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On the Capitulation of the 2-ideal Classes of the Field $\mathbb{Q}(\sqrt{p_1p_2q}, i)$ of Type $(2, 2, 2)$

Abdelmalek Azizi, Abdelkader Zekhnini and Mohammed Taous

ABSTRACT: We study the capitulation of the 2-ideal classes of the field $\Bbbk = \mathbb{Q}(\sqrt{p_1p_2q}, \sqrt{-1})$, where $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ are different primes, in its three quadratic extensions contained in its absolute genus field $\Bbbk^*$ whenever the 2-class group of $\Bbbk$ is of type $(2, 2, 2)$.

Key Words: Absolute genus fields, Fundamental systems of units, 2-class group, Capitulation, Quadratic fields, Biquadratic fields, Multiquadratic CM-fields.

Contents

1 Introduction **127**

2 Preliminary results **129**

3 Main results **131**

1. Introduction

Let k be an algebraic number field, and denote by $\mathbf{C}l_2(k)$ its 2-class group, that is the 2-Sylow subgroup of the ideal class group, $\mathbf{C}l(k)$, of k. We denote by k^* the absolute genus field of k. Suppose F is a finite extension of k, then we say that an ideal class of k capitulates in F if it is in the kernel of the homomorphism

$$J_F : \mathbf{C}l(k) \longrightarrow \mathbf{C}l(F)$$

induced by extension of ideals from k to F. An important problem in Number Theory is to explicitly determine the kernel of J_F, which is usually called the capitulation kernel. If F is the relative genus field of a cyclic extension K/k, which we denote by $(K/k)^*$ and that is the maximal unramified extension of K which is obtained by composing K and an abelian extension over k, F. Terada states in [11] that all the ambiguous ideal classes of K/k, which are classes of K fixed under any element of $\mathrm{G}al(K/k)$, capitulate in $(K/k)^*$. In [12], H. Furuya confirmed that if F is the absolute genus field of an abelian extension $K/\mathbb{Q}$, then every strongly ambiguous class of $K/\mathbb{Q}$, that is an ambiguous ideal class containing at least one ideal invariant under any element of $\mathrm{G}al(K/\mathbb{Q})$, capitulates in F.

Let $\Bbbk = \mathbb{Q}(\sqrt{d}, i)$ and $\mathbb{K}$ be a quadratic extension of $\Bbbk$ contained in its genus field $\Bbbk^*$. In [5], we studied the capitulation problem in the field $\mathbb{K}$ for the radicand $d = 2pq$ where $p \equiv q \equiv 1 \pmod 4$ are different primes, and in [6], we have

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dealt with the same problem assuming $p \equiv -q \equiv 1 \pmod 4$. In [7,8,9] under the assumption $\mathbf{C}l_2(\Bbbk) \simeq (2,2,2)$, we studied the capitulation problem of the 2-ideal classes of $\Bbbk$ in its fourteen unramified extensions, within the first Hilbert 2-class field of $\Bbbk$, and we gave the abelian type invariants of the 2-class groups of these fourteen fields. In these series of papers, we also determined the structure of the metabelian Galois group $\Bbbk_2^{(2)}/\Bbbk$ of the second Hilbert 2-class field $\Bbbk_2^{(2)}$ of $\Bbbk$.

Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ be different primes and $\Bbbk = \mathbb{Q}(\sqrt{p_1p_2q}, i)$. In [10], we studied the capitulation of the 2-ideal classes of $\Bbbk$ in its three quadratic extensions $\mathbb{K}$ contained in $\Bbbk^*$ without any conditions on the type of $\mathbf{C}l_2(\Bbbk)$, the 2-class group of $\Bbbk$. In the present note, we apply these results to compute the capitulation kernel of $\mathbb{K}/\Bbbk$ whenever $\mathbf{C}l_2(\Bbbk)$ is of type $(2,2,2)$.

