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Infinitely Many Solutions for Nonlocal Problems with Variable Exponent and Nonhomogeneous Neumann Condition

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abstract: In this article we will provide new multiplicity results of the solutions for nonlocal problems with variable exponent and nonhomogeneous Neumann conditions. We investigate the existence of infinitely many solutions for perturbed nonlocal problems with variable exponent and nonhomogeneous Neumann conditions. The approach is based on variational methods and critical point theory.

Key Words: Infinitely many solutions; Variable exponent Sobolev spaces, $p(x)$-Laplacian, Nonhomogeneous neumann condition, Variational methods, Critical point theory.

Contents

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary $\partial\Omega$. The aim of this paper is to investigate the existence of infinitely many solutions for the following nonlocal problem

$$\begin{cases} T(u) = \lambda f(x,u(x)) + \mu h(x,u(x)), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} = \lambda g(\vartheta u(x)), & \text{on } \partial\Omega \end{cases} \tag{$P_g^{f,h}$}$$

where

$$T(u) = M\Big(\int_\Omega \frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\big)\mathrm{d}x\Big)(-\Delta_{p(x)}u + \alpha(x)|u|^{p(x)-2}u)$$

in which $M : [0,+\infty[\to \mathbb{R}$ is a continuous function such that there are two positive constants m_0 and m_1 with $m_0 \le M(t) \le m_1$ for all $t \ge 0$, $p \in C(\bar\Omega)$, $\Delta_{p(x)}u := \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ is the $p(x)$-Laplacian operator and $\alpha \in L^\infty(\Omega)$ with $ess\inf_\Omega \alpha > 0$, $\lambda \in [0,+\infty)$, $f,h : \Omega\times\mathbb{R} \to \mathbb{R}$ are two L^1-Carathéodory functions, $\lambda > 0$ and $\mu \ge 0$ are two parameters, v is the outer unit normal to $\partial\Omega$, $g : \mathbb{R} \to \mathbb{R}$ is a non-negative continuous function and $\vartheta : W^{1,p(x)}(\Omega) \to L^{p(x)}(\partial\Omega)$ is the trace

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operator. If $\Omega =]a, b[$ and $k : [a, b] \to \mathbb{R}$ is a continuous function, then $\int_{\partial\Omega} k(x)\mathrm{d}x$ reads $k(b) + k(a)$.

Problems like ($P_g^{f,h}$) are usually called nonlocal problems because of the presence of the integral over the entire domain, and this implies that the first equation in ($P_g^{f,h}$) is no longer a pointwise identity.

The problem ($P_g^{f,h}$) is related to the nonstationary problem

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 \mathrm{d}x\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

for $0 < x < L$, $t \geq 0$, where $u = u(x, t)$ is the lateral displacement at the space coordinate x and the time t, E the Young modulus, ρ the mass density, h the cross-section area, L the length and ρ_0 the initial axial tension, proposed by Kirchhoff [38] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example in [3,17]. On the other hand, Kirchhoff-type boundary value problems model several physical and biological systems where u describes a process which depends on the average of itself, for example the population density. It received great attention only after Lions [40] proposed an abstract framework for the problem. The solvability of the Kirchhoff type problems has been under various authors' attentions. Some early classical investigations of Kirchhoff equations can be seen in the papers [2,30,32,33,41,44,45,54] and the references therein.

Differential equations and variational problems including $p(x)$-growth conditions due to their applications have been studied deeply by many researchers. It varies from nonlinear elasticity theory, electro-rheological fluids, and so on (see [55,59]). The necessary framework for the study of these problems is represented by the function spaces with variable exponent $\mathrm{L}^{p(x)}(\Omega)$ and $\mathrm{W}^{m,p(x)}(\Omega)$. Materials which require such advanced theories have been under experimental studies from the 1950s onwards. The first important discovery on electro-rheological fluids was contributed by Willis Winslow in 1949. The viscosity of these fluids depends on the electric field of the fluids. He discovered that the viscosity of such fluids as instance lithium polymetachrylate in an electrical field is an inverse relation to the strength of the field. The field causes string-like formations in the fluid, parallel to the field. They can increase the viscosity five orders of magnitude. This event is called the Winslow effect. For a general account of the underlying physics see [31] and for some technical applications [48]. Electro-rheological fluids also have functions in robotics and space technology. Many experimental researches have been done chiefly in the USA, as in NASA laboratories. For more information on properties, modeling and the application of variable exponent spaces to these fluids we refer to [50,51,52]. The study of various mathematical problems with variable exponent has received considerable attention in recent years. For background and recent results, we refer the reader to [1,9,10,14,19,27,36,42,47,49,52,57,58] and the references therein for details. For example, Yao in [57] by using the variational method, under appropriate assumptions on f and g, obtained a number of results on existence and multiplicity of solutions for the nonlinear Neumann boundary value problem of the form

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \lambda f(x,u), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} = \mu g(x,u), & \text{on } \partial\Omega \end{cases}$$

where $\lambda, \mu \in \mathbb{R}$, $p \in \mathrm{C}(\bar{\Omega})$ and $p(x) > 1$. Moschetto [47] under suitable assumptions on the functions α, f, p and g, based on the Ricceri two-local-minima theorem, together with the Palais-Smale property, studied the existence of at least three solutions for the following Neumann problem:

$$\begin{cases} -\Delta_{p(x)}u + \alpha(x)|u|^{p(x)-2}u = \alpha(x)f(u) + \lambda g(x,u), & \text{in } \Omega, \\ \frac{\partial u}{\partial v} = 0, & \text{on } \partial\Omega. \end{cases}$$

Bonanno and Chinnì in [10] by applying variational methods under appropriate growth conditions on the nonlinearity, obtained the existence of multiple solutions for nonlinear elliptic Dirichlet problems with variable exponent. Yin in [58] based on three critical points theorem due Ricceri, obtained the existence of three solutions to a Neumann problem with nonstandard growth conditions. D'Aguì and Sciammetta in [19] established the existence of an unbounded sequence of weak solutions for a class of differential equations with $p(x)$-Laplacian and subject to small perturbations of nonhomogeneous Neumann conditions. Qian et al. in [49] by the theory of the generalized Lebesgue-Sobolev space $\mathrm{W}^{1,p(x)}(\Omega)$, a nonsmooth Mountain Pass theorem and the Weierstrass theorem, studied the nonhomogeneous Neumann problem of $p(x)$-Laplacian equations, where the weighted function $V(x)$ is indefinite and the potential is only locally Lipschitz.

