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Operational Shifted Hybrid Gegenbauer Functions Method for Solving Multi-term Time Fractional Differential Equations

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ABSTRACT: In this paper, we propose an efficient operational formulation of spectral tau method for solving multi-term time fractional differential equations with initial-boundary conditions. The shifted hybrid Gegenbauer functions(SHGFs) operational matrices of Riemann-Liouville fractional integral and Caputo fractional derivatives are presented. By using these operational matrices, the shifted hybrid Gegenbauer functions tau method for both temporal and spatial discretization are presented, which allow us to introduce an efficient spectral method for solving such problems. Finally, numerical results show good deal with the theoretical analysis.

Key Words: Operational matrix, Multi-term FDEs, Shifted Hybrid Gegenbauer.

Contents

1. Introduction

Many phenomena in fluid mechanics, physics and other sciences can be described successfully by models using mathematical tools from fractional calculus [2,4,7,19,28]. Theory of derivatives and integrals with fractional order and some applications are given in [20]. In this paper, the following multi-term time fractional partial differential equation will be considered:

$$P(D_t)u(x,t) = \kappa\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t),\ 0 < x < L, t > 0, \tag{1.1}$$

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with the following initial and boundary conditions:

$$u(x,0) = \phi_1(x), \; \frac{\partial u(x,0)}{\partial t} = \phi_2(x), \tag{1.2}$$

$$u(0,t) = \rho_1(t), \; u(L,t) = \rho_2(t). \tag{1.3}$$

where $P(D_t)u(x,t) = (D_t^\alpha + \sum_{j=0}^{m} d_j D_t^{\alpha_j})u(x,t)$, with $1 < \alpha_m < \cdots < \alpha_1 < \alpha < 2$, $d_j \geq 0,\ j = 0,1,\ldots,m$. $D_t^{\alpha_j}$ is the Caputo fractional derivative operator of order α_j with respect to variable t and the functions $\phi_1(x), \phi_2(x), \rho_1(t)$ and $\rho_2(t)$ are given. The multi-term fractional differential equations have been widely studied in rheology, mechanical models, and in many other areas [28]. In [7] an analytical solution of multi-term time-space fractional advection diffusion equations with mixed boundary conditions on a finite domain has been given. Atanackovic et.al. [2], analyzed diffusion wave equation with two fractional derivatives of different order on bounded and unbounded spatial domains. A finite difference scheme in time has been proposed for solving this equation in [1]. Also authors in [19], have presented a numerical solution of linear multi-term fractional differential equations by using piecewise polynomial collocation methods.
Operational matrices are used in several areas of numerical analysis, they also hold particular importance for solving different kinds of problems in various subjects such as differential equations, integro-differential equations, ordinary and partial fractional differential equations, optimal control problems and etc [16,17,18,22,23, 24]. The present method in this paper is based on shifted hybrid Gegenbauer functions.

In this section SHGFs are introdued by using Gegenbauer polynomials together with block pulse function(BPFs), then they were used to constract operational matrix of fractional integration and were applied to solve the Eqs.(1.1-1.3).

2. Preliminaries

In this section, first some basic properties of fractional calculus theory are recalled, then the shifted hybrid Gegenbauer function is introducted.

2.1. Fractional calculus

Definition 2.1. *The Rimann-Liouville fractional integral of order* $\alpha \geq 0$ *of function* $u(x,t)$ *with respect to variable* t *is defined as:*

$$I_x^\alpha u(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^x u(\xi,t)(x-\xi)^{\alpha-1} d\xi, & if \quad \alpha > 0, \\ u(x,t), & if \quad \alpha = 0. \end{cases} \tag{2.1}$$

Definition 2.2. *The Caputo fractional derivatives of order* α *are respectively given as:*

$$D_x^\alpha u(x,t) = \frac{1}{\Gamma(n-\alpha)} \int_0^x \frac{u^{(n)}(\xi,t)}{(x-\xi)^{\alpha-n+1}} d\xi, \tag{2.2}$$

where $n-1 < \alpha \leq n, n \in N$.

The operator D_x^{α} satisfies the following properties:

1. $D_x^{\nu}(c_1 u(x,t) \pm c_2 w(x,t)) = c_1 D^{\nu} u(x,t) \pm c_2 D^{\nu} w(x,t)$,
2. $D_x^{\nu} I_x^{\nu} u(x,t) = u(x,t)$,
3. $I_x^{\nu} D_x^{\nu} u(x,t) = u(x,t) - \sum_{i=0}^{\lceil \nu \rceil -1} u^{(i)}(0^+,t) \frac{x^i}{i!}$,
4. $$D_x^{\nu} x^{\beta} = \begin{cases} 0, & \beta < \nu, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)} x^{\beta-\nu}, & \beta \geq \nu. \end{cases} \tag{2.3}$$

