{(\beta)}$ transform can be simply extracted as follows.

Theorem 5. Let $f \in F_{p,\mu}$ and $\varphi \in C_0^\infty$. Then, we have

$$L_{\gamma,\sigma}^{(\beta)}(f \circ \varphi)(x) = \left(\left(L_{\gamma,\sigma}^{(\beta)} f\right) \diamond \varphi\right)(x) \quad (x > 0)$$

where $\beta > 0$; $\mathrm{Re}(\gamma) > \frac{1}{\beta} - 1; \sigma \in \mathbb{R}$.

Proof. Let $x > 0, \beta > 0$; $\mathrm{Re}(\gamma) > \frac{1}{\beta} - 1$; $\sigma \in \mathbb{R}$ be established for $f \in F_{p,\mu}$. Then, by making use of (2) and (6) and the Fubini's theorem we get

$$L_{\gamma,\sigma}^{(\beta)}(f \circ \varphi)(x) = \int_0^\infty \lambda_{\gamma,\sigma}^{(\beta)}(xt) \int_0^\infty \frac{f\left(y^{-1}t\right)}{y} \varphi(t)\, dydt. \tag{7}$$

By interchanging the variables in (7) by $y^{-1}t = \zeta$ we get

$$L_{\gamma,\sigma}^{(\beta)}(f \circ \varphi)(x) = \int_0^\infty \varphi(y) \int_0^\infty \lambda_{\gamma,\sigma}^{(\beta)}(xy\zeta)\, f(\zeta)\, d\zeta dy.$$

Hence the proof of the theorem is completed.

Lemma 6. Let $f \in F_{p,\mu}$ and $\psi, \varphi \in C_0^\infty$. Then, we have

$(i)\, f \circ (\varphi \circ \psi) = (f \circ \varphi) \circ \psi$.

$(ii)\, f \circ \varphi = \varphi \circ f$.

$(iii)\, \alpha (f \circ \varphi) = (\alpha f) \circ \varphi$.

$(iv)\, f \circ (\varphi + \psi) = f \circ \varphi + f \circ \psi$.

(v) If $\{f_n\}_1^\infty \in F_{p,\mu}$, $\lim\{f_n\} = f$ in $F_{p,\mu}$, then $\lim\{f_n \circ \varphi\} = f \circ \varphi$ in $F_{p,\mu}$.

Proof of Parts (i) and (ii) follows from the properties of $\circ$. Proof of Parts (iii) and (iv) follows from simple integral calculus. The limit in (v) similarly follows from integral calculus techniques. Hence, to complete the proof it suffices to show that

$$f \circ \varphi \in F_{p,\mu}. \tag{8}$$

Using the Jensen's inequality and the hypothesis that $f \in F_{p,\mu}$ we write (8) as

$$\left\| x^k \frac{d^k}{dx^k} \left(x^{-\mu} (f \circ \varphi)(x)\right) \right\|_p^p = \int_0^\infty \left| \frac{\varphi(t)}{t} \right| \int_0^\infty \left| x^p \frac{d^k}{dx^k} \left(x^{-\mu} f\left(t^{-1}x\right)\right) \right|^p dxdt$$

Or,

$$\left\| x^k \frac{d^k}{dx^k} \left(x^{-\mu} \left(f \circ \varphi \right) (x) \right) \right\|_p^p \leq \gamma_k^{p,\mu} (f) \int_0^\infty \frac{|\varphi (t)|}{t} dt \tag{9}$$

Since $\varphi \in C_0^\infty$, we have $\varphi (t) = 0$, $\forall t \notin (a, b)$, $0 < a < b < \infty$. Hence, (9) yields

$$\left\| x^k \frac{d^k}{dx^k} \left(x^{-\mu} \left(f \circ \varphi \right) (x) \right) \right\|_p^p \leq \left\| x^k \frac{d^k}{dx^k} \left(x^{-\mu} (f) \right) \right\|_p^p \int_a^b \frac{|\varphi (t)|}{t} dt < \infty.$$

Thus, we have completed the proof of the theorem.