On the other hand, $p(x)$-Kirchhoff problems which are investigated on function spaces with variable exponents, have been studied by many researchers, see [15,16,20,21,23,24,34,35,37,56] and the references therein. For example, Dai and Hao in [20] by means of a direct variational approach and the theory of the variable exponent Sobolev spaces, established conditions ensuring the existence and multiplicity of solutions for the $p(x)$-Kirchhoff-type problem with Dirichlet boundary data. Viasi in [56] used variational techniques to prove an eigenvalue theorem for a stationary $p(x)$-Kirchhoff problem, and provided an estimate for the range of such eigenvalues. He employed a specific family of test functions in variable exponent Sobolev spaces. His approach permits to handle both non-degenerate and degenerate Kirchhoff coefficients. In [16], Cammaroto and Vilasi by using variational nature and weak formulation takes place in suitable variable exponent Sobolev spaces, established the existence of three weak solutions for a nonlinear transmission problem involving degenerate nonlocal coefficients of $p(x)$-Kirchhoff type. In [24] multiplicity results for the problem ($P_g^{f,h}$), in the case $\mu = 0$ were established. In fact, using variational methods and critical point theory the existence results for the problem under algebraic conditions with the classical Ambrosetti-Rabinowitz (AR) condition on the nonlinear term were ensured. Furthermore, by combining two conditions on the nonlinear term which guarantees the existence of two solutions, applying the mountain pass theorem given by Pucci and Serrin the existence of third solution for the problem was proved while in [23] based on variational methods the existence of at least one weak solution for the same problem was discussed. In [34] using two kinds of three critical point theorem the existence of at least three weak solutions for a class of differential equations with $p(x)$-Kirchhoff-type and subject to perturbations of nonhomogeneous Neumann conditions was studied.

The existence and multiplicity of solutions for stationary higher order problems of Kirchhoff type (in n-dimensional domains, $n \geq 1$) were also treated in some recent papers, via variational methods like the symmetric mountain pass theorem in [18] and a three critical point theorem in [6]. Moreover, in [4,5], some evolutionary higher order Kirchhoff type problems were treated, mainly focusing on the qualitative properties of the solutions.

We refer to the recent monograph by Molica Bisci, Rădulescu and Servadei [46] for related problems concerning the variational analysis of solutions of some classes of nonlocal problems.

Motivated by the above works, in the present paper, by employing a smooth version of [13, Theorem 2.1], which is more precise version of Ricceri's Variational Principle [53, Theorem 2.5] under some hypotheses on the behavior of the nonlinear terms at infinity, we prove the existence of definite intervals about λ and μ in which the problem $(P_g^{f,h})$ admits a sequence of solutions which is unbounded in the generalized Lebesgue-Sobolev space $\mathrm{W}^{1,p(x)}(\Omega)$ which will be introduced later (Theorem 3.1). Furthermore, some consequences of Theorem 3.1 are listed. Replacing the conditions at infinity on the nonlinear terms, by a similar one at zero, we obtain a sequence of pairwise distinct solutions strongly converging at zero; see Theorem 3.12. Three examples of applications are pointed out (see Examples 3.5, 3.11 and 3.15).

For a discussion of the existence of infinitely many solutions for some differential and difference equations, applying Ricceri's Variational Principle [53] and its variants we refer to the paper [7,11,12,22,33,43].

The paper is organized as follows. In Section 2, we recall some basic definitions and our main tool, while Section 3 is devoted to our abstract results.

2. Preliminaries

Our main tool to ensure the existence of infinitely many solutions for the problem $(P_g^{f,h})$ is a smooth version of Theorem 2.1 of [13] which is a more precise version of Ricceri's Variational Principle [53] that we now recall here.

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r])} \Psi(v) - \Psi(u)}{r - \Phi(u)}$$

and

$$\gamma := \liminf_{r\to+\infty} \varphi(r), \qquad \delta := \liminf_{r\to(\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(*a*) *for every* $r > \inf_X \Phi$ *and every* $\lambda \in]0, \frac{1}{\varphi(r)}[$, *the restriction of the functional* $I_\lambda = \Phi - \lambda\Psi$ *to* $\Phi^{-1}(]-\infty, r[)$ *admits a global minimum, which is a critical point (local minimum) of* I_λ *in* X*.*

(*b*) *If* $\gamma < +\infty$ *then, for each* $\lambda \in]0, \frac{1}{\gamma}[$, *the following alternative holds: either*

(b_1) I_λ *possesses a global minimum,*

or

(b_2) *there is a sequence* $\{u_n\}$ *of critical points (local minima) of* I_λ *such that*

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(*c*) *If* $\delta < +\infty$ *then, for each* $\lambda \in]0, \frac{1}{\delta}[$, *the following alternative holds: either*

(c_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ*,*

or

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of* I_λ *which weakly converges to a global minimum of* Φ*.*

Here and in the sequel, $\mathrm{meas}(\Omega)$ denotes the Lebesgue measure of the set Ω, and we also assume that $p \in \mathrm{C}(\bar{\Omega})$ verifies the following condition:

$$N < p^- := \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := \sup_{x\in\Omega} p(x) < +\infty \tag{2.1}$$

and is globally log-Hölder continuous on Ω (see Definition 4.1.1 and Remark 4.1.5 of [26]). Define the variable exponent Lebesgue space by

$$\mathrm{L}^{p(x)}(\Omega) := \Big\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|^{p(x)} \mathrm{d}x < +\infty\Big\},$$

$$\mathrm{L}^{p(x)}(\partial\Omega) := \Big\{u : \partial\Omega \to \mathbb{R} \text{ measurable and } \int_{\partial\Omega} |u(x)|^{p(x)} \mathrm{d}\sigma < +\infty\Big\}.$$

On $\mathrm{L}^{p(x)}(\Omega)$ and $\mathrm{L}^{p(x)}(\partial\Omega)$ we consider the norms respectively

$$\|u\|_{\mathrm{L}^{p(x)}(\Omega)} = \inf\Big\{\eta > 0 : \int_\Omega \Big|\frac{u(x)}{\eta}\Big|^{p(x)} \mathrm{d}x \le 1\Big\},$$

$$\|u\|_{\mathrm{L}^{p(x)}(\partial\Omega)} = \inf\Big\{\eta > 0 : \int_{\partial\Omega} \Big|\frac{u(x)}{\eta}\Big|^{p(x)} \mathrm{d}\sigma \le 1\Big\},$$

where $d\sigma$ is the surface measure on $\partial\Omega$.

Consider the generalized Lebesgue-Sobolev space

$$\mathrm{W}^{1,p(x)}(\Omega) = \Big\{ u \in \mathrm{L}^{p(x)}(\Omega) : \ |\nabla u| \in \mathrm{L}^{p(x)}(\Omega) \Big\}$$

endowed with the following norm

$$\|u\|_{\mathrm{W}^{1,\mathrm{p(x)}}(\Omega)} := \|u\|_{\mathrm{L}^{p(x)}(\Omega)} + \||\nabla u|\|_{\mathrm{L}^{p(x)}(\Omega)}. \tag{2.2}$$

It is well known (see [29]) that, in view of (2.1), both $\mathrm{L}^{p(x)}(\Omega)$ and $\mathrm{W}^{1,p(x)}(\Omega)$, with the respective norms, are separable, reflexive and uniformly convex Banach spaces. Moreover, since $\alpha \in \mathrm{L}^\infty(\Omega)$, with $\alpha^- := ess\inf_{x\in\Omega} \alpha(x) > 0$ is assumed, then the following norm

$$\|u\|_\alpha = \inf\Big\{\sigma > 0 : \int_\Omega \Big(\alpha(x)\Big|\frac{u(x)}{\sigma}\Big|^{p(x)} + \Big|\frac{\nabla u(x)}{\sigma}\Big|^{p(x)}\Big) \mathrm{d}x \le 1\Big\},$$

on $\mathrm{W}^{1,p(x)}(\Omega)$ is equivalent to that introduce in (2.2). Since $\mathrm{W}^{1,p(x)}(\Omega)$ is continuously embedded in $\mathrm{W}^{1,p^-}(\Omega)$ (see [29] or [39]) and $p^- > N$, $\mathrm{W}^{1,p(x)}(\Omega)$ is continuously embedded in $\mathrm{C}^0(\bar{\Omega})$ and one has

$$\|u\|_{\mathrm{C}^0(\bar{\Omega})} \le k_{p^-}\|u\|_{\mathrm{W}^{1,p^-}(\Omega)}.$$