2.2. Shifted hybrid Gegenbauer functions

We denote the orthogonal set of SHGFs by $L h_{ij}(x), i = 1,2,\ldots,n, j = 0,1,\ldots,m-1$ and define:

$$_L h_{ij}(x) = \begin{cases} C_j^{\lambda}(2n\frac{x}{L} - 2i + 1), & L\frac{i-1}{n} \leq x < L\frac{i}{n}, \\ 0, & otherwise, \end{cases} \tag{2.4}$$

where $C_j^{\lambda}(x)$ are the Gegenbauer polynomials which are defined in $[-1,1]$ by the following recursive relation:

$C_0^{\lambda}(x) = 1, \qquad C_1^{\lambda}(x) = 2\lambda x,$

$C_m^{\lambda}(x) = \frac{1}{m}[2x(m+\lambda-1)C_{m-1}^{\lambda}(x) - (m+2\lambda-2)C_{m-2}^{\lambda}(x)], \quad m = 2,3,\ldots.$

The set of $\{C_m(x) : m = 0,1,...\}$ in Hilbert space $L^2[-1,1]$ is a complete orthogonal system with respect to the weight function $\omega^{\lambda}(z) = (1-z^2)^{\lambda-\frac{1}{2}}$, i.e.:

$$\int_{-1}^{1} C_j^{\lambda}(z) C_k^{\lambda}(z) \omega^{\lambda}(z) dz = h_k^{\lambda} \delta_{jk}, \qquad h_k^{\lambda} = \frac{2^{2\lambda}\Gamma^2(\lambda+\frac{1}{2})\Gamma(k+2\lambda)}{(2k+2\lambda)k!\Gamma^2(2\lambda)}. \tag{2.5}$$

Lemma 2.3. *The set of SHGFs* $h_{ij}(x)$, *is a complete orthogonal system in* $L^2_{\omega_L^{\lambda}}[0,1] := \{u : [0,L] \to \mathbb{R} | \int_0^L u^2(x) \omega_L^{\lambda}(x) dx < \infty\}$ *with respect to weight function,* $\omega_L^{\lambda}(x) = (1-(2n\frac{x}{L} - 2i+1)^2)^{\lambda-\frac{1}{2}}$, *i.e:*

$$\int_0^1 {}_L h_{ij}(x) {}_L h_{pq}(x) \omega_L^{\lambda} dx = \begin{cases} \frac{L\pi 2^{-2\lambda}\Gamma(j+2\lambda)}{nj!(j+\lambda)\Gamma^2(\lambda)}, & j = q, \\ 0, & j \neq q. \end{cases}$$

Proof: *By using the change of variables* $t = 2n\frac{x}{L} - 2i + 1$ *and applying the orthogonal property in* (2.5), *we will have:*

$$\begin{aligned}
\int_0^1 {}_L h_{ij}(x) {}_L h_{pq}(x) \omega_L^{\lambda}(x) dx &= \int_{L\frac{i-1}{n}}^{L\frac{i}{n}} C_j^{\lambda}(2n\frac{x}{L} - 2i+1) C_q^{\lambda}(2n\frac{x}{L} - 2i + 1) \\
&\quad \times (1-(2n\frac{x}{L} - 2i+1)^2)^{\lambda-\frac{1}{2}} dx \\
&= \frac{L}{2n} \int_{-1}^{1} C_j^{\lambda}(t) C_q^{\lambda}(t) \omega(t) dt \\
&= \begin{cases} \frac{L\pi 2^{-2\lambda}\Gamma(j+2\lambda)}{nj!(j+\lambda)\Gamma^2(\lambda)}, & j = q, \\ 0, & j \neq q. \end{cases}
\end{aligned}$$

□

The function $u(x) \in L^2_{\omega_L^\lambda}[0,L]$, can be estimated by a unique function in the space, ${}_{n,m}G_L^\lambda$, where:

$$ {}_{n,m}G_L^\lambda := span\{{}_L h_{ij}^\lambda(x), i=0,1,\dots,n, j=0,1,\dots,m\} \tag{2.6}$$

as:

$$u(x) \simeq \sum_{i=1}^{n}\sum_{j=0}^{m-1} c_{ij\,L}h_{ij}(x) = \mathbf{c^T}\Phi_{n,m}(x), \tag{2.7}$$

where:

$$c_{ij} = \frac{\langle u(x), {}_L h_{ij}(x)\rangle_{\omega_L^\lambda}}{\langle {}_L h_{ij}(x), {}_L h_{ij}(x)\rangle_{\omega_L^\lambda}}, \qquad i=1,2,\dots,\infty, j=0,1,\dots,\infty, \tag{2.8}$$

$$\mathbf{c} = [c_{10},\dots,c_{1,m-1},c_{20},\dots,c_{2,m-1},\dots,c_{n0},\dots,c_{n,m-1}]^T, \tag{2.9}$$

$$ {}_L\Phi_{n,m}(x) = [{}_L h_{10}(x),\dots,{}_L h_{1,m-1}(x),{}_L h_{20}(x),\dots, {}_L h_{2,m-1}(x),\dots,{}_L h_{n0}(x),\dots,{}_L h_{n,m-1}(x)]. \tag{2.10}$$