Theorem 7. If $\{\delta_n\}$ and $\{\epsilon_n\}$ be in Δ, then $\{\delta_n \circ \epsilon_n\} \in \Delta$.
Proof of this theorem follows from the properties of $\circ$, it has also been seen in citations of the author.

Finally, to have our space β_1 established, we have to prove the following axiom.

Theorem 8. If $\{\delta_n\} \in \Delta$ and $f \in F_{p,\mu}$, then $\lim\{f \circ \delta_n\} = f$ in $F_{p,\mu}$.
Proof. Let $\{\delta_n\} \in \Delta$ and $f \in F_{p,\mu}$ be given. Then, by Jensen's inequality and the identity (ii) of Δ we write

$$\left\| x^k \frac{d^k}{dx^k} \left(x^{-\mu} \left(f \circ \delta_n - f \right) (x) \right) \right\|_p^p \leq \int_0^\infty |\delta_n (t)| \left| x^k \frac{d^k}{dx^k} \left(x^{-\mu} f_t(x) \right) \right|^p dx dt. \tag{10}$$

Since the mapping $f_t (x) = \dfrac{f \left(x t^{-1} \right)}{t} - f (x)$ is a member of $F_{p,\mu}$ for every $x > 0$, (10) can be rewritten as

$$\left\| x^k \frac{d^k}{dx^k} \left(x^{-\mu} \left(f \circ \delta_n - f \right) (x) \right) \right\|_p^p \leq M \int_{a_n}^{b_n} |\delta_n (t)| \, dt \tag{11}$$

where $\{a_n\}$ and $\{b_n\}$ are real numbers such that $\operatorname{supp} \delta_n (t) \subseteq [a_n, b_n]$, $\lim\{a_n\} = \lim\{b_n\} = 0$. Hence, (11) reveals

$$\gamma_k^{p,\mu} \left(f \circ \delta_n - f \right) = \gamma_k^{p,\mu} \left(f \circ \delta_n \right) \gamma_k^{p,\mu} (f) \subseteq MN \left(a_n, b_n \right), \tag{12}$$

where N is some positive integer. Hence, we have shown that

$$\gamma_k^{p,\mu} \left(f \circ \delta_n \right) \to \gamma_k^{p,\mu} (f) \text{ as } n \to \infty.$$

This completes the proof of the theorem.

Hence, the space β_1 of the given sets $F_{p,\mu}, C_0^\infty$, Δ and the operation $\circ$ is obtained.

In that manner, we establish the space β_2, by the sets $F_{p,\frac{2}{p}-\mu-1}, C_0^\infty, \Delta$ and the products $\circ$ and $\diamond$. Proof of the following results are analogous to the corresponding ones used in generating the space β_1. Hence details are deleted.

Theorem 9. The following truly hold.

(i)Let $g \in F_{p,\frac{2}{p}-\mu-1}$and $\varphi \in C_0^\infty$, then $g \diamond \varphi \in F_{p,\frac{2}{p}-\mu-1}$.

(ii)Let $g \in F_{p,\frac{2}{p}-\mu-1}$and $\varphi, \psi \in C_0^\infty$, then $g \diamond (\varphi + \psi) = g \diamond \varphi + g \diamond \psi$.

(iii) Let $\{g_n\} \in F_{p,\frac{2}{p}-\mu-1}$ and $\lim \{g_n\} = g$ in $F_{p,\frac{2}{p}-\mu-1}$, then for $\varphi \in C_0^\infty$, $\lim\{g_n \diamond \varphi\} = g \diamond \varphi$.

(iv) Let $\alpha \in \mathbb{C}$, then $\alpha (g \diamond \varphi) = (\alpha g) \diamond \varphi$.

(v) Let $\{\delta_n\} \in \Delta$ and $g \in F_{p,\frac{2}{p}-\mu-1}$, then $\lim\{g \diamond \delta_n\} = g$ in $F_{p,\frac{2}{p}-\mu-1}$.

Hence it is sufficient to establish the following distributive law.