When Ω is convex, an explicit upper bound for the constant k_{p^-} is

$$k_{p^-} \le 2^{\frac{p^- -1}{p^-}} \max\Bigg\{ \Big(\frac{1}{\|\alpha\|_{\mathrm{L}^1(\Omega)}}\Big)^{\frac{1}{p^-}}, \frac{D}{N^{\frac{1}{p^-}}}\Big(\frac{p^- -1}{p^- - N}\mathrm{meas}(\Omega)\Big)^{\frac{p^- -1}{p^-}} \frac{\|\alpha\|_\infty}{\|\alpha\|_{\mathrm{L}^1(\Omega)}} \Bigg\},$$

where $D = \mathrm{diam}(\Omega)$ and $\mathrm{meas}(\Omega)$ is the Lebesgue measure of Ω (see [8, Remark 1]). On the other hand, taking into account that $p^- \le p(x)$, [39, Theorem 2.8] ensures that $\mathrm{L}^{p(x)}(\Omega) \hookrightarrow \mathrm{L}^{p^-}(\Omega)$ and the constant of such embedding does not exceed $1 + \mathrm{meas}(\Omega)$. So, one has

$$\|u\|_{\mathrm{W}^{1,p^-}(\Omega)} \le (1 + \mathrm{meas}(\Omega))\|u\|_{\mathrm{W}^{1,p(x)}(\Omega)} \le (1 + \mathrm{meas}(\Omega))\|\alpha\|_{\mathrm{L}^1(\Omega)}.$$

In conclusion, put

$$\varrho = k_{p^-}(1 + \mathrm{meas}(\Omega)),$$

it results

$$\|u\|_{\mathrm{C}^0(\bar{\Omega})} \le \varrho\|u\|_\alpha \tag{2.3}$$

for each $u \in \mathrm{W}^{1,p(x)}(\Omega)$.

Put

$$F(x,t) := \int_0^t f(x,\xi)\mathrm{d}\xi \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

$$H(x,t) := \int_0^t h(t,\xi)\mathrm{d}\xi \quad \text{for all } (t,x) \in \Omega \times \mathbb{R},$$

$$G(t) := \int_0^t g(\xi)\mathrm{d}\xi \quad \text{for all } t \in \mathbb{R}$$

and

$$\widehat{M}(t) = \int_0^t M(\xi)\mathrm{d}\xi \quad \text{for all } t \geq 0.$$

Definition 2.2. *We mean by a (weak) solution of the problem* ($P_g^{f,h}$)*, any function* $u \in \mathrm{W}^{1,p(x)}(\Omega)$ *such that*

$$M\Big(\int_\Omega \frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\big)\mathrm{d}x\Big)$$
$$\int_\Omega \Big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x) + \alpha(x)|u(x)|^{p(x)-2}u(x)v(x)\Big)\mathrm{d}x$$
$$-\lambda\int_\Omega f(x,u(x))v(x)\mathrm{d}x - \lambda\int_{\partial\Omega} g(\vartheta u(x))\vartheta v(x)\mathrm{d}\sigma - \mu\int_\Omega h(x,u(x))v(x)\mathrm{d}x = 0$$

for every $v \in \mathrm{W}^{1,p(x)}(\Omega)$.

Proposition 2.3 ([25, Proposition 2.4])**.** *Let*

$$\rho_\alpha(u) = \int_\Omega \Big(|\nabla u|^{p(x)} + \alpha(x)|u|^{p(x)}\Big)\,\mathrm{d}x$$

for $u \in \mathrm{W}^{1,p(x)}(\Omega)$*, we have*
(1) $\|u\|_\alpha \geq 1 \Longrightarrow \|u\|_\alpha^{p^-} \leq \rho_\alpha(u) \leq \|u\|_\alpha^{p^+}$,
(2) $\|u\|_\alpha \leq 1 \Longrightarrow \|u\|_\alpha^{p^+} \leq \rho_\alpha(u) \leq \|u\|_\alpha^{p^-}$.

A special case of our main result is the following theorem.

Theorem 2.4. *Let* $\mathrm{meas}(\Omega) = \int_{\partial\Omega}\mathrm{d}\sigma = 1$, $f : \mathbb{R} \longrightarrow \mathbb{R}$ *be a continuous function and put* $F(t) = \int_0^t f(\xi)\mathrm{d}\xi$ *for all* $t \in \mathbb{R}$*. Assume that*

$$\liminf_{\xi\to+\infty}\frac{(F+G)(\xi)}{\xi^p} = 0 \quad \textit{and} \quad \limsup_{\xi\to+\infty}\frac{(F+G)(\xi)}{\xi^p} = +\infty.$$

Then, for every continuous function $h : \mathbb{R} \longrightarrow \mathbb{R}$ *whose* $H(t) = \int_0^t h(\xi)\mathrm{d}\xi$ *for every* $t \in \mathbb{R}$*, is a nonnegative function satisfying in the condition*

$$h_\star := \frac{\varrho^p p}{m_0}\lim_{\xi\to+\infty}\frac{\sup_{|t|\leq\xi} H(t) + G(\xi)}{\xi^p} < +\infty$$

and for every $\mu \in [0, \mu_{\star,\lambda})$ *where* $\mu_{\star,\lambda} := \frac{1}{h_\star}\Big(1 - \lambda\varrho^p p \liminf_{\xi\to+\infty}\frac{F(\xi)}{\xi^p}\Big)$*, the problem*

$$\begin{cases} M\Big(\int_\Omega \frac{1}{p}\big(|\nabla u(x)|^p + |u(x)|^p\big)\mathrm{d}x\Big)(-\Delta_p u(x) + |u(x)|^{p-2}u(x)) & \\ \quad = \lambda f(u(x)) + \mu h(u(x)), & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial v} = \lambda g(\vartheta u(x)), & \text{on } \partial\Omega \end{cases}$$

has an unbounded sequence of solutions.

3. Main results

We present our main result as follows.

Put

$$\Gamma(\partial\Omega)=\int_{\partial\Omega}\mathrm{d}\sigma \quad \text{and} \quad \mathcal{B}^{\infty}=\limsup_{\xi\to+\infty}\frac{\int_\Omega F(x,\xi)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}}.$$

Theorem 3.1. *Assume that there exist two real sequences $\{a_n\}$ and $\{b_n\}$ with $a_n>1$ for all $n\in\mathbb{N}$ and $\lim_{n\to+\infty}b_n=+\infty$, such that*

(*A1*) $a_n^{p^+}<\frac{m_0p^-b_n^{p^-}}{m_1p^+\varrho^{p^-}\|\alpha\|_{L^1(\Omega)}}$;

(*A2*) $\mathcal{A}_\infty:=\lim_{n\to+\infty}\frac{\int_\Omega\sup_{|t|\le b_n}F(x,t)\mathrm{d}x-\int_\Omega F(x,a_n)\mathrm{d}x+\Gamma(\partial\Omega)(G(b_n)-G(a_n))}{m_0p^-b_n^{p^-}-m_1p^+\varrho^{p^-}\|\alpha\|_{L^1(\Omega)}a_n^{p^+}}$
$<\frac{m_0p^-}{m_1\|\alpha\|_{L^1(\Omega)}\varrho^{p^-}p^+}\mathcal{B}^\infty$.