Similarly, a function of two independent variables $u(x,t) \in {}_{\omega_L^\lambda}L^2[0,L] \times {}_{\omega_L^\lambda}L^2[0,T]$ can be expaned in terms of SHGFs as:

$$u(x,t) \simeq \sum_{i_1=1}^{n}\sum_{j_1=0}^{m-1}\sum_{i_2=1}^{n}\sum_{j_2=0}^{m-1} k_{i_1j_1i_2j_2\,L}h_{i_1,j_1}(x)_T h_{i_2,j_2}(t) = {}_L\,\Phi_{n,m}^T(x) K_T \Phi_{n,m}(t), \tag{2.11}$$

where $\mathbf{K} = [k_{i_1j_1i_2j_2}]_{(nm)\times(nm)}$ is the coefficient matrix and:

$$k_{i_1j_1i_2j_2} = \frac{\langle\langle u(x,t), {}_L h_{i_1,j_1}(x)\rangle_{\omega_L^\lambda}, {}_T h_{i_2,j_2}(t)\rangle_{\omega_L^\lambda}}{\langle {}_L h_{i_1,j_1}(x), {}_L h_{i_1,j_1}(x)\rangle_{\omega_L^\lambda}\langle {}_T h_{i_2,j_2}(t) {}_T h_{i_2,j_2}(t)\rangle_{\omega_L^\lambda}},$$
$$i_1,i_2 = 1,2,\dots,n, \quad j_1,j_2 = 0,1,\dots,m-1.$$

3. SHGFs operational matrix of fractional integration

In this section, the operational matrix of Riemann-Liouville fractional integration for SHGFs will be interoduced.

Theorem 3.1. *If ${}_L\Phi_{n,m}(x)$ be the SHGFs vector and $\alpha > 0$, then $I^\alpha {}_L\Phi_{n,m}(x) \simeq \boldsymbol{P}^\alpha {}_L\Phi_{n,m}(x)$, where the $(mn)\times(mn)$-matrix $\boldsymbol{P}^\alpha = [P_{p,q}]$ is called SHGFs operational matrix of fractional integration of order α and:*

$$P_{p,q} = \sum_{r=0}^{\lceil \frac{j}{2}\rceil}\sum_{s=0}^{j-2r} E_{r,s,p,q}(\gamma_1 - \gamma_2), \qquad i=1,2,\dots,n,\ j=0,1,\dots,m-1, \tag{3.1}$$

such that:

$$E_{r,s,p,q} := \frac{(-1)^r \Gamma(\lambda+j-r) 2^{j-2r+2\lambda+s} n^{s+1} q!(q+\lambda)\Gamma(\lambda)}{r!(j-2r-s)!\Gamma(s+1+\alpha) L\pi\Gamma(q+2\lambda)},$$

$$\gamma_1 = \int_a^b (-1)^{j-2r-s}(t-\frac{i-1}{n})^{s+\alpha} C_q^\lambda (2n\frac{x}{L}-2p+1)\omega_p(x),$$

$$\gamma_2 = \int_c^d (t-\frac{i}{n})^{s+\alpha} C_q^\lambda (2n\frac{x}{L}-2p+1)\omega_p(x),$$

$$a := max\{\frac{p-1}{n}, \frac{i-1}{n}\}, \quad b := max\{\frac{p}{n}, \frac{i-1}{n}\}, \quad c := max\{\frac{p-1}{n}, \frac{i}{n}\}, \quad d := max\{\frac{p}{n}, \frac{i}{n}\}.$$

Proof: *Suppose that* $h_{ij}(x)$ *is the* l*-th element of the vector* ${}_L\Phi_{n,m}(x)$*. By applying the operator* I^α*, we have:*

$$\begin{aligned} I^\alpha h_{ij}(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} h_{ij}(t)dt \\ &= \frac{1}{\Gamma(\alpha)} \{x^{\alpha-1} * h_{ij}(x)\}. \end{aligned}$$

