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The Distance From the Holomorphically Decomposable Fredholm Spectrum

A. Tajmouati, A. El Bakkali and M.B. Mohamed Ahmed

abstract: The class of holomorphically decomposable Fredholm operators was introduced by A. Tajmouati and A. El Bakkali since subclass of Saphar operators. Several authors have been interested in the distance of spectrum associated with the different classes. Especially C. Schmöeger has calculated the distance of Saphar spectrum. For this reason, in this paper, we establish the distance of the holomorphically decomposable Fredholm spectrum.

Key Words: Fredholm operator, Distance spectrum, Holomorphically decomposable Fredholm operator.

Contents

1. Introduction

Let X be a complex Banach space and let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on X. For $T \in \mathcal{B}(X)$, we use $N(T)$, $R(T)$, $R^\infty(T) = \bigcap_{k\geq 1} R(T^k)$ and r(T) to denote the kernel, the range, the hyper-range and the spectral radius of T. We define the reduced minimum of T by

$$\gamma(T) = \inf\{\|Tx\| : x \in X, dist\{x, N(T)\} = 1\} \; and \, \gamma(0) = \infty.$$

An operator $T \in \mathcal{B}(X)$ is called Kato operator if $R(T)$ is closed and $N(T) \subseteq R^\infty(T)$ [1].The Kato spectrum of T is defined by

$$\sigma_K(T) := \{\mu \in \mathbb{C} \setminus \; \mu - T \;\text{ is not Kato operator}\}.$$

We know that if $T \in \mathcal{B}(X)$ is Kato operator, then we have

$$\Gamma(T) := dist\{0, \sigma_k(T)\} = \lim_{n\to\infty} [\gamma(T^n)]^{\frac{1}{n}} = \sup_{n\geq 1} [\gamma(T^n)]^{\frac{1}{n}}.$$

We say that $T \in \mathcal{B}(X)$ is relatively regular if there exists an operator $S \in \mathcal{B}(X)$ for which $TST = T$, S is called a pseudo-inverse of T. $\mathcal{R}(X)$ will denote the set of all relatively regular operators.
An operator $T \in \mathcal{B}(X)$ is called Saphar operator if $T \in \mathcal{R}(X)$ and $N(T) \subseteq R^\infty(T)$.

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We write $\mathcal{S}(X)$ for the set of Saphar operators. For $T \in \mathcal{B}(X)$ we denote the Saphar resolvent set and Saphar spectrum respectively by $\rho_{rr}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \in \mathcal{S}(X)\}$ and $\sigma_{rr}(T) = \mathbb{C} \setminus \rho_{rr}(T)$. The class $\mathcal{S}(X)$ has been studied by P. Saphar in [7]. It is known that $\lambda_0 \in \rho_{rr}(T)$ if and only if there exist a neighbourhoud U of λ_0 and an analytic function $F : U \to B(X)$ such that $(T-\lambda I)F(\lambda)(T-\lambda I) = (T-\lambda I)$ for all $\lambda \in U$. C. Schmöeger in [8, Theorem 3] proved that if $0 \in \rho_{rr}(T)$, then the distance $dist\{0, \sigma_{rr}(T)\}$ is given by

$$dist\{0, \sigma_{rr}(T)\} = \lim_{n\to\infty}[\delta_n(T)]^{\frac{1}{n}} = \sup_{n\geq 1}[\delta_n(T)]^{\frac{1}{n}},$$

where $\delta_n(T) = \sup\{r(S)^{-1} : T^n S T^n = T^n,\ S \in \mathcal{B}(X)\}$ and $\delta(T) = \sup_{n\geq 1}[\delta_n(T)]^{\frac{1}{n}}$. An operator $T \in \mathcal{B}(X)$ is called Fredholm operator, in symbol $T \in \Phi(X)$, if $dimN(T)$ and $codimR(T)$ are finite. The holomorphically decomposable Fredholm resolvent set of $T \in \mathcal{B}(X)$ is defined by:
$\rho_{hF}(T) = \{\lambda \in \mathbb{C} :$ there exist a neighbourhood U of λ and an analytic $F : U \to \mathcal{B}(X)$ such that $(T - \mu I)F(\mu)(T - \mu I) = T - \mu I$ and $F(\mu) \in \Phi(X)$ for all $\mu \in U\}$.
Also, $\sigma_{hF}(T) := \mathbb{C}\setminus\rho_{hF}(T)$ is called the holomorphically decomposable Fredholm spectrum of T. T is called holomorphically decomposable Fredholm operator if $0 \in \rho_{hF}(T)$. The class of this operators is denoted by $\mathcal{H}\Phi(X)$. In [2] and [3], the authors introduced and studied this class. Particulary, if $T \in \mathcal{B}(X)$, then $f(\sigma_{hF}(T)) = \sigma_{hF}(f(T))$ for all injective $f \in \mathcal{H}(\sigma(T))$ where $\mathcal{H}(\sigma(T))$ is the algebra of all complex-valued functions which are analytic in some neighbourhood of $\sigma(T)$ and $f(T) \in \mathcal{B}(X)$ defined by Riesz-Dunford functional calculus.
In this paper, we study the holomorphically decomposable Fredholm operators. Moreover, for $T \in \mathcal{H}\Phi(X)$ we estimate the distance $dist\{0, \sigma_{hF}(T)\}$.