Notations

Let k be a number field, during this paper, we adopt the following notations:

- $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ are different primes.
- $\Bbbk$: denotes the field $\mathbb{Q}(\sqrt{p_1p_2q}, \sqrt{-1})$.
- κ_K: the capitulation kernel of an unramified extension $K/\Bbbk$.
- $\mathcal{O}_k$: the ring of integers of k.
- E_k: the unit group of $\mathcal{O}_k$.
- W_k: the group of roots of unity contained in k.
- F.S.U : the fundamental system of units.
- k^+: the maximal real subfield of k.
- $Q_k = [E_k : W_k E_{k^+}]$ is Hasse's unit index, if k is a CM-field.
- $q(k/\mathbb{Q}) = [E_k : \prod_{i=1}^{s} E_{k_i}]$ is the unit index of k, if k is multiquadratic, where k_1, ..., k_s are the quadratic subfields of k.
- k^*: the absolute genus field of k.
- $\mathbf{C}l_2(k)$: the 2-class group of k.
- $i = \sqrt{-1}$.
- ϵ_m: the fundamental unit of $\mathbb{Q}(\sqrt{m})$, if $m > 1$ is a square-free integer.
- $N(a)$: denotes the absolute norm of a number a, i.e., $N_{k/\mathbb{Q}}(a)$, where $k = \mathbb{Q}(\sqrt{a})$.
- $x \pm y$ means $x + y$ or $x - y$ for some numbers x and y.

2. Preliminary results

Let us first collect some results that will be useful in what follows.

Lemma 2.1 ([1], Lemma 5)**.** *Let $d > 1$ be a square-free integer and $\epsilon_d = x + y\sqrt{d}$, where x, y are integers or semi-integers. If $N(\epsilon_d) = 1$, then $2(x+1)$, $2(x-1)$, $2d(x+1)$ and $2d(x-1)$ are not squares in $\mathbb{Q}$.*

Lemma 2.2 ([2], 3.(1) p.19)**.** *Let $d > 2$ be a square-free integer and $k = \mathbb{Q}(\sqrt{d}, i)$, put $\epsilon_d = x + y\sqrt{d}$.*

1. *If $N(\epsilon_d) = -1$, then $\{\epsilon_d\}$ is a* F.S.U *of k.*
2. *If $N(\epsilon_d) = 1$, then $\{\sqrt{i\epsilon_d}\}$ is a* F.S.U *of k if and only if $x \pm 1$ is a square in $\mathbb{N}$, i.e., $2\epsilon_d$ is a square in $\mathbb{Q}(\sqrt{d})$. Else $\{\epsilon_d\}$ is a* F.S.U *of k.*

Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ be different primes, and consider the imaginary bicyclic biquadratic number field $\Bbbk = \mathbb{Q}(\sqrt{p_1p_2q}, i)$, so $\Bbbk$ admits three quadratic extensions contained in its genus field $\Bbbk^* = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2}, \sqrt{q}, i)$ (they are unramified quadratic extensions of $\Bbbk$ abelian over $\mathbb{Q}$), which are $\mathbb{K}_1 = \Bbbk(\sqrt{p_1}) = \mathbb{Q}(\sqrt{p_1}, \sqrt{p_2q}, i)$, $\mathbb{K}_2 = \Bbbk(\sqrt{p_2}) = \mathbb{Q}(\sqrt{p_2}, \sqrt{p_1q}, i)$ and $\mathbb{K}_3 = \Bbbk(\sqrt{q}) = \mathbb{Q}(\sqrt{q}, \sqrt{p_1p_2}, i)$. Put $\epsilon_{p_1p_2q} = x + y\sqrt{p_1p_2q}$.

Theorem 2.3 ([10])**.** *Let $\mathbb{K}_j$, $1 \le j \le 3$, be the three unramified quadratic extensions of $\Bbbk$ defined above.*

1. *Let $\epsilon_{p_2q} = a + b\sqrt{p_2q}$.*
   - (a) *If $x \pm 1$ is a square in $\mathbb{N}$ and $a+1$, $a-1$ are not, then $|\kappa_{\mathbb{K}_1}| = 8$.*
   - (b) *If $a \pm 1$ and ($2p_1(x \pm 1)$ or $p_2(x \pm 1)$) are squares in $\mathbb{N}$, then $|\kappa_{\mathbb{K}_1}| = 2$.*
   - (c) *For the other cases $|\kappa_{\mathbb{K}_1}| = 4$.*
2. *Let $\epsilon_{p_1q} = a + b\sqrt{p_1q}$.*
   - (a) *If $x \pm 1$ is a square in $\mathbb{N}$ and $a+1$, $a-1$ are not, then $|\kappa_{\mathbb{K}_2}| = 8$.*
   - (b) *If $a \pm 1$ and ($2p_1(x \pm 1)$ or $p_2(x \pm 1)$) are squares in $\mathbb{N}$, then $|\kappa_{\mathbb{K}_2}| = 2$.*
   - (c) *For the other cases $|\kappa_{\mathbb{K}_2}| = 4$.*
3. *Let $\epsilon_{p_1p_2} = a + b\sqrt{p_1p_2}$.*
   - (a) *If $N(\epsilon_{p_1p_2}) = 1$, then*
     - i. *If $x \pm 1$ is a square in $\mathbb{N}$, then $|\kappa_{\mathbb{K}_3}| = 4$.*
     - ii. *Else $|\kappa_{\mathbb{K}_3}| = 2$.*
   - (b) *If $N(\epsilon_{p_1p_2}) = -1$, then*
     - i. *If $q(x \pm 1)$ or $2q(x \pm 1)$ is a square in $\mathbb{N}$, then $|\kappa_{\mathbb{K}_3}| = 2$.*
     - ii. *Else $|\kappa_{\mathbb{K}_3}| = 4$.*