By taking Laplace transform from both sides of the above equation, we will have:

$$\begin{aligned} \mathcal{L}[I^\alpha h_{ij}(x)] &= \frac{1}{\Gamma(\alpha)} \mathcal{L}[x^{\alpha-1}]\mathcal{L}[h_{ij}(x)] \\ &= \frac{1}{s^\alpha}\mathcal{L}[h_{ij}(x)], \\ &= \frac{1}{s^\alpha}\mathcal{L}\left[C_j^\lambda(2n\frac{x}{L}-2i+1)\left(U(x-L\frac{i-1}{n}) - U(x-L\frac{i}{n})\right)\right], \end{aligned}$$

where $U(x)$ *is the unit step function. thus:*

$$\begin{aligned} \mathcal{L}[I^\alpha h_{ij}(x)] &= \frac{1}{s^\alpha}\mathcal{L}[C_j^\lambda(\frac{2n}{L}x-1)]e^{-\frac{i-1}{n}s} - \frac{1}{s^\alpha}\mathcal{L}[C_j^\lambda(\frac{2n}{L}x+1)]e^{-\frac{i}{n}s} \\ &= \sum_{r=0}^{\lceil \frac{j}{2} \rceil} \sum_{m=0}^{j-2r} \frac{(-1)^m \Gamma(\lambda+j-r) 2^{j-2r+m} n^m}{\Gamma(\lambda) r!(j-2r)!(j-2r-m)! L^m} \left(\frac{(-1)^{j-2r-m} e^{-\frac{i-1}{n}s}}{s^{\alpha+m+1}} - \frac{e^{-\frac{i}{n}s}}{s^{\alpha+m+1}} \right). \end{aligned}$$

Now, by taking the inverse Laplace transform, we get:

$$\begin{aligned} I^\alpha h_{ij}(x) = \sum_{r=0}^{\lceil \frac{j}{2} \rceil} \sum_{m=0}^{j-2r} [&\frac{(-1)^m \Gamma(\lambda+j-r) 2^{j-2r+m} n^m}{\Gamma(\lambda) r!(j-2r)!(j-2r-m)!\Gamma(m+\alpha+1) L^m} \\ &\times \left((-1)^{j-2r+m}(x-\frac{i-1}{n})^{m+\alpha} U(x-L\frac{i-1}{n}) - (x-\frac{i}{n})^{m+\alpha} U(x-L\frac{i}{n}) \right)]. \end{aligned}$$

However, for $i=1,\ldots,n$ and $j=0,\ldots,m-1$ by approximating $I^{\alpha}h_{ij}(x)$, via SHGFs, we have:

$$I^{\alpha}h_{ij}(x) \simeq \sum_{p=1}^{n}\sum_{q=0}^{m-1} P_{p,q}h_{p,q}(x), \tag{3.2}$$

so that:

$$\begin{aligned}
P_{p,q} &= \frac{\langle I^{\alpha}h_{ij}(x), h_{p,q}(x)\rangle_{\omega_p(x)}}{\langle h_{p,q}(x), h_{p,q}(x)\rangle_{\omega_p(x)}} \\
&= \frac{\int_0^1 C_q^{\lambda}(2n\frac{x}{L}-2p+1)\left(U(x-L\frac{p-1}{n})-U(x-L\frac{p}{n})\right) I^{\alpha}h_{ij}(x)\omega_p(x)}{\frac{L\pi 2^{-2\lambda}\Gamma(q+2\lambda)}{nq!(q+2\lambda)\Gamma^2(\lambda)}} \\
&= \sum_{r=0}^{\lceil\frac{j}{2}\rceil}\sum_{m=0}^{j-2r} E_{r,s,p,q}\left(\int_a^b f_1(x)dx - \int_c^d f_2(x)dx\right).
\end{aligned}$$

$\square$

4. Description of the method

Consider Eq.(1.1). Suppose that:

$$f(x,t) \simeq {}_L\Phi_{n,m}^T(x)F_T\Phi_{n,m}(t), \tag{4.1}$$

and:

$$\frac{\partial^2 u(x,t)}{\partial x^2} \simeq {}_L\Phi_{n,m}^T(x)O_T\Phi_{n,m}(t), \tag{4.2}$$

where $\Phi_{L,m}(x)$ and $\Phi_{T,m}(t)$ are SHGFs in terms of x and t, respectively. F and O are known and unknown coefficients matrices. By applying the operator I_x^{α} on (4.2) and using property (2.3), we get:

$$u(x,t)-u(0,t)-\frac{\partial u(0,t)}{\partial x}x \simeq {}_L\Phi_{n,m}^T(x)({}_xP_2)^TO_T\Phi_{n,m}(t). \tag{4.3}$$

Utilizing the boundary conditions (1.3), we have:

$$\begin{aligned}
u(x,t) \simeq& {}_L\Phi_{n,m}^T(x)({}_xP_2)^TO_T\Phi_{n,m}(t)+p_1(t) \\
&+\frac{x}{L}\left(p_1(t)-p_2(t)-{}_L\Phi_{n,m}^T(L)({}_xP_2)^TO_T\Phi_{n,m}(t)\right).
\end{aligned} \tag{4.4}$$