Then, for each $\lambda\in(\lambda_1,\lambda_2)$ with $\lambda_1:=\frac{m_1\|\alpha\|_{L^1(\Omega)}}{p^-\mathcal{B}^\infty}$ and $\lambda_2:=\frac{m_0}{\varrho^{p^-}p^+\mathcal{A}_\infty}$, for every continuous function $h:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ whose $H(x,t)=\int_0^t g(x,\xi)\mathrm{d}\xi$ for every $(x,t)\in\Omega\times\mathbb{R}$, is a nonnegative function satisfying the condition

$$h_{b_n}:=\frac{\varrho^{p^-}p^+}{m_0}\times\lim_{n\to\infty}\frac{\int_\Omega\sup_{|t|\le b_n}H(x,t)\mathrm{d}x-\int_\Omega H(x,a_n)\mathrm{d}x+\Gamma(\partial\Omega)\left(G(b_n)-G(a_n)\right)}{m_0p^-b_n^{p^-}-m_1p^+\varrho^{p^-}\|\alpha\|_{L^1(\Omega)}a_n^{p^+}}<+\infty \tag{3.1}$$

and for every $\mu\in\left[0,\mu_{h,\lambda}\right)$ with $\mu_{h,\lambda}:=\frac{m_0-\lambda p^+\varrho^{p^-}\mathcal{A}_\infty}{m_0h_{b_n}}$, the problem $(P_g^{f,h})$ has an unbounded sequence of solutions in $W^{1,p(x)}(\Omega)$.

Proof: Fix $\overline{\lambda}\in(\lambda_1,\lambda_2)$ and let h be a function satisfying the condition (3.1). Since, $\overline{\lambda}<\lambda_2$, one has $\mu_{h,\overline{\lambda}}>0$. Fix $\overline{\mu}\in[0,\mu_{h,\overline{\lambda}}[$ and put $\nu_1:=\lambda_1$ and $\nu_2:=\frac{\lambda_2}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2h_{b_n}}$. If $h_{b_n}=0$, clearly, $\nu_1=\lambda_1$, $\nu_2=\lambda_2$ and $\overline{\lambda}\in]\nu_1,\nu_2[$. If $h_{b_n}\neq0$, since $\overline{\mu}<\mu_{g,\overline{\lambda}}$, we obtain $\frac{\overline{\lambda}}{\lambda_2}+\overline{\mu}h_{b_n}<1$, and so $\frac{\lambda_2}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_2h_{b_n}}>\overline{\lambda}$, namely, $\overline{\lambda}<\nu_2$. Hence, since $\overline{\lambda}>\lambda_1=\nu_1$, one has $\overline{\lambda}\in]\nu_1,\nu_2[$. Now, set $Q(x,t)=F(x,t)+\frac{\overline{\mu}}{\overline{\lambda}}H(x,t)$ for all $(x,t)\in\Omega\times\mathbb{R}$. Take $X=W^{1,p(x)}(\Omega)$ and define on X two functionals Φ and Ψ by setting, for each $u\in X$, as follows

$$\Phi(u)=\widehat{M}\Big(\int_\Omega\frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)}+\alpha(x)|u(x)|^{p(x)}\big)\mathrm{d}x\Big)$$

and

$$\Psi(u)=\int_\Omega F(x,u(x))\mathrm{d}x+\int_{\partial\Omega}G(\vartheta u(x))\mathrm{d}\sigma+\frac{\overline{\mu}}{\overline{\lambda}}\int_\Omega H(x,u(x))\mathrm{d}x.$$

Since X is a finite dimensional Banach space Ψ is a Gâteaux differentiable functional and sequentially weakly upper semi-continuous whose Gâteaux derivative at the point $u \in X$ is the functional $\Psi'(u) \in X^*$, given by

$$\Psi'(u)(v) = \int_\Omega f(x,u(x))v(x)\mathrm{d}x + \int_{\partial\Omega} g(\vartheta u(x))\vartheta v(x)\mathrm{d}\sigma + \frac{\overline{\mu}}{\overline{\lambda}}\int_\Omega h(x,u(x))v(x)\mathrm{d}x$$

for every $v \in X$, and $\Psi' : X \to X^*$ is a compact operator. Moreover, Φ is a Gâteaux differentiable functional which Gâteaux derivative at the point $u \in X$ is the functional $\Phi'(u) \in X^*$, given by

$$\begin{aligned}\Phi'(u)(v) =& M\Big(\int_\Omega \frac{1}{p(x)}\big(|\nabla u(x)|^{p(x)} + \alpha(x)|u(x)|^{p(x)}\big)\mathrm{d}x\Big)\\ &\times \int_\Omega \big(|\nabla u(x)|^{p(x)-2}\nabla u(x)\nabla v(x) + \alpha(x)|u(x)|^{p(x)-2}u(x)v(x)\big)\mathrm{d}x\end{aligned}$$

for every $v \in X$. Furthermore, Φ is sequentially weakly lower semi-continuous (see [8, Remark 2.2 and Remark 2.3]). Put $I_{\overline{\lambda}} := \Phi - \overline{\lambda}\Psi$. We observe that the weak solutions of the problem $(P_g^{f,h})$ are exactly the solutions of the equation $I'_{\overline{\lambda}}(u) = 0$. So, our end is to apply Theorem 2.1 to Φ and Ψ. Now, we wish to prove that $\gamma < +\infty$, where γ is defined in Theorem 2.1. Put $r_n = \frac{m_0}{p^+}(\frac{b_n}{\varrho})^{p^-}$ for all $n \in \mathbb{N}$. For all $u \in X$ with $\Phi(u) < r_n$, owing to [14, Poroposition 2.2], one has

$$\|u\|_\alpha \le \max\{(p^+r_n)^{\frac{1}{p^+}}, (p^+r_n)^{\frac{1}{p^1}}\} = \frac{b_n}{\varrho}.$$

So, due to the embedding $X \hookrightarrow \mathrm{C}^0(\Omega)$ (see (2.3)), one has $\|u\|_\infty \le \varrho\|u\|_\alpha < b_n$. It follows that

$$\Phi^{-1}(-\infty, r_n) = \{u \in X; \Phi(u) < r_n\} \subseteq \{u \in X; |u| \le b_n\}.$$

Now, for each $n \in \mathbb{N}$, let w_n be defined by $w_n(x) = a_n$ for every $x \in \Omega$, Clearly, $w_n \in X$,

$$\begin{aligned}\Phi(w_n) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}\big(|\nabla a_n|^{p(x)} + \alpha(x)|a_n|^{p(x)}\big)\mathrm{d}x\Big)\\ &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}\alpha(x)a_n^{p(x)}\mathrm{d}x\Big)\end{aligned}$$

and since $a_n > 1$ for all $n \in \mathbb{N}$,

$$\frac{m_0 a_n^{p^-}}{p^+}\|\alpha\|_{\mathrm{L}^1(\Omega)} \le \Phi(w_n) \le \frac{m_1 a_n^{p^+}}{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)}.$$