Theorem 2.4 ([10]). *Keep the notations and hypotheses previously mentioned, and put* $\epsilon_{p_2q} = a + b\sqrt{p_2q}$ *and* $\epsilon_{p_1p_2q} = x + y\sqrt{p_1p_2q}$.

1. *If* $x \pm 1$ *is a square in* $\mathbb{N}$ *and* $a+1$, $a-1$ *are not, then* $\kappa_{\mathbb{K}_1} = \langle[\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3\mathcal{H}_4]\rangle$.
2. *If* $a \pm 1$ *and* $(p_1(x \pm 1)$ *or* $2p_1(x \pm 1))$ *are squares in* $\mathbb{N}$, *then* $\kappa_{\mathbb{K}_1} = \langle[\mathcal{H}_1]\rangle$.
3. *If* $a + 1$, $a - 1$ *are not squares in* $\mathbb{N}$ *and* $p_1(x \pm 1)$ *or* $2p_1(x \pm 1)$ *is, then* $\kappa_{\mathbb{K}_1} = \langle[\mathcal{H}_1], [\mathcal{H}_3\mathcal{H}_4]\rangle$.
4. *In the other cases we have:* $\kappa_{\mathbb{K}_1} = \langle[\mathcal{H}_1], [\mathcal{H}_2]\rangle$.

Theorem 2.5 ([10]). *Keep the notations and hypotheses previously mentioned and put* $\epsilon_{p_1q} = a + b\sqrt{p_1q}$.

1. *If* $x \pm 1$ *is a square in* $\mathbb{N}$ *and* $a+1$, $a-1$ *are not, then* $\kappa_{\mathbb{K}_2} = \langle[\mathcal{H}_3], [\mathcal{H}_4], [\mathcal{H}_1\mathcal{H}_2]\rangle$.
2. *If* $a \pm 1$ *and* $(p_2(x \pm 1)$ *or* $2p_2(x \pm 1))$ *are squares in* $\mathbb{N}$, *then* $\kappa_{\mathbb{K}_2} = \langle[\mathcal{H}_3]\rangle$.
3. *If* $a + 1$ *and* $a - 1$ *are not squares in* $\mathbb{N}$ *and* $p_2(x \pm 1)$ *or* $2p_2(x \pm 1)$ *is, then* $\kappa_{\mathbb{K}_2} = \langle[\mathcal{H}_3], [\mathcal{H}_1\mathcal{H}_2]\rangle$.
4. *In the other cases,* $\kappa_{\mathbb{K}_2} = \langle[\mathcal{H}_3], [\mathcal{H}_4]\rangle$.

Theorem 2.6 ([10]). *Keep the notations and hypotheses previously mentioned and assume* $N(\epsilon_{p_1p_2}) = 1$.

1. *If* $x \pm 1$ *is a square in* $\mathbb{N}$, *then* $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_1\mathcal{H}_2], [\mathcal{H}_3\mathcal{H}_4]\rangle$.
2. *If* $p_1(x \pm 1)$ *or* $2p_1(x \pm 1)$ *is a square in* $\mathbb{N}$, *then* $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_3\mathcal{H}_4]\rangle$.
3. *If* $p_2(x \pm 1)$ *or* $2p_2(x \pm 1)$ *is a square in* $\mathbb{N}$, *then* $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_1\mathcal{H}_2]\rangle$.
4. *If* $q(x \pm 1)$ *or* $2q(x \pm 1)$ *is a square in* $\mathbb{N}$, *then* $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_1\mathcal{H}_2]\rangle = \langle[\mathcal{H}_3\mathcal{H}_4]\rangle$.

Theorem 2.7 ([10]). *Keep the notations and hypotheses previously mentioned and assume* $N(\epsilon_{p_1p_2}) = -1$.