If we assume that:

$$p_1(t)\frac{x}{L}\left(p_1(t)-p_2(t)\right) \simeq {}_L\Phi_{n,m}^T(x)S_T\Phi_{n,m}(t), \qquad \frac{x}{L} = {}_L\Phi_{n,m}^T(x)X, \tag{4.5}$$

then, Eq.(4.4) can be rewritten as:

$$u(x,t) \simeq {}_L\Phi_{n,m}^T(x)\left(P_{x2}^TO+S-X{}_L\Phi_{n,m}^T(L)P_{x2}^TO\right){}_T\Phi_{n,m}(t). \tag{4.6}$$

By substituting Eqs.(4.1),(4.2) and Eq.(4.6) into Eq.(1.1) and applying the operator I_t^α on it, we get:

$$u(x,t)-u(x,0)-\frac{\partial u(x,0)}{\partial t}t+\sum_{i=1}^{n}d_i I^{\alpha-\alpha_i}\left(u(x,t)-u(x,0)-\frac{\partial u(x,0)}{\partial t}t\right)$$

$$\simeq I^{\alpha}({}_L\Phi_{n,m}^T(x)O_T\Phi_{n,m}(t))+I^{\alpha}({}_L\Phi_{n,m}^T(x)F_T\Phi_{n,m}(t)). \tag{4.7}$$

Using initial condition (1.2) and substituting Eq.(4.6) into Eq.(4.7), we have:

$${}_L\Phi_{n,m}^T(x)\left(P_{x2}^T O+S-X{}_L\Phi_{n,m}^T(L)P_{x2}^T O\right){}_T\Phi_{n,m}(t)$$

$$\simeq_L \Phi_{n,m}^T(x)\left(OP_\alpha+FP_\alpha-\sum_{i=1}^{n}d_i OP_{\alpha-\alpha_i}+G\right){}_T\Phi_{n,m}(t), \tag{4.8}$$

where G is the coefficients matrix of the function $g(x,t)$, which is defined as follows:

$$g(x,t)=\phi_{L,m}(x)+\psi(x)t-\sum_{i=1}^{n}d_i\left(\phi_{L,m}(x)\frac{t^{\alpha-\alpha_i}}{\Gamma(\alpha-\alpha_i+1)}+\psi(x)\frac{t^{\alpha-\alpha_i+1}}{\Gamma(\alpha-\alpha_i++2)}\right). \tag{4.9}$$

By applying the orthogonal property of SHGFs, we achieve the following linear matrix system of equations:

$$P_{x2}^T O+S-X\Phi_{L,m}^T(L)P_{x2}^T O-OP_\alpha-FP_\alpha+\sum_{i=1}^{n}d_i(P_{x2}^T O+S-X\Phi_{L,m}^T(L)P_{x2}^T O)P_{\alpha-\alpha_i}-G=0. \tag{4.10}$$

Actually, equation (4.10) is a Sylvester system as $AO+OB+C=0$, where A, B and C are as the following:

$$\begin{aligned}&A=P_{x2}^T-X\Phi^T(L)P_{x2}^T,\\&B=-P_\alpha(I+\textstyle\sum_{i=1}^{n}P_{\alpha-\alpha_i})^{-1},\\&C=\left(\textstyle\sum_{i=1}^{n}SP_{\alpha-\alpha_i}-FP_\alpha-G\right)\left(I+\textstyle\sum_{i=1}^{n}P_{\alpha-\alpha_i}\right)^{-1}.\end{aligned}$$

Some algorithms for the numerical solution of Sylvester equations can be find in [10,11,12].

5. Error analysis

In this section, an upper bound of the shifted hybrid Gegenbauer approximation will be given. Also, by using the presented equations in previous section, we introduce a process for estimating the error function when the method is used to solve the main problem,(1.1)-(1.3).

5.1. Error bound

suppose that:

$$\sqcap_{m,n}^{\lambda} := span\{{}_L h_{i_1,j_1}(x)_T h_{i_2,j_2}(t), \quad i_1, 1_2 = 1,2,\dots,n, \quad j_1, j_2 = 0,1,\dots,m-1\},$$

We assume that $u_{m,n}(x,t) \in \sqcap_{m,n}^{\lambda}$ is the best approximation of $u(x,t)$, i.e:

$$\forall v_{m,n}(x,t) \in \sqcap_{m,n}^{\lambda}, \|u(x,t) - u_{m,n}(x,t)\|_\infty \leq \|u(x,t) - v_{m,n}(x,t)\|_\infty. \tag{5.1}$$