Moreover, from the assumption $(A1)$ one has $\Phi(w_n) < r_n$. Hence, for every n large enough, one has

$$\varphi(r_n) = \inf_{u\in\Phi^{-1}(]-\infty,r_n[)} \frac{(\sup_{v\in\Phi^{-1}(]-\infty,r_n])}\Psi(v)) - \Psi(u)}{r_n - \Phi(u)} \leq \frac{\sup_{v\in\Phi^{-1}(]-\infty,r_n])}\Psi(v)}{r_n - \Phi(u)} \tag{3.2}$$

$$\begin{aligned}
&\leq \frac{\int_\Omega \sup_{|t|\leq b_n} Q(x,t)\mathrm{d}x + \int_{\partial\Omega}\sup_{|t|\leq b_n} G(\vartheta(t))\mathrm{d}\sigma - \Psi(w_n)}{\frac{m_0}{p^+}(\frac{b_n}{\varrho})^{p^-} - \Phi(w_n)} \\
&= \frac{\int_\Omega \sup_{|t|\leq b_n}\big(F(x,t) + \frac{\overline{\mu}}{\lambda}H(x,t)\big)\mathrm{d}x + \int_{\partial\Omega}\sup_{|t|\leq b_n} G(\vartheta(b_n))\mathrm{d}\sigma - \Psi(a_n)}{\frac{m_0}{p^+}(\frac{b_n}{\varrho})^{p^-} - \Phi(a_n)} \\
&\leq \frac{\varrho^{p^-}p^+}{m_0}\Bigg(\frac{\int_\Omega \sup_{|t|\leq b_n} F(x,t)\mathrm{d}x - \int_\Omega F(x,a_n)\mathrm{d}x + \Gamma(\partial\Omega)(G(b_n) - G(a_n))}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}} \\
&\quad + \frac{\overline{\mu}}{\lambda}\frac{\int_\Omega \sup_{|t|\leq b_n} H(x,t)\mathrm{d}x - \int_\Omega H(x,a_n)\mathrm{d}x}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}}\Bigg).
\end{aligned}$$

Now by $(A2)$ we have

$$\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\leq b_n} F(x,t)\mathrm{d}x - \int_\Omega F(x,a_n)\mathrm{d}x + \Gamma(\partial\Omega)(G(b_n) - G(a_n))}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}} \tag{3.3}$$
$$\leq \mathcal{A}_\infty < +\infty.$$

Then, in view of (3.1) and (3.3), we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\int_\Omega \sup_{|t|\leq b_n} F(x,t)\mathrm{d}x - \int_\Omega F(x,a_n)\mathrm{d}x + \Gamma(\partial\Omega)(G(b_n) - G(a_n))}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}} \\
&\quad + \lim_{n\to\infty}\frac{\overline{\mu}}{\lambda}\frac{\int_\Omega \sup_{|t|\leq b_n} H(x,t)\mathrm{d}x - \int_\Omega H(x,a_n)\mathrm{d}x}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}} < +\infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma &\leq \liminf_{n\to+\infty}\varphi(r_n) \qquad (3.4) \\
&\leq \lim_{n\to\infty} \frac{\int_\Omega \sup_{|t|\leq b_n} F(x,t)\mathrm{d}x - \int_\Omega F(x,a_n)\mathrm{d}x + \Gamma(\partial\Omega)(G(b_n) - G(a_n))}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}} \\
&\quad + \lim_{n\to\infty}\frac{\overline{\mu}}{\lambda}\frac{\int_\Omega \sup_{|t|\leq b_n} H(x,t)\mathrm{d}x - \int_\Omega H(x,a_n)\mathrm{d}x}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}} < +\infty.
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{\int_\Omega \sup_{|t|\leq b_n} Q(x,t)\mathrm{d}x - \int_\Omega Q(x,a_n)\mathrm{d}x}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}} \leq \frac{\int_\Omega \sup_{|t|\leq b_n} F(x,t)\mathrm{d}x - \int_\Omega F(x,a_n)\mathrm{d}x}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}} \\
&+ \frac{\overline{\mu}}{\lambda}\frac{\sum_\Omega \sup_{|t|\leq b_n} H(x,t)\mathrm{d}x + \int_\Omega H(x,a_n)\mathrm{d}x}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}},
\end{aligned}$$

taking (3.1) into account, one has

$$\frac{\varrho^{p^-}p^+}{m_0}\lim_{n\to+\infty}\frac{\int_\Omega \sup_{|t|\le b_n} Q(x,t)\mathrm{d}x - \int_\Omega Q(x,a_n)\mathrm{d}x}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}} \tag{3.5}$$
$$\le \frac{\varrho^{p^-}p^+}{m_0}\lim_{n\to+\infty}\frac{\int_\Omega \sup_{|t|\le b_n} F(x,t)\mathrm{d}x - \int_\Omega F(x,a_n)\mathrm{d}x}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}} + \frac{\overline{\mu}}{\overline{\lambda}}h_{b_n}.$$

Moreover, since H is nonnegative, we have

$$\limsup_{|\xi|\to+\infty}\frac{\int_\Omega Q(x,\xi)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}} \ge \limsup_{|\xi|\to+\infty}\frac{\int_\Omega F(x,\xi)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}}. \tag{3.6}$$

Therefore, from (3.5) and (3.6), and from Assumption $(A1)$ and (3.4) one has

$$\overline{\lambda} \in (\nu_1,\nu_2) \subseteq \Big(\frac{m_1\|\alpha\|_{\mathrm{L}^1(\Omega)}}{p^-\mathcal{B}^\infty}, \frac{m_0}{\varrho^{p^-}p^+\mathcal{A}_\infty}\Big) \subseteq \Big(0,\frac{1}{\gamma}\Big).$$

For the fixed $\overline{\lambda}$, the inequality (3.4) assures that the condition (b) of Theorem 2.1 can be used and either $I_{\overline{\lambda}}$ has a global minimum or there exists a sequence $\{u_n\}$ of solutions of the problem $(P_g^{f,h})$ such that $\lim_{n\to\infty}\|u\|_\alpha = +\infty$.
The other step is to verify that the functional $I_{\overline{\lambda}}$ has no global minimum. Since

$$\frac{1}{\overline{\lambda}} < \frac{p^-}{m_1\|\alpha\|_{\mathrm{L}^1(\Omega)}}\limsup_{|\xi|\to+\infty}\frac{\int_\Omega F(x,\xi)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}},$$

we can consider a real sequence $\{c_n\}$ with $c_n > 1$ for all $n \in \mathbb{N}$ and a positive constant τ such that $c_n \to +\infty$ as $n \to \infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{p^-}{m_1\|\alpha\|_{\mathrm{L}^1(\Omega)}}\frac{\int_\Omega F(x,c_n)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{c_n^{p^+}} \tag{3.7}$$

for each $n \in \mathbb{N}$ large enough. Thus, if we consider a sequence $\{y_n\}$ in X defined by setting

$$y_n(x) = c_n \quad \text{for all } x \in \Omega. \tag{3.8}$$

Thus $y_n \in X$ and

$$\frac{m_0 c_n^{p^-}}{p^+}\|\alpha\|_{\mathrm{L}^1(\Omega)} \le \Phi(y_n) \le \frac{m_1 c_n^{p^+}}{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)}. \tag{3.9}$$