Theorem 10. Let $f \in F_{p,\mu}$ and $\varphi, \psi \in C_0^\infty$. Then, we have

$$(f \diamond (\varphi \circ \psi))(y) = ((f \diamond \varphi) \diamond \psi)(y) \quad (y > 0),$$

Proof. Let $f \in F_{p,\mu}$ and $\varphi, \psi \in C_0^\infty$ and $y > 0$, then, on account of (5) and (6), we write

$$(f \diamond (\varphi \circ \psi))(y) = \int_0^\infty f(xy)(\varphi \circ \psi)(x)\, dx = \int_0^\infty \frac{\psi(t)}{t} \int_0^\infty f(xy)\, \varphi\left(t^{-1}x\right) dxdt.$$

By changing variables the above integral reveals

$$(f \diamond (\varphi \circ \psi))(y) = \int_0^\infty \psi(t) \int_0^\infty (tzy)\, \varphi(z)\, dzdt$$

Once again making use of (5) gives

$$(f \diamond (\varphi \circ \psi))(y) = \int_0^\infty (f \diamond \varphi)(ty)\, \psi(t)\, dt.$$

To complete the proof of this theorem, it suffices to show that $f \diamond \phi \in F_{p,\frac{2}{p}-\mu-1}$, for every $f \in F_{p,\frac{2}{p}-\mu-1}$ and $\phi \in C_0^\infty$. Therefore, by Jensen's inequality we have

$$\left\| x^k \frac{d^k}{dx^k} \left(x^{-\left(\frac{2}{p}-\mu-1\right)} (f \diamond \phi)(x) \right) \right\|_p^p \leq \int_0^\infty |\phi(t)| \int_0^\infty \left| x^p \frac{d^k}{dx^k} x^{\mu+1-\frac{2}{p}} f(xt) \right|^p dxdt,$$

$(k \in \mathbb{N}_0)$. The hypothesis that $f \in F_{p,\frac{2}{p}-\mu-1}$ implies

$$\left\| x^k \frac{d^k}{dx^k} \left(x^{-\left(\frac{2}{p}-\mu-1\right)} (f \diamond \phi)(x) \right) \right\|_p^p \leq M \int_0^\infty |\phi(t)|\, dt.$$

The hypothesis that $\phi \in C_0^\infty$ implies that $\phi(t) = 0$ for all $t \notin (a, b)$, $0 < a < b < \infty$. Hence

$$\left\| x^k \frac{d^k}{dx^k} x^{\mu+1-\frac{2}{p}} (f \diamond \phi)(x) \right\|_p^p \leq M \int_a^b |\phi(t)|\, dt < \infty$$

for every $x > 0$.

Addition, scalar multiplication, convolution, differentiation and convergence in the spaces β_1 and β_2 can be defined in the natural way as in [10]. Hence, we omit the same details.
Therefore, we define the generalized $L^{(\beta)}_{\gamma,\sigma}$ transformation of $\frac{f_n}{\delta_n} \in \beta_1$ to be the mapping in β_2 given by

$$L^{g(\beta)}_{\gamma,\sigma} \frac{f_n}{\delta_n} = \frac{L^{(\beta)}_{\gamma,\sigma} f_n}{\delta_n}. \tag{13}$$

4. Extended properties

In this chapter we will prove some properties of the extended integral. There are of course more interesting facts, in particular, there are connections between extended integral and the classical integral. We confine ourselves to study few basic properties of the extended integral as follows.

Theorem 11 The operator $L^{g(\beta)}_{\gamma,\sigma} : \beta_1 \to \beta_2$ is well - defined and linear.