If $v_{m,n}(x,t)$ denotes the interpolating polynomial for $u(x,t)$ at points (x_r, t_s) where x_r are the roots of ${}_L h_{i_1,m}(x)$, while t_s are the roots of ${}_L h_{i_2,m}(t)$, then:

$$\begin{aligned} u(x,t) - v_{m,n}(x,t) =& \frac{\partial^m u(\eta,t)}{\partial x^m (m)!} \prod_{r=0}^{m-1}(x - x_i) + \frac{\partial^m u(x,\mu)}{\partial x^m (m)!} \prod_{s=0}^{m-1}(t - t_j) \\ & - \frac{\partial^{2m} u(\eta',\mu')}{\partial x^m \partial t^m (m)!(m)!} \prod_{r=0}^{m-1}(x - x_r) \prod_{s=0}^{m-1}(t - t_s), \end{aligned}$$

where $\eta, \eta' \in [0, L]$ and $\mu, \mu' \in [0, T]$, and we obtain:

$$\begin{aligned} \|u(x,t) - v_{m,n}(x,t)\|_\infty \leq & \, max_{(x,t)\in I} |\frac{\partial^m u(\eta,t)}{\partial x^m}| \frac{\| \prod_{r=0}^{m-1}(x - x_r)\|_\infty}{(m)!} \\ & + max_{(x,t)\in I} |\frac{\partial^m u(x,\mu)}{\partial x^m}| \frac{\| \prod_{s=0}^{m-1}(t - t_s)\|_\infty}{(m)!} \\ & + max_{(x,t)\in I} |\frac{\partial^{2m} u(\eta',\mu')}{\partial x^m \partial t^m}| \frac{\| \prod_{r=0}^{m-1}(x - x_r)\|_\infty \| \prod_{s=0}^{m-1}(t - t_s)\|_\infty}{m!m!}. \end{aligned}$$

Since $u(x,t)$ is a smooth function on $u(x,t) \in [0, L] \times [0, T]$, then there exist the constants c_1, c_2 and c_3, such that:

$$max_{(x,t)\in I} |\frac{\partial^m u(\eta,t)}{\partial x^m}| \leq c_1, \; max_{(x,t)\in I} |\frac{\partial^m u(x,\mu)}{\partial x^m}| \leq c_2, \; max_{(x,t)\in I} |\frac{\partial^{2m} u(\eta',\mu')}{\partial x^m \partial t^m}| \leq c_3. \tag{5.2}$$

Now, Let $x := zL$, then $z \in [0, L]$ and we get:

$$\begin{aligned} min_{x_r \in [0,L]} max_{0 \leq x \leq L} | \prod_{r=0}^{m} (x - x_i)| =& \, min_{z_i \in [0,1]} max_{0 \leq z \leq 1} | \prod_{r=0}^{m} \mathrm{L}(z - z_r)| \\ =& \, L^m min_{z_i \in [0,1]} max_{0 \leq z \leq 1} | \prod_{r=0}^{m-1} (z - z_r)| \\ \leq & \, L^m, \end{aligned} \tag{5.3}$$

where z_is are the roots of $h_{i_1,m}^{\lambda}(z)$. Then by using Eqs.(5.2),(5.3), we have:

$$\|u(x,t) - v_{m,n}(x,t)\|_\infty \leq c_1 \left(\frac{L^m}{m!}\right) + c_2 \left(\frac{T^m}{m!}\right) + c_3 \left(\frac{L^m T^m}{(m!)^2}\right).$$

5.2. Estimation of the error function

Suppos that the function $u_{m,n}(x,t) \in \sqcap^{\lambda}_{m,n}$ is the approximate solution of the main problem (1.1)-(1.3), which is obtained by using the presented method. First, we name:

$$e_{m,n}(x,t) := u(x,t) - u_{m,n}(x,t), \tag{5.4}$$

as the error function where $u(x,t)$ is the exact solution of Eq.(1.1). Hence, $u_{m,n}(x,t)$ satisfies the following problem:

$$P(D_t)u_{m,n}(x,t) - k\frac{\partial^2 u_{m,n}(x,t)}{\partial x^2} = f(x,t) + R_{m,n}(x,t), \tag{5.5}$$

with the following initial and boundary conditions:

$$u_{m,n}(x,0) \simeq \phi(x), \quad \frac{\partial u_{m,n}(x,t)}{\partial t}|_{t=0} \simeq \psi(x), \qquad u_{m,n}(0,t) \simeq p_1(t), \quad u_{m,n}(L,t) \simeq p_2(t), \tag{5.6}$$

Here, $R_{m,n}(x,t)$ is the residual function which is obtained by substituting the approximate solution $u_{m,n}(x,t)$ into Eq.(1.1).