2. Main results

We start with the following two lemmas.

Lemma 2.1. *[5, Theorem 3.9] let Ω be a non-void connected open subset of $\mathbb{C}$ and $F : \Omega \longrightarrow \mathcal{B}(X)$ be analytic, then the following statements are equivalent:*

1. *The function F has a local analytic pseudo-inverse on Ω;*
2. *The function F has a global analytic pseudo-inverse on Ω.*

Lemma 2.2. *[4, Lemma 4] Let $T \in \mathcal{B}(X)$ such that $TST = T$ for some $S \in \mathcal{B}(X)$, then we have*

$$\|S\|^{-1} \leq \gamma(T).$$

Next proposition contains the useful properties.

Proposition 2.3. *Let $T \in \mathcal{H}\Phi(X)$, then there exists $S \in \Phi(X)$ such that $TST = T$. Moreover, we have:*

1. *$T^n \in \mathcal{H}\Phi(X)$ and $T^n S^n T^n = T^n$ for all $n \in \mathbb{N}$;*

2. $(T-\lambda I)(I-\lambda S)^{-1}S(T-\lambda I) = (T-\lambda I)$ *for all* $\lambda \in \mathbb{C}$ *such that* $|\lambda| < r(S)^{-1}$;
3. $\{\lambda \in \mathbb{C} : |\lambda| < r(S)^{-1}\} \subseteq \rho_{hF}(T)$.

Proof:

1. By [3, Proposition 2.1].
2. From [10, Corollary 1.5].
3. It is immediately, because $(T-\lambda I)(I-\lambda S)^{-1}S(T-\lambda I) = (T-\lambda I)$ and $(I-\lambda S)^{-1}S$ is Fredholm for all $\lambda \in \mathbb{C}$ satisfying $|\lambda| < r(S)^{-1}$.

□

Now, inspired by the paper of C. Schmöeger [8], we define the new distance.

Definition 2.4. *Let* $T \in \mathcal{B}(X)$. *We define for all* $n \in \mathbb{N}$ *the following quantities:*

$$\delta_{\Phi,n}(T) = \sup\{r(S)^{-1} : T^nST^n = T^n, S \in \Phi(X)\} \text{ and } \delta_\Phi(T) = \sup_{n\geq 1}[\delta_{\Phi,n}(T)]^{\frac{1}{n}}.$$

We give in the subsequent proposition an surround of the distance of holomorphically decomposable Fredholm spectrum.

Proposition 2.5. *Let* $T \in \mathcal{H}\Phi(X)$, *then we have for all* $k \geq 1$

$$[\delta_{\Phi,k}(T)]^{\frac{1}{k}} \leq \delta_\Phi(T) \leq \Gamma(T).$$

Proof: By the first assertion of Proposition 2.3, then $T^k \in \mathcal{H}\Phi(X)$ for all $k \in \mathbb{N}$. Let $k \geq 1$ and $S \in \Phi(X)$ such that

$$T^kST^k = T^k,$$

then from (1)Proposition 2.3 for all $n \in \mathbb{N}$ we have

$$[T^k]^nS^n[T^k]^n = [T^k]^n.$$

Applying Lemma 2.2, it follows immediately that

$$\|S^n\|^{-1} \leq \gamma((T^k)^n),$$

thus

$$[\|S^n\|^{\frac{1}{n}}]^{-1} \leq [[\gamma(T^{kn})]^{\frac{1}{nk}}]^k.$$

Hence, by extending n to infinity, we conclude

$$r(S)^{-1} \leq [\Gamma(T)]^k.$$

Therefore, since $\delta_{\Phi,k}(T) = \sup\{r(S)^{-1} : T^kST^k = T^k, S \in \Phi(X)\}$, then

$$\delta_{\Phi,k}(T) \leq [\Gamma(T)]^k,$$

so
$$[\delta_{\Phi,k}(T)]^{\frac{1}{k}} \leq \Gamma(T).$$
Finally, we obtain for all $k \geq 1$
$$[\delta_{\Phi,k}(T)]^{\frac{1}{k}} \leq \sup_{k\geq 1}[\delta_{\Phi,k}(T)]^{\frac{1}{k}} = \delta_{\Phi}(T) \leq \Gamma(T).$$

□

For an holomorphically decomposable Fredholm operator, we study in the next proposition its global analytic pseudo-inverse.