1. *If* $q(x \pm 1)$ *or* $2q(x \pm 1)$ *is a square in* $\mathbb{N}$, *then* $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_1\mathcal{H}_3]\rangle$ *or* $\langle[\mathcal{H}_1\mathcal{H}_4]\rangle$.
2. *If* $x \pm 1$ *is a square in* $\mathbb{N}$, *then* $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_1\mathcal{H}_3], [\mathcal{H}_2\mathcal{H}_4]\rangle$ *or* $\langle[\mathcal{H}_1\mathcal{H}_4], [\mathcal{H}_2\mathcal{H}_3]\rangle$.
3. *If* $p_1(x \pm 1)$ *or* $2p_1(x \pm 1)$ *is a square in* $\mathbb{N}$, *then* $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_1\mathcal{H}_3], [\mathcal{H}_1\mathcal{H}_4]\rangle$.
4. *If* $p_2(x \pm 1)$ *or* $2p_2(x \pm 1)$ *is a square in* $\mathbb{N}$, *then* $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_1\mathcal{H}_3], [\mathcal{H}_2\mathcal{H}_3]\rangle$.

3. Main results

Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ be different primes such that $\mathbf{C}l_2(\mathbb{k})$ is of type $(2,2,2)$. According to [3], $\mathbf{C}l_2(\mathbb{k})$ is of type $(2,2,2)$ if and only if p_1, p_2 and q satisfy the following conditions:

1. $p_1 \equiv 5$ or $p_2 \equiv 5 \pmod 8$.
2. Two at least of the elements of $\left\{\left(\frac{p_1}{p_2}\right), \left(\frac{p_1}{q}\right), \left(\frac{p_2}{q}\right)\right\}$ are equal to -1.

These conditions are detailed in three types I, II and III, and each type consists of three cases (a), (b) and (c) (see [4]). To continue we need the following results.

Lemma 3.1. *Let $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ be different primes as above.*

1. *If p_1, p_2 and q are of type I, then $p_1(x \pm 1)$ is a square in $\mathbb{N}$.*
2. *If p_1, p_2 and q are of type II, then $p_2(x \pm 1)$ is a square in $\mathbb{N}$.*
3. *If p_1, p_2 and q are of type III, then $q(x \pm 1)$, i.e., $p_1p_2(x \mp 1)$ is a square in $\mathbb{N}$.*

Proof. As $p_1 \equiv 5$ or $p_2 \equiv 5 \pmod 8$, so the unit index of $\mathbb{k}$ is 1 (see [3, Corollary 3.2]). On the other hand, $N(\varepsilon_d) = 1$, i.e., $x^2 - 1 = y^2 p_1 p_2 q$, hence by Lemma 2.2, $x \pm 1$ is not a square in $\mathbb{N}$. Thus by the unique prime factorization in $\mathbb{Z}$ and by Lemma 2.1, there exist y_1, y_2 in $\mathbb{Z}$ such that:

$$(1)\begin{cases} x \pm 1 = p_1 y_1^2, \\ x \mp 1 = p_2 q y_2^2; \end{cases} \text{or } (2)\begin{cases} x \pm 1 = 2p_1 y_1^2, \\ x \mp 1 = 2p_2 q y_2^2; \end{cases} \text{or } (3)\begin{cases} x \pm 1 = p_2 y_1^2, \\ x \mp 1 = p_1 q y_2^2; \end{cases} \text{or}$$

$$(4)\begin{cases} x \pm 1 = 2p_2 y_1^2, \\ x \mp 1 = 2p_1 q y_2^2; \end{cases} \text{or } (5)\begin{cases} x \pm 1 = q y_1^2, \\ x \mp 1 = p_1 p_2 y_2^2; \end{cases} \text{or } (6)\begin{cases} x \pm 1 = 2q y_1^2, \\ x \mp 1 = 2p_1 p_2 y_2^2; \end{cases}$$

1. Suppose p_1, p_2 and q are of type I, then this contradicts systems (2), (3), (4), (5) and (6), since:

1. system (2) implies that $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_1}{q}\right) = 1$
2. system (3) implies that $\left(\frac{p_1}{p_2}\right) = \left(\frac{2}{p_1}\right)$ and $\left(\frac{p_1 q}{p_2}\right) = \left(\frac{2}{p_2}\right)$
3. system (4) implies that $\left(\frac{p_1}{p_2}\right) = \left(\frac{p_2}{q}\right) = 1$
4. system (5) implies that $\left(\frac{p_1}{q}\right) = \left(\frac{2}{p_1}\right) = 1$
5. system (6) implies that $\left(\frac{p_1}{q}\right) = 1$.