Proof Assume $\dfrac{\psi_n}{\delta_n} = \dfrac{\varphi_n}{\epsilon_n} \in \beta_1$ and the hypothesis of the theorem satisfies. Then by the notion of equivalence classes of β_1 it holds that

$$\varphi_n \circ \delta_m = \psi_m \circ \epsilon_n = \psi_n \circ \epsilon_m \;\; (\forall n, m \in \mathbb{N}).$$

Employing $L^{(\beta)}_{\gamma,\sigma}$ transform for both sides of the previous equation and investing Lemma 2 suggest to write

$$L^{(\beta)}_{\gamma,\sigma} \varphi_n \diamond \delta_m = L^{(\beta)}_{\gamma,\sigma} \psi_n \diamond \epsilon_m \;\; (\forall n, m \in \mathbb{N}).$$

Thus, the quotients $\dfrac{L^{(\beta)}_{\gamma,\sigma}\varphi_n}{\epsilon_n}$ and $\dfrac{L^{(\beta)}_{\gamma,\sigma}\psi_n}{\delta_n}$ are equivalent in the sense of β_2 and, consequently,

$$\frac{L^{(\beta)}_{\gamma,\sigma}\varphi_n}{\epsilon_n} = \frac{L^{(\beta)}_{\gamma,\sigma}\psi_n}{\delta_n} \;\; (\forall n \in \mathbb{N}).$$

Proof of linearity of $L^{g(\beta)}_{\gamma,\sigma}$ is as follows. Let $\dfrac{\varphi_n}{\epsilon_n}, \dfrac{\psi_n}{\delta_n} \in \beta_1 \; (\forall n \in \mathbb{N})$. Then, by Lemma 2 and the convolution theorem we have

$$\begin{aligned} L^{g(\beta)}_{\gamma,\sigma} \left(\frac{\varphi_n}{\epsilon_n} + \frac{\psi_n}{\delta_n} \right) &= L^{g(\beta)}_{\gamma,\sigma} \left(\frac{\varphi_n \circ \delta_n + \psi_n \circ \epsilon_n}{\epsilon_n \circ \delta_n} \right) \\ \text{i.e.} \qquad &= \frac{L^{(\beta)}_{\gamma,\sigma} \left(\varphi_n \circ \delta_n + \psi_n \circ \epsilon_n \right)}{\epsilon_n \circ \delta_n} \\ \text{i.e.} \qquad &= \frac{L^{(\beta)}_{\gamma,\sigma}\varphi_n \diamond \delta_n + L^{(\beta)}_{\gamma,\sigma}\psi_n \diamond \epsilon_n}{\epsilon_n \circ \delta_n} \; (\forall n \in \mathbb{N}). \end{aligned}$$

Which can be expressed to give

$$L^{g(\beta)}_{\gamma,\sigma} \left(\frac{\varphi_n}{\epsilon_n} + \frac{\psi_n}{\delta_n} \right) = L^{g(\beta)}_{\gamma,\sigma} \frac{\varphi_n}{\epsilon_n} + L^{g(\beta)}_{\gamma,\sigma} \frac{\psi_n}{\delta_n} \;\; (\forall n \in \mathbb{N}).$$

Let $\Omega \in \mathbb{C}$, then of course $\Omega L^{g(\beta)}_{\gamma,\sigma} \frac{\varphi_n}{\epsilon_n} = \Omega \frac{L^{(\beta)}_{\gamma,\sigma}\varphi_n}{\epsilon_n} = \frac{L^{(\beta)}_{\gamma,\sigma}\Omega\varphi_n}{\epsilon_n}$. Hence, we lead to write

$$\Omega L^{g(\beta)}_{\gamma,\sigma} \frac{\varphi_n}{\epsilon_n} = L^{g(\beta)}_{\gamma,\sigma} \left(\Omega \frac{\varphi_n}{\epsilon_n}\right) \;\; (\forall n \in \mathbb{N}).$$

We have finished the proof of the theorem.

Theorem 12 The operator $L^{g(\beta)}_{\gamma,\sigma} : \beta_1 \to \beta_2$ is an injective mapping.