On the other hand, since H and G are nonnegative functions, we observe

$$\Psi(y_n) \ge \int_\Omega F(x,c_n)\mathrm{d}x. \tag{3.10}$$

So, from (3.7), (3.9) and (3.10) we conclude

$$I_{\overline{\lambda}}(y_n) = \Phi(y_n) - \overline{\lambda}\Psi(y_n) \le \frac{m_1\|\alpha\|_{\mathrm{L}^1(\Omega)}}{p^-}c_n^{p^+} - \overline{\lambda}\int_\Omega F(x,c_n)\mathrm{d}x$$
$$< \frac{m_1\|\alpha\|_{\mathrm{L}^1(\Omega)}(1-\overline{\lambda}\tau)}{p^-}c_n^{p^+},$$

for every $n \in \mathbb{N}$ large enough. Hence, the functional $I_{\overline{\lambda}}$ is unbounded from below, and it follows that $I_{\overline{\lambda}}$ has no global minimum. Therefore, Theorem 2.1 assures that there is a sequence $\{u_n\} \subset X$ of critical points of $I_{\overline{\lambda}}$ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$, which it follows that $\lim_{n\to\infty} \|u_n\|_\alpha = +\infty$. Hence, we have the conclusion. □

Remark 3.2. *If $\{a_n\}$ and $\{b_n\}$ are two real sequences with $a_n > 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} b_n = +\infty$, such that the assumption (A_1) in Theorem 3.1 is satisfied. Then, under the conditions $\mathcal{A}_\infty = 0$ and $\mathcal{B}^\infty = +\infty$, Theorem 3.1 assures that for every $\lambda > 0$ and for each $\mu \in [0, \frac{1}{h_{b_n}})$ the problem $(P_g^{f,h})$ admits infinitely many solutions. Moreover, if $h_{b_n} = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.*

Remark 3.3. *If f is non-negative, then the strong maximum principle ensures that the weak solutions the problem $(P_g^{f,h})$ are non-negative (see [28, Lemma 1.1]).*

Theorem 3.4. *Assume that*

$(A3)$ $\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\leq\xi} F(x,t)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}$
$< \frac{m_0 p^-}{m_1\|\alpha\|_{L^1(\Omega)}\varrho^{p^-}p^+} \limsup_{\xi\to+\infty} \frac{\int_\Omega F(x,\xi)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}}$.

Then, for each

$$\lambda \in \left(\frac{m_1\|\alpha\|_{L^1(\Omega)}}{p^- \limsup_{\xi\to+\infty} \frac{\int_\Omega F(x,\xi)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}}}, \frac{m_0}{\varrho^{p^-}p^+ \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\leq\xi} F(x,t)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}}\right),$$

for every continuous function $h : \Omega\times\mathbb{R} \to \mathbb{R}$ whose $H(x,t) = \int_0^t h(x,\xi)\mathrm{d}\xi$ for every $(x,t) \in \Omega\times\mathbb{R}$, is a nonnegative function satisfying the condition

$$h_\infty := \frac{\varrho^{p^-}p^+}{m_0} \lim_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\leq\xi} H(x,t)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}} < \infty \tag{3.11}$$

and for every $\mu \in [0, \mu'_{h,\lambda})$ where

$$\mu'_{h,\lambda} := \frac{1}{h_\infty}\left(1 - \frac{\lambda\varrho^{p^-}p^+}{m_0} \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\leq\xi} F(x,t)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}\right), \tag{3.12}$$

the problem $(P_g^{f,h})$ has an unbounded sequence of solutions in $W^{1,p(x)}(\Omega)$.

Proof: We choose the sequence $\{b_n\}$ of positive numbers such that goes to infinity and

$$\lim_{n\to+\infty} \frac{\int_\Omega \sup_{|t|\leq b_n} F(x,t)\mathrm{d}x + \Gamma(\partial\Omega)G(b_n)}{b_n^{p^-}} = \liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\leq\xi} F(x,t)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}.$$

Now, since $\Phi(0) = \Psi(0) = 0$ we can taking $a_n = 0$ for every $n \in \mathbb{N}$ in (3.2), from Theorem 2.1 the conclusion follows. □

Now, we give an application of Theorem 3.4.

Example 3.5. *Let $N = 1$, $\Omega = (0, \frac{\pi}{2}) \subset \mathbb{R}$, $M(t) = 1 + \frac{1}{\cosh t}$ for all $t \in \mathbb{R}$, $p(x) = 2(1 + \sin x)$ for all $x \in [0, \frac{\pi}{2}]$, $\alpha(x) = \frac{\sin x}{(\pi+2)^2}$ for all $x \in [0, \frac{\pi}{2}]$, $g(t) = 1 + t^{10}$ for all $t \in \mathbb{R}$ and a_n be a sequence defined by*

$$\begin{cases} a_1 = 4, \\ a_{n+1} = 1 + 10\sqrt{22}a_n^2 \quad \text{for } n \geq 2 \end{cases}$$

and b_n be a sequence such that $b_1 = 4^4$ and $b_n = a_n^4 - a_{n-1}^4$ for all $n \geq 2$. Consider the problem

$$\begin{cases} \left\{1 + \cosh\left[\left(\int_0^{\frac{\pi}{2}} \frac{1}{3+2\sin x}\left(|u'(x)|^{3+2\sin x} + \frac{\sin x}{(\pi+2)^2}|u(x)|^{3+2\sin x}\right)\mathrm{d}x\right)^{-1}\right]\right\} \times \\ \left(-\left(|u'(x)|^{1+2\sin x}u'(x)\right)' + \frac{\sin x}{(\pi+2)^2}|u(x)|^{1+2\sin x}u(x)\right) \\ = \lambda f(x,t) + \mu e^x u(x), \quad \text{in } (0, \frac{\pi}{2}), \\ |u'(0)|u'(0) = \lambda\left(1 + (\vartheta u(0))^{10}\right), \ |u'(\frac{\pi}{2})|^3 u'(\frac{\pi}{2}) = \lambda\left(1 + (\vartheta u(\frac{\pi}{2}))^{10}\right) \end{cases} \tag{3.13}$$

where $f(x,t) = e^x k(t)$ for all $(x,t) \in (0, \frac{\pi}{2}) \times \mathbb{R}$ with

$$k(t) = \sum_{n=1}^{\infty} 2b_n\left(1 - 2\left|t - a_n + \frac{1}{2}\right|\right)\chi_{[a_n-1,a_n]}(t) \quad \text{for all } t \in \mathbb{R}$$