Thus only the system (1) occurs, which yields that $p_1(x \pm 1)$ is a square in $\mathbb{N}$ and $p_2(x \pm 1)$, $2p_2(x \pm 1)$ are not.

Similarly one can check 2. and 3. □

Using the similar argument, one can prove the following lemma.

Lemma 3.2. *Let* $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ *be different primes and put* $\varepsilon_{p_2q} = a + b\sqrt{p_2q}$.

1. *If* p_1, p_2 *and* q *are of type* $I(a)$, *then* $p_2(a \pm 1)$ *or* $2p_2(a \pm 1)$ *is a square in* $\mathbb{N}$.
2. *If* p_1, p_2 *and* q *are of type* $I(b)$, *then* $p_2(a \pm 1)$ *is a square in* $\mathbb{N}$.
3. *If* p_1, p_2 *and* q *are of type* $I(c)$ *or* $II(a)$, *then* $a \pm 1$ *is a square in* $\mathbb{N}$.
4. *If* p_1, p_2 *and* q *are of type* $II(c)$ *or* $III(a)$ *or* $III(b)$, *then* $p_2(a \pm 1)$ *is a square in* $\mathbb{N}$.
5. *If* p_1, p_2 *and* q *are of type* $II(b)$ *or* $III(c)$, *then* $2p_2(a \pm 1)$ *is a square in* $\mathbb{N}$.

Denote by $\mathcal{H}_1$ and $\mathcal{H}_2$ (resp. $\mathcal{H}_3$ and $\mathcal{H}_4$) the prime ideals of $\Bbbk$ above p_1 (resp. p_2), then we have:

Lemma 3.3 ([4]). *Let* $p_1 \equiv p_2 \equiv -q \equiv 1 \pmod 4$ *be different primes and assume* $\mathbf{C}l_2(\Bbbk) \simeq (2, 2, 2)$.

1. *If* p_1, p_2 *and* q *are of type* I, *then* $\mathbf{C}l_2(\Bbbk) = \langle [\mathcal{H}_1], [\mathcal{H}_3], [\mathcal{H}_4] \rangle$.
2. *If* p_1, p_2 *and* q *are of type* II *or* III, *then* $\mathbf{C}l_2(\Bbbk) = \langle [\mathcal{H}_1], [\mathcal{H}_2], [\mathcal{H}_3] \rangle$.

Theorem 3.4. *Let* $\Bbbk = \mathbb{Q}(\sqrt{d}, i)$, *where* $d = p_1p_2q$ *with* p_1, p_2 *and* q *are different primes such that* $\mathbf{C}l_2(\Bbbk)$, *the* 2*-class groupe of* $\Bbbk$, *is of type* $(2, 2, 2)$.

1. *If* p_1, p_2 *and* q *are of type* $I(c)$, *then* $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1] \rangle$.
2. *If* p_1, p_2 *and* q *are of type* $I(a)$ *or* $I(b)$, *then* $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_3\mathcal{H}_4] \rangle$.
3. *If* p_1, p_2 *and* q *are of type* II *or* III, *then* $\kappa_{\mathbb{K}_1} = \langle [\mathcal{H}_1], [\mathcal{H}_2] \rangle$.

Proof. From Lemmas 3.1, 3.2 and 3.3 we get:

1. If p_1, p_2 and q are of type $I(c)$, then $a \pm 1$ and $p_1(x \pm 1)$ are squares in $\mathbb{N}$. Hence Theorem 2.4 implies the result.
2. If p_1, p_2 and q are of type $I(a)$ or $I(b)$, then $p_2(a \pm 1)$ or $2p_2(a \pm 1)$ is a square in $\mathbb{N}$, and since $p_1(x \pm 1)$ is a square in $\mathbb{N}$, hence Theorem 2.4 implies the result.
3. a. If p_1, p_2 and q are of type II, then $a \pm 1$ or $p_2(a \pm 1)$ or $2p_2(a \pm 1)$ is a square in $\mathbb{N}$, and as in this case $p_2(x \pm 1)$ is also a square in $\mathbb{N}$, hence Theorem 2.4 implies the result.
   b. If p_1, p_2 and q are of type III, then $p_2(a \pm 1)$ or $2p_2(a \pm 1)$ is a square in $\mathbb{N}$, and since $q(x \pm 1)$ is also a square in dans $\mathbb{N}$, hence Theorem 2.4 implies the result.