Proof Assume $L^{g(\beta)}_{\gamma,\sigma} \frac{\psi_n}{\delta_n} = L^{g(\beta)}_{\gamma,\sigma} \frac{\varphi_n}{\epsilon_n} \; (\forall n \in \mathbb{N})$. Using the concept of quotients in β_2 implies

$$L^{(\beta)}_{\gamma,\sigma}\psi_n \diamond \epsilon_m = L^{(\beta)}_{\gamma,\sigma}\varphi_m \diamond \delta_n \;\; (\forall m, n \in \mathbb{N}).$$

Lemma 2 and the convolution theorem imply

$$L^{(\beta)}_{\gamma,\sigma} (\psi_n \circ \epsilon_m) = L^{(\beta)}_{\gamma,\sigma} (\varphi_m \circ \delta_n) \;\; (\forall m, n \in \mathbb{N})$$

Hence $\psi_n \circ \epsilon_m = \varphi_m \circ \delta_n$. Therefore

$$\frac{\psi_n}{\delta_n} = \frac{\varphi_n}{\epsilon_n} \;\; (\forall n \in \mathbb{N}).$$

This finishes the proof of our theorem.

Theorem 13 The operator $L^{g(\beta)}_{\gamma,\sigma} : \beta_1 \to \beta_2$ is surjective.

Proof of this theorem is obvious by Theorem 1. Hence, we omit the proof details.

Definition 14 Let $\{\hat{\psi}_n\} \in F_{p,\frac{2}{p}-\mu-1}$ and $\frac{\hat{\psi}_n}{\delta_n} \in \beta_2$. Then, for each $\{\delta_n\} \in \Delta$ and some $\{\hat{\psi}_n\} = L^{(\beta)}_{\gamma,\sigma}\{\psi_n\}$, for some $\{\psi_n\} \in F_{p,\mu}$ we define the inverse $L^{g(\beta)}_{\gamma,\sigma}$ transform as

$$I^{g(\beta)}_{\gamma,\sigma} \frac{\hat{\psi}_n}{\delta_n} = \frac{\left(L^{(\beta)}_{\gamma,\sigma}\right)^{-1} \hat{\psi}_n}{\delta_n} = \frac{\psi_n}{\delta_n}. \tag{14}$$

Theorem 15 The mapping $I^{g(\beta)}_{\gamma,\sigma} : \beta_2 \to \beta_1$ is well-defined.

Proof Assume $\frac{\hat{\psi}_n}{\delta_n} = \frac{\hat{\varphi}_n}{\epsilon_n}$ in β_1. Then, $\hat{\psi}_n \circ \epsilon_m = \hat{\varphi}_m \circ \delta_n$ $(\forall m, n \in \mathbb{N})$ in the sense of β_1. Therefore, applying the $\left(L^{(\beta)}_{\gamma,\sigma}\right)^{-1}$ transform then investing Lemma 2 yield

$$\left(L^{(\beta)}_{\gamma,\sigma}\right)^{-1} \hat{\psi}_n \diamond \epsilon_m = \left(L^{(\beta)}_{\gamma,\sigma}\right)^{-1} \hat{\varphi}_m \diamond \delta_n.$$

In β_1, it means

$$\frac{\left(L^{(\beta)}_{\gamma,\sigma}\right)^{-1} \hat{\psi}_n}{\delta_n} = \frac{\left(L^{(\beta)}_{\gamma,\sigma}\right)^{-1} \hat{\varphi}_n}{\epsilon_n} \;\; (\forall n \in \mathbb{N}).$$

This finishes the proof of the theorem.

Theorem 16 The mapping $I_{\gamma,\sigma}^{g(\beta)}:\beta_2\to\beta_1$ is linear.

Proof of this theorem can be followed similarly as in the citations of the same author. Hence, we avoid to repeat the similar proofs.

Theorem 17 The Boehmian $\frac{\psi_n}{\delta_n}\in\beta_1$ is in the range $L_{\gamma,\sigma}^{g(\beta)}$ transform when ψ_n is in the range of $L_{\gamma,\sigma}^{(\beta)}$ transform for every $n\in\mathbb{N}$.