Now, let us subtract Eqs.(1.1),(1.2) from Eqs.(5.5),(5.6), respectively. Hence, we obtain the error problem as follows:

$$P(D_t)e_{m,n}(x,t) - k\frac{\partial^2 e_{m,n}(x,t)}{\partial x^2} = -R_{m,n}(x,t), \tag{5.7}$$

with the homogeneous conditions:

$$e_{m,n}(x,0) = \frac{\partial e_{m,n}(x,t)}{\partial t}|_{t=0} = 0, \qquad e_{m,n}(0,t) = e_{m,n}(L,t) = 0. \tag{5.8}$$

Finally, we solve the error problem Eqs.(5.7),(5.8) in the same way as presented in Section (4) and thus we will find the following approximation:

$$\epsilon_{m,n}(x,t) = \sum_{i_1=1}^{n}\sum_{j_1=0}^{m-1}\sum_{i_2=1}^{n}\sum_{j_2=0}^{m-1} a^*_{i_1,j_1,i_2,j_2}\, {}_L h_{i_1,j_1}(x)\, {}_T h_{i_2,j_2}(t) \tag{5.9}$$

$$= H^T(x)A^*H(t), \tag{5.10}$$

for the error function $e_{m,n}(x,t)$. We note that if the exact solution of the problem (1.1) is unknown, then the maximum absolute error can be estimated approximately by using:

$$E_{m,n}(x,t) = max\{\epsilon_{m,n}(x,t), 0 \le x \le L, 0 \le t \le T\}. \tag{5.11}$$

6. Numerical results and comparisons

In this section we examine the method described in the previous sections by some examples.

Example1. Consider the following time fractional partial differential equation:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{\partial^2 u(x,t)}{\partial x^2} + \left(\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}\right)x^2 - 2t^2, \;\; 0 < x < 1, \; 0 < t < 1,$$

$$u(0,t)=0, u(1,t)=t^2 \;\;\&\;\; u(x,0)=0, u_t(x,0)=0,$$

where $u(x,t)=x^2t^2$ is the exact solution, and $1\le\alpha<2$ and $\beta=2$. The approximate solutions are obtained β, $m=4, n=1$ and $\lambda=0.5$ for a fixed α in a specified time. Figure 2 shows the error for $m=4, n=1$ and $\lambda=0.5$ for $\alpha=1.5$ and $\beta=2$. Table 1 shows the error in $t=0.2$. Figure 1 shows the approximate solutions at moment $t=0.2$ for $\beta=2$, $\beta=1.8$ and $\beta=1.4$ that $m=4, n=1$ and $\lambda=0.5$. Actully, if $\beta\longrightarrow 2$, then we get closer to the exact solution.

x	*error*	*error*	*error*	*EOC*
x	$\beta=1.4$	$\beta=1.8$	$\beta=2$	$\beta=2$
0	0.5e-3	0.3e-3	-0.090e-10	18.3466
0.1	0.5e-3	1e-3	0.141e-10	18.0228
0.2	1.1e-3	0.6e-3	0.288e-10	17.5076
0.3	0.9e-3	0.5e-3	0.364e-10	17.3386
0.4	0.9e-3	0.4e-3	0.382e-10	17.3618
0.5	0.6e-3	0.2e-3	0.355e-10	17.3567
0.6	0.2e-3	-0.0e-3	0.296e-10	17.4878
0.7	-0.3e-3	-0.2e-3	0.218e-10	17.7084
0.8	-0.7e-3	-0.2e-3	0.134e-10	17.0595
0.9	-0.6e-3	-0.2e-3	0.057e-10	18.6761

Table 1: The eror function for $m=4, n=1$ and different β in example 1.

Therefore EOC in the table1 is the estimated order of convergence such that $EOC=\frac{\log|e|}{log\frac{1}{M}}$, we expect that $e\propto(\frac{1}{M})^r$, so $r=\frac{\log|e|}{log\frac{1}{M}}$.

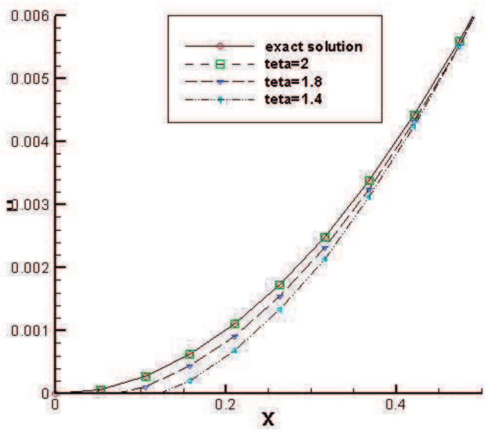

Figure 1: Approximate solution for $m=4, n=1$ and $\alpha=1.5$ in ex. 1.