□

As p_1 and p_2 play symmetric roles, so with a similar argument to that used in the previous theorem, we deduce the following theorem. Note that in this case $\mathcal{H}_3$ and $\mathcal{H}_4$ always capitulate in $\mathbb{K}_2$ (see [10, Proposition 14]). Note also that whenever p_1, p_2 and q are of type II, then $[\mathcal{H}_3] = [\mathcal{H}_4]$ since in this case $p_2(x \pm 1)$ is a square in $\mathbb{N}$ and the result is guaranteed by [4, Proposition 1]. Finally, note that if p_1, p_2 and q are of type III, then $\mathcal{Q}$, the prime ideal of $\Bbbk$ lies above q, is principal in $\Bbbk$; hence $\mathcal{H}_1\mathcal{H}_2\mathcal{H}_3\mathcal{H}_4$ is too.

Theorem 3.5. *Keep the hypotheses and notations mentioned in Theorem 3.4.*

1. *If p_1, p_2 and q are of type $II(c)$, then $\kappa_{\mathbb{K}_2} = \langle[\mathcal{H}_3]\rangle$.*
2. *If p_1, p_2 and q are of type $II(a)$ or $II(b)$ or III, then $\kappa_{\mathbb{K}_2} = \langle[\mathcal{H}_3], [\mathcal{H}_1\mathcal{H}_2]\rangle$.*
3. *If p_1, p_2 and q are of type I, then $\kappa_{\mathbb{K}_2} = \langle[\mathcal{H}_3], [\mathcal{H}_4]\rangle$.*

Finally, we compute the 2-idea classes of $\Bbbk$ that capitulate in $\mathbb{K}_3 = \mathbb{Q}(\sqrt{q}, \sqrt{p_1p_2}, i)$.

Theorem 3.6. *Keep the hypotheses and notations mentioned in Theorem 3.4 and assume $N(\varepsilon_{p_1p_2}) = 1$.*

1. *If p_1, p_2 and q are of type I, then $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_3\mathcal{H}_4]\rangle$.*
2. *If p_1, p_2 and q are of type II or III, then $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_1\mathcal{H}_2]\rangle$.*

Proof. From Lemmas 3.1 and 3.3 we get:

1. If p_1, p_2 and q are of type $I(c)$, then then $p_1(x \pm 1)$ is a square in $\mathbb{N}$. Hence Theorem 2.6 implies the result.
2. a. If p_1, p_2 and q are of type II, then $p_2(x \pm 1)$ is a square in $\mathbb{N}$, hence Theorem 2.6 implies the result.
   b. If p_1, p_2 and q are of type III, then $q(x \pm 1)$, i.e., $p_1p_2(x \pm 1)$ is a square in $\mathbb{N}$, hence Theorem 2.6 implies the result.

□

Theorem 3.7. *Keep the hypotheses and notations mentioned in Theorem 3.4 and assume $N(\varepsilon_{p_1p_2}) = -1$.*

1. *If p_1, p_2 and q are of type III, then $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_1\mathcal{H}_3]\rangle$ or $\langle[\mathcal{H}_2\mathcal{H}_3]\rangle$.*
2. *If p_1, p_2 and q are of type II, then $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_1\mathcal{H}_3], [\mathcal{H}_2\mathcal{H}_3]\rangle$.*
3. *If p_1, p_2 and q are of type I, then $\kappa_{\mathbb{K}_3} = \langle[\mathcal{H}_1\mathcal{H}_3], [\mathcal{H}_1\mathcal{H}_4]\rangle$.*

Proof. It is a simple deduction from Theorem 2.7 and Lemma 3.1. □

From Theorems 3.4, 3.5, 3.6 and 3.7, we deduce the following result.

Theorem 3.8. *Keep the hypotheses and notations mentioned in Theorem 3.4. Then all the classes of $\mathbf{Cl}_2(\Bbbk)$ capitulate in $\Bbbk^*$, i.e., $\kappa_{\Bbbk^*} = \mathbf{Cl}_2(\Bbbk)$.*

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Abdelmalek Azizi,
Mohammed First University
Sciences Faculty
Mathematics Department
Oujda, Morocco.
E-mail address: abdelmalekazizi@yahoo.fr

and

Abdelkader Zekhnini,
Mohammed First University
Pluridisciplinary Faculty of Nador
Mathematics Department
Morocco.
E-mail address: zekha1@yahoo.fr

and

Mohammed Taous,
Moulay Ismail University
Sciences and Techniques Faculty
Mathematics Department
Errachidia, Morocco.
E-mail address: taousm@hotmail.com