Proof Let $\frac{\psi_n}{\delta_n}$ be in the range of $L_{\gamma,\sigma}^{(\beta)}$ in the sense of β_2 then ofcourse ψ_n belongs to the range of $L_{\gamma,\sigma}^{(\beta)},\forall n\in\mathbb{N}$. For the converse, let ψ_n be in the range of $L_{\gamma,\sigma}^{(\beta)},\forall n\in\mathbb{N}$. Then there is $\hat{\psi}_n\in F_{p,\frac{2}{p}-\mu-1}$ such that $L_{\gamma,\sigma}^{(\beta)}\hat{\psi}_n=\psi_n, n\in\mathbb{N}$. Since $\frac{\psi_n}{\delta_n}\in\beta_2$ we get

$$\psi_n\diamond\delta_m=\psi_m\diamond\delta_n,$$

$\forall m,n\in\mathbb{N}$. Therefore, the convolution theorem yields

$$L_{\gamma,\sigma}^{(\beta)}\left(\hat{\psi}_n\circ\delta_m\right)=L_{\gamma,\sigma}^{(\beta)}\left(\hat{\psi}_m\circ\delta_n\right),\forall m,n\in\mathbb{N},$$

where $\hat{\psi}_n\in F_{p,\frac{2}{p}-\mu-1}$ and $\delta_n\in\Delta,\forall n\in\mathbb{N}$. Thus, we have

$$\hat{\psi}_n\circ\delta_m=\hat{\psi}_m\circ\delta_n,\forall m,n\in\mathbb{N}.$$

Hence, $\frac{\hat{\psi}_n}{\delta_n}\in\beta_2$ and $L_{\gamma,\sigma}^{g(\beta)}\left(\frac{\hat{\psi}_n}{\delta_n}\right)=\frac{\psi_n}{\delta_n}$.
The theorem is therefore completely proved.
Following is the convolution theorem of the generalized transform.

Theorem 18 (Convolution theorem) If $\frac{\psi_n}{\delta_n}\in\beta_1$ and $\vartheta\in F_{\rho,\mu}$, then

$$L_{\gamma,\sigma}^{g(\beta)}\left(\left(\frac{\psi_n}{\delta_n}\right)\circ\vartheta\right)=\left(\frac{L_{\gamma,\sigma}^{(\beta)}\psi_n}{\delta_n}\right)\diamond\vartheta.$$

Proof Assume $\frac{\psi_n}{\delta_n}\in\beta_1$ and $\vartheta\in F_{\rho,\mu}$ then

$$L_{\gamma,\sigma}^{g(\beta)}\left(\left(\frac{\psi_n}{\delta_n}\right)\circ\vartheta\right)=\frac{L_{\gamma,\sigma}^{(\beta)}\left(\vartheta_n\circ\vartheta\right)}{\delta_n}.$$

Hence the convolution theorem yields

$$L_{\gamma,\sigma}^{g(\beta)}\left(\left(\frac{\psi_n}{\delta_n}\right)\circ\vartheta\right)=\frac{L_{\gamma,\sigma}^{(\beta)}\psi_n\diamond\vartheta}{\delta_n}=\frac{L_{\gamma,\sigma}^{(\beta)}\psi_n}{\delta_n}\diamond\vartheta.$$

The proof of the theorem is completed.

5. Discussion and future research

We have studied a class of modified Bessel-type integrals and investigated their various properties, including convolution theorems and convolution products associated with this integral. Those given properties are pointed out to be the extension of the related results associated with the cassical integral. We have also presented an inverse problem in our investigation.
As the modified Bessel-type integral integral can be expressed in terms of H-fuctions as

$$\left(L_{\gamma,\sigma}^{(\beta)}f\right)(x)=\int_0^\infty H_{1,2}^{2,0}\left(xt\left|\begin{array}{c}\left(1-\frac{\sigma+1}{\beta},\frac{1}{\beta}\right)\\(0,1),\left(-\gamma-\frac{\sigma}{\beta},\frac{1}{\beta}\right)\end{array}\right.\right)f(t)\,dt\ \ (x>0).$$

then using similar techniques and general properties of differentiation and shifting results of H functions we in view of above extension claim that our generalized integral of the above new form can be extended to some moderate class of Boehmians under certain restriction on the kernel function of the integral.

6. Conclusion

The results presented here are sure to be new and potentially useful. Since the research subject here and its related ones are competitive, the content of this paper may attract interested readers who have been interested in this and related research subjects.

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