Proposition 2.6. *Let $T \in \mathcal{H}\Phi(X)$ and $\Omega = \{\lambda \in \mathbb{C} : |\lambda| < d(T) := dist\{0, \sigma_{hF}(T)\}\}$, then there exists an analytic function $F : \Omega \to \Phi(X)$ such that for all $\lambda \in \Omega$*
$$(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I.$$
Moreover, we have
$$F(\lambda) = \sum_{n=0}^{\infty} \lambda^n F(0)^{n+1} \text{ and } d(T) = [limsup\|F(0)^{n+1}\|^{\frac{1}{n}}]^{-1}.$$

Proof: By definition of $d(T)$, we obtain that
$$\Omega \subseteq \rho_{hF}(T),$$
hence for all $\lambda \in \Omega$, there exist a neighborhood U_λ of λ and a local analytic function $F : U_\lambda \to \Phi(X) \subseteq \mathcal{B}(X)$ satisfying for all $\mu \in U_\lambda$
$$(T - \mu I)F(\mu)(T - \mu I) = T - \mu I.$$
Thus, From Lemma 2.1, there exists an analytic function $F : \Omega \to \mathcal{B}(X)$ such that for all $\lambda \in \Omega$
$$(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I.$$
Since $TF(0)T = T$ and by (2)Proposition 2.3, then we can write
$$F(\lambda) = (I - \lambda F(0))^{-1}F(0) = \sum_{n=0}^{\infty} \lambda^n F(0)^{n+1}.$$
It is clear that $F(\lambda)$ is Fredholm for all $\lambda \in \Omega$, then $F : \Omega \to \Phi(X)$ is analytic. On other hand the radius of convergence of the series $\sum_{n=0}^{\infty} \lambda^n F(0)^{n+1}$ is
$$R = [limsup\|F(0)^{n+1}\|^{\frac{1}{n}}]^{-1}.$$
It is clearly that
$$R \geq d(T).$$

Now, assume that

$$R > d(T),$$

hence by definition of $d(T)$, we obtain

$$\sigma_{hF}(T) \cap \{\lambda \in \mathbb{C} : |\lambda| < R\} \neq \varnothing. \tag{2.1}$$

We consider for all λ such that $|\lambda| < R$ the function G defined by

$$G(\lambda) = (T - \lambda I)F(\lambda)(T - \lambda I) - (T - \lambda I).$$

It follows that for all $\lambda \in \Omega$

$$G(\lambda) = 0.$$

Consequently, since G is analytic, then for all λ such that $|\lambda| < R$

$$G(\lambda) = 0.$$

Therefore, by definition of $\rho_{hF}(T)$, we conclude that

$$\{\lambda \in \mathbb{C} : |\lambda| < R\} \subseteq \rho_{hF}(T).$$

Thus $\sigma_{hF}(T) \cap \{\lambda \in \mathbb{C} : |\lambda| < R\} = \varnothing$ and this is a contradiction with 2.1.
Finally, we conclude that

$$d(T) = R = [limsup\|F(0)^{n+1}\|^{\frac{1}{n}}]^{-1}.$$

□

The following theorem establishes the distance $dist\{0, \sigma_{hF}(T)\}$.

Theorem 2.7. *Let $T \in \mathcal{H}\Phi(X)$, then we have*

1. $\delta_\Phi(T) \leq d(T) \leq \Gamma(T)$;
2. $d(T) = \lim_{n\to\infty}[\delta_{\Phi,n}(T)]^{\frac{1}{n}} = \delta_\Phi(T)$.