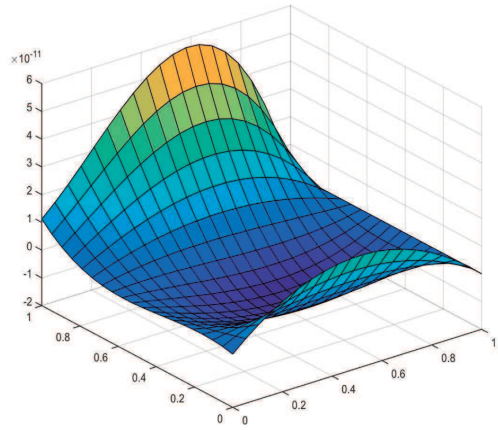

Figure 2: The eror function for $m = 4, n = 1$ and $\alpha = 1.4$ in ex. 1.

Example2. Consider the following time fractional partial differential equation:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + \frac{\partial^{\alpha_1} u(x,t)}{\partial t^{\alpha_1}} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \qquad 0 \leq t, x \leq 1,$$

$$u(x,0) = 0, \qquad \frac{\partial u(x,0)}{\partial t} = 0 \ , \ u(0,t) = 0, \qquad u(1,t) = 0,$$

Such that $u(x,t) = sin(\pi x)t^3$ is the exact solution, where $1 \leq \alpha < 2$ and $f(x,t)$ is as follows:

$$f(x,t) = \left(\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha} + \frac{6}{\Gamma(4-\alpha_1)} t^{3-\alpha_1} \right) sin(\pi x) + \pi^2 t^3 sin(\pi x).$$

The procedure for $\alpha = 1.5, \lambda = 0.5$ presents the error function value at moment $t = 0.2$ for $m = 4$ and different n. The results show that the larger n is the smaller error we have, so we get closer to the exact solution.

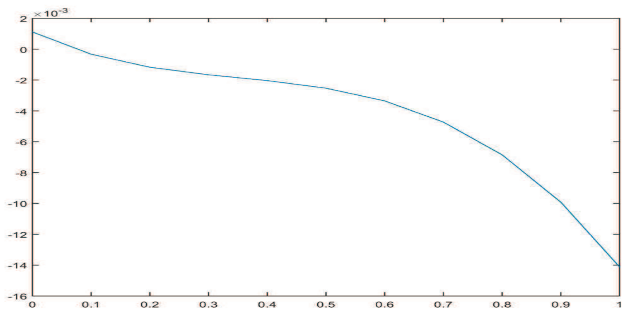

Figure 3: The error function for $m = 4, n = 1$ in moment $t = 0.2$ in ex.2.

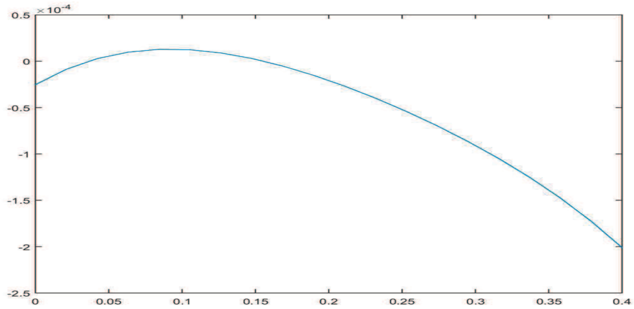

Figure 4: The error function for $m = 4, n = 2$ in moment $t = 0.2$ in ex.2.

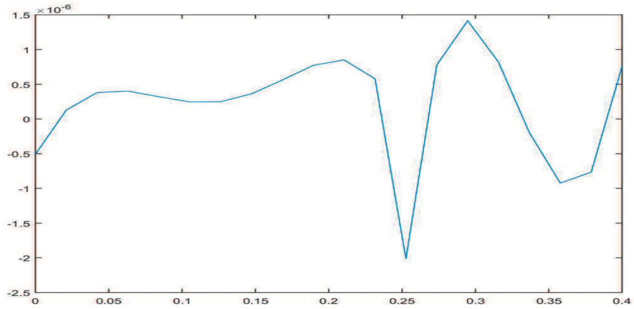

Figure 5: The error function for $m = 4, n = 4$ in moment $t = 0.2$ in ex.2.

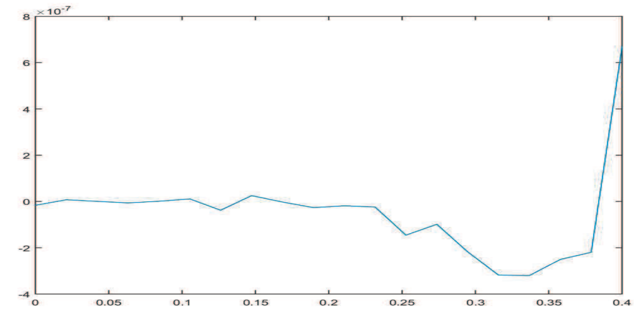

Figure 6: The error function for $m = 4, n = 8$ in moment $t = 0.2$ in ex.2.

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