where $\chi_{[\alpha,\beta]}$ denotes the characteristic function of the interval $[\alpha, \beta]$. According to the above data we have, $m_0 = 1$, $m_1 = 2$, $p^- = 2$, $p^+ = 4$, $\operatorname{meas}(\Omega) = \frac{\pi}{2}$, $D = \frac{\pi}{2}$, $\Gamma(\partial\Omega) = 0$, $\|\alpha\|_{L^1(\Omega)} = \|\alpha\|_\infty = \frac{1}{(\pi+2)^2}$, $k_{p^-} = k_2 \leq \sqrt{2}(\pi + 2)$ and thus, $\varrho \leq \frac{\sqrt{2}}{2}(\pi + 2)^2$, $H(x,t) = \frac{e^x t^2}{2}$ for all $(x,t) \in (0, \frac{\pi}{2}) \times \mathbb{R}$ and it is easy to verify that $a_{n+1} - 1 > a_n$ and $\int_{a_n-1}^{a_n} k(t)\mathrm{d}t = b_n$ for all $n \in \mathbb{N}$. Then, one has $F(x, a_n) = e^x \int_0^{a_n} k(\xi)\mathrm{d}\xi = e^x a_n^4$ and

$$\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\leq\xi} F(x,t)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}$$
$$\leq \lim_{n\to+\infty} \frac{\int_0^{\frac{\pi}{2}} e^x F(a_{n+1}-1)\mathrm{d}x}{(a_{n+1}-1)^2} = \frac{1}{2200} \lim_{n\to\infty} \frac{a_n^4 \int_0^{\frac{\pi}{2}} e^x \mathrm{d}x}{a_n^4} = \frac{e^{\frac{\pi}{2}} - 1}{2200}$$

and

$$\limsup_{\xi\to+\infty} \frac{\int_\Omega F(x,t)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}} = \lim_{n\to+\infty} \frac{\int_0^{\frac{\pi}{2}} F(x,a_n)\mathrm{d}x}{a_n^4} = \int_0^{\frac{\pi}{2}} e^x \mathrm{d}x = e^{\frac{\pi}{2}} - 1.$$

Hence, using Theorem 3.4, since

$$h_\infty := \frac{\varrho^{p^-} p^+}{m_0} \lim_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\le\xi} H(x,t)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}$$
$$\le 2(\pi+2)^4 \lim_{n\to+\infty} \frac{a_n^2 \int_0^{\frac{\pi}{2}} e^x \mathrm{d}x}{a_n^2} = 2(\pi+2)^4 (e^{\frac{\pi}{2}}-1) < +\infty,$$

the problem (3.13) for every

$$\lambda \in \left(\frac{1}{(\pi+2)^2(e^{\frac{\pi}{2}}-1)}, \frac{1100}{(\pi+2)^4(e^{\frac{\pi}{2}}-1)} \right)$$

and $\mu \in \big[0, \frac{1}{2(\pi+2)^4(e^{\frac{\pi}{2}}-1)}(1-\lambda\frac{(\pi+2)^4(e^{\frac{\pi}{2}}-1)}{1100})\big)$ *has an unbounded sequence of solutions in the space* $\mathrm{W}^{1,2(1+\sin x)}(0,\frac{\pi}{2})$.

Here, we point out two simple consequences of Theorems 3.1 and 3.4, respectively.

Corollary 3.6. *Assume that there exist two real sequences* $\{a_n\}$ *and* $\{b_n\}$ *with* $a_n > 1$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to+\infty} b_n = +\infty$, *such that the assumption (A1) in Theorem 3.1 holds,* $\mathcal{A}_\infty < \frac{m_0}{\varrho^{p^-}p^+}$ *and* $\mathcal{B}_\infty > \frac{m_1\|\alpha\|_{\mathrm{L}^1(\Omega)}}{p^-}$. *Then, for every arbitrary function* $h \in \mathrm{C}(\Omega\times\mathbb{R},\mathbb{R})$ *whose* $H(x,t) = \int_0^t h(x,\xi)\mathrm{d}\xi$ *for every* $(x,t) \in \Omega\times\mathbb{R}$ *is a nonnegative function satisfying the condition (3.1) and for every* $\mu \in [0, \mu_{h,1}[$ *with* $\mu_{h,1} := \frac{m_0 - p^+\varrho^{p^-}\mathcal{A}_\infty}{m_0 h_{b_n}}$, *the problem*

$$\begin{cases} T(u) = f(x,u(x)) + \mu h(x,u(x)), & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} = g(\vartheta u(x)), & \text{on } \partial\Omega \end{cases} \tag{3.14}$$

has an unbounded sequence of solutions.

Corollary 3.7. *Assume that* $\mathcal{B}_\infty > \frac{m_1\|\alpha\|_{\mathrm{L}^1(\Omega)}}{p^-}$ *and*

$$\liminf_{\xi\to+\infty} \frac{\int_\Omega \sup_{|t|\le\xi} F(x,t)\mathrm{d}x + \Gamma(\partial\Omega)F(\xi)}{\xi^{p^-}} < \frac{m_0}{\varrho^{p^-}p^+}.$$

Then, for every arbitrary function $h \in \mathrm{C}(\Omega\times\mathbb{R},\mathbb{R})$ *whose* $H(x,t) = \int_0^t h(x,\xi)d\xi$ *for every* $(x,t) \in \Omega\times\mathbb{R}$ *is a nonnegative function satisfying the condition (3.11) and for every* $\mu \in [0, \mu'_{h,1}[$ *where* $\mu'_{h,1}$ *given by (3.12) with* $\lambda = 1$, *the problem (3.14) has an unbounded sequence of solutions.*

Remark 3.8. *Theorem 2.4 is an immediately consequence of Corollary 3.7.*

We here give the following two consequences of the main result.

Corollary 3.9. *Assume that there exist two real sequences* $\{a_n\}$ *and* $\{b_n\}$ *with* $a_n > 1$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to+\infty} b_n = +\infty$*, such that the assumption* $(A1)$ *in Theorem 3.1 holds,* $f_1 \in \mathrm{C}(\Omega \times \mathbb{R}, \mathbb{R})$ *and* $F_1(x,t) = \int_0^t f_1(x,\xi)\mathrm{d}t$ *for all* $t \in \mathbb{R}$*. Moreover, assume that*

$(A3)$ $\lim_{n\to+\infty} \frac{\int_\Omega \sup_{|t|\le b_n} F_1(x,t)\mathrm{d}x - \int_\Omega F_1(x,a_n)\mathrm{d}x + \Gamma(\partial\Omega)(G(b_n)-G(a_n))}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}} < +\infty$;

$(A4)$ $\limsup_{\xi\to+\infty} \frac{\int_\Omega F_1(x,\xi)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}} = +\infty.$