Proof:

1. Let $|\lambda| < \delta_\Phi(T)$, since $\delta_\Phi(T) = \lim_{n\to\infty}[\delta_{\Phi,n}(T)]^{\frac{1}{n}}$, then there exists $k \in \mathbb{N}$ such that for all $n \geq k$

   $$|\lambda| < [\delta_{\Phi,n}(T)]^{\frac{1}{n}}.$$

   Hence

   $$|\lambda^n| < \delta_{\Phi,n}(T).$$

   By definition of $\delta_{\Phi,n}(T)$, then for all $S \in \Phi(X)$ satisfy $T^nST^n = T^n$ we have

   $$|\lambda^n| < r(S)^{-1}.$$

   By Proposition 2.3 we obtain

   $$\lambda^n \in \rho_{hF}(T^n).$$

Since the analytic function $f : z \to z^n$ is injective and by [2, Theorem 3.2], then

$$\lambda \in \rho_{hF}(T).$$

Therefore, we obtain

$$\{\lambda \in \mathbb{C} : |\lambda| < \delta_\Phi(T)\} \subseteq \rho_{hF}(T),$$

and consequently

$$\delta_\Phi(T) \leq d(T).$$

Now let $|\lambda| < d(T)$, then there exists an analytic function F such that

$$(T - \lambda I)F(\lambda)(T - \lambda I) = T - \lambda I.$$

By Proposition 2.6, we have

$$F(\lambda) = \sum_{n=0}^{\infty} \lambda^n F(0)^{n+1} \textit{ and } T^{n+1}F(0)^{n+1}T^{n+1} = T^{n+1}.$$

Consequently by Lemma 2.2, we obtain

$$\|F(0)^{n+1}\|^{-1} \leq \gamma(T^{n+1}).$$

Therefore

$$[\|F(0)^{n+1}\|^{\frac{1}{n}}]^{-1} \leq [[\gamma(T^{n+1})]^{\frac{1}{n+1}}]^{\frac{n+1}{n}}.$$

Going to the limit and again applying Proposition 2.6, then

$$d(T) \leq \Gamma(T).$$

2. Let $\epsilon \in]0, d(T)[$ and put $\eta = d(T) - \epsilon$ with $d(T) = dist\{0, \sigma_{hF}(T)\}$. By Proposition 2.6 for all $|\lambda| < d(T)$, we have

$$(T - \lambda I)F(\lambda)(T - \lambda I) = (T - \lambda I) \text{ with } F(\lambda) = \sum_{n=0}^{\infty} \lambda^n F(0)^{n+1}.$$

Since $\limsup \|F(0)^{n+1}\|^{\frac{1}{n}} = \frac{1}{d(T)}$, then there is an integer k_1 such that

$$\|F(0)^{n+1}\|^{\frac{1}{n}} < \frac{1}{d(T)} + \frac{\epsilon}{d(T)\eta} = \frac{1}{\eta} \textit{ for all } n \geq k_1.$$

On the other hand $TF(0)T = T$, it follows that

$$T^{n+1}F(0)^{n+1}T^{n+1} = T^{n+1}.$$

Therefore for all $n \geq k_1$, we conclude that

$$\begin{array}{rcl} \eta^n & < & \|F(0)^{n+1}\|^{-1}; \\ & \leq & [r(F(0)^{n+1})]^{-1}; \\ & \leq & \delta_{\Phi,n+1}(T); \\ & \leq & [\delta_\Phi(T)]^{n+1}. \end{array}$$

Another, by previous assertion of this Theorem 2.7, we know that

$$\delta_\Phi(T) \leq d(T).$$

Hence for all $n \geq k_1$, we obtain

$$\begin{aligned} d(T) - \epsilon = \eta &< [[\delta_\Phi(T)]^{n+1}]^{\frac{1}{n}}; \\ &\leq [\delta_\Phi(T)]^{\frac{1+n}{n}}; \\ &\leq d(T)^{\frac{1+n}{n}}. \end{aligned}$$

Since $\lim_{n\to\infty} d(T)^{\frac{1+n}{n}} = d(T)$, then there is $k_2 \in \mathbb{N}$ such that for all $n \geq k_2$

$$d(T)^{\frac{1+n}{n}} < d(T) + \epsilon.$$

Thus, we obtain for all $n \geq \max\{k_1, k_2\}$

$$d(T) - \epsilon < [\delta_\Phi(T)]^{\frac{1+n}{n}} < d(T) + \epsilon.$$

Finally, we conclude that

$$d(T) = \delta_\Phi(T) = \lim_{n\to\infty} [\delta_\Phi(T)]^{\frac{1+n}{n}}.$$

□

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A. Tajmouati and M.B. Mohamed Ahmed,
Sidi Mohamed Ben Abdellah Univeristy,
Faculty of Sciences Dhar Al Mahraz, Fez, Morocco.
E-mail address: abdelaziz.tajmouati@usmba.ac.ma
E-mail address: bbaba2012@gmail.com

and

A. El Bakkali,
Department of Mathematics
University Chouaib Doukkali,
Faculty of Sciences, Eljadida, Morocco.
E-mail address: aba0101q@yahoo.fr