Then, for every function $f_i \in \mathrm{C}(\Omega\times\mathbb{R},\mathbb{R})$*, denoting* $F_i(k,t) = \int_0^t f_i(k,\xi)\mathrm{d}\xi$ *for all* $t \in \mathbb{R}$ *for* $2 \le i \le n$*, satisfying*

$$\max\left\{\sup_{\xi\in\mathbb{R}} \left(F_i(x,\xi) + \Gamma(\partial\Omega)G(\xi)\right);\ 2 \le i \le n\right\} \le 0$$

and

$$\min\left\{\liminf_{\xi\to+\infty} \frac{F_i(x,\xi)}{\xi^{p^+}};\ 2 \le i \le n\right\} > -\infty,$$

for each

$$\lambda \in \left(0,\ \frac{m_0}{\varrho^{p^-} p^+ \lim_{n\to+\infty} \frac{\int_\Omega \sup_{|t|\le b_n} F_1(x,t)\mathrm{d}x - \int_\Omega F_1(x,a_n)\mathrm{d}x + \Gamma(\partial\Omega)(G(b_n)-G(a_n))}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}}}\right),$$

for every arbitrary function $h \in \mathrm{C}(\Omega\times\mathbb{R},\mathbb{R})$ *whose* $H(x,t) = \int_0^t h(x,\xi)\mathrm{d}\xi$ *for every* $(x,t) \in \Omega \times \mathbb{R}$*, is a non-negative function satisfying the condition* (3.1) *and for every* $\mu \in [0, \mu_{h,\lambda,1})$ *where*

$$\mu_{h,\lambda,1} := \frac{1}{h_{b_n}}\left(1 - \frac{\lambda \varrho^{p^-} p^+}{m_0} \times \lim_{n\to+\infty} \frac{\int_\Omega \sup_{|t|\le b_n} F_1(x,t)\mathrm{d}x - \int_\Omega F_1(x,a_n)\mathrm{d}x + \Gamma(\partial\Omega)(G(b_n)-G(a_n))}{m_0 p^- b_n^{p^-} - m_1 p^+ \varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)} a_n^{p^+}}\right),$$

the problem

$$\begin{cases} T(u) = \lambda \sum_{i=1}^n f_i(x,u(x)) + \mu h(x,u(x)), & \text{in } \Omega,\\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial v} = \lambda g(\vartheta u(x)), & \text{on } \partial\Omega \end{cases} \tag{3.15}$$

has an unbounded sequence of solutions in $\mathrm{W}^{1,p(x)}(\Omega)$.

Proof: Set $F(x,\xi) = \sum_{i=1}^n F_i(x,\xi)$ for all $\xi \in \mathbb{R}$. Assumption $(A4)$ along with the condition

$$\min\left\{\liminf_{\xi\to+\infty} \frac{F_i(x,\xi)}{\xi^{p^-}};\ 2 \le i \le n\right\} > -\infty$$

ensures

$$\begin{aligned}&\limsup_{\xi\to+\infty}\frac{\int_\Omega F(x,\xi)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}}\\&=\limsup_{\xi\to+\infty}\frac{\sum_{i=1}^n\int_\Omega F_i(x,\xi)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}}=+\infty.\end{aligned}$$

Moreover, Assumption $(A3)$ together with the condition

$$\max\left\{\sup_{\xi\in\mathbb{R}}F_i(x,\xi);\ 2\le i\le n\right\}\le 0,$$

implies

$$\begin{aligned}&\lim_{n\to+\infty}\frac{\int_\Omega\sup_{|t|\le b_n}F(x,t)\mathrm{d}x-\int_\Omega F(x,a_n)\mathrm{d}x+\Gamma(\partial\Omega)(G(b_n)-G(a_n))}{m_0p^-b_n^{p^-}-m_1p^+\varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)}a_n^{p^+}}\\&\le\lim_{n\to+\infty}\frac{\int_\Omega\sup_{|t|\le b_n}F_1(x,t)\mathrm{d}x-\int_\Omega F_1(x,a_n)\mathrm{d}x+\Gamma(\partial\Omega)(G(b_n)-G(a_n))}{m_0p^-b_n^{p^-}-m_1p^+\varrho^{p^-}\|\alpha\|_{\mathrm{L}^1(\Omega)}a_n^{p^+}}\\&<+\infty.\end{aligned}$$

Hence, the conclusion follows from Theorem 3.1. □

Corollary 3.10. *Let $f_1\in\mathrm{C}(\Omega\times\mathbb{R},\mathbb{R})$ and let $F_1(x,t)=\int_0^t f_1(x,\xi)\mathrm{d}t$ for all $t\in\mathbb{R}$. Assume that*

$$\liminf_{\xi\to+\infty}\frac{\int_\Omega\sup_{|t|\le\xi}F_1(x,t)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}<+\infty$$

and

$$\limsup_{\xi\to+\infty}\frac{\int_\Omega F_1(x,\xi)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}}=+\infty.$$

Then, for every function $f_i\in\mathrm{C}(\Omega\times\mathbb{R},\mathbb{R})$, denoting $F_i(k,t)=\int_0^t f_i(k,\xi)\mathrm{d}\xi$ for all $t\in\mathbb{R}$ for $2\le i\le n$, satisfying

$$\max\left\{\sup_{\xi\in\mathbb{R}}(F_i(x,\xi)+\Gamma(\partial\Omega)F(\xi))\,;\ 2\le i\le n\right\}\le 0$$

and

$$\min\left\{\liminf_{\xi\to+\infty}\frac{F_i(x,\xi)+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}};\ 2\le i\le n\right\}>-\infty,$$

for each

$$\lambda\in\left(0,\ \frac{m_0}{\varrho^{p^-}p^+\liminf_{\xi\to+\infty}\frac{\int_\Omega\sup_{|t|\le\xi}F_1(x,t)dx+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}}\right),$$

for every arbitrary function $h \in \mathrm{C}(\Omega\times\mathbb{R},\mathbb{R})$ *whose* $H(x,t)=\int_0^t h(x,\xi)\mathrm{d}\xi$ *for every* $(x,t)\in\Omega\times\mathbb{R}$*, is a non-negative function satisfying the condition* (3.1) *and for every* $\mu\in[0,\mu'_{h,\lambda,1}[$ *where*

$$\mu'_{h,\lambda,1} := \frac{1}{h_\infty}\left(1-\frac{\lambda\varrho^{p^-}p^+}{m_0}\liminf_{\xi\to+\infty}\frac{\int_\Omega \sup_{|t|\le\xi}F_1(x,t)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}\right),$$

the problem (3.15) *has an unbounded sequence of solutions in* $\mathrm{W}^{1,p(x)}(\Omega)$.

Now, we give an application of Corollary 3.10.

Example 3.11. *Let* $N=2$, $\Omega=\{(x_1,x_2)\in\mathbb{R}^2;\ x_1^2+x_2^2<1\}\subset\mathbb{R}^2$, $M(t)=2+\sin t$ *for all* $t\in\mathbb{R}$, $p(x_1,x_2)=e^{1+x_1^2+x_2^2}$ *for all* $(x_1,x_2)\in\overline{\Omega}$, $\alpha(x_1,x_2)=\frac{1}{1+x_1^2+x_2^2}$ *for all* $(x_1,x_2)\in\overline{\Omega}$, $g(t)=e|t|^{e-2}t$ *for all* $t\in\mathbb{R}$*, Let* $f_1,f_2,h:\Omega\times\mathbb{R}\to\mathbb{R}$ *are defined by*

$$f_1(x_1,x_2,t)=$$
$$\begin{cases} k(x_1,x_2)t^9\big(10+20\sin^2(\ln t)+2\sin(2\ln t)\big) & \text{if } (x_1,x_2,t)\in\Omega\times(0,+\infty),\\ 0 & \text{if } (x_1,x_2,t)\in\Omega\times(-\infty,0],\end{cases}$$

$$f_2(x_1,x_2,t)=-l(x_1,x_2)\frac{2t}{(1+t^2)^2}$$

and

$$h(x_1,x_2,t)=(x_1^2+x_2^2)(2t+\sin t)$$

respectively, where $k,l:\Omega\to\mathbb{R}$ *are two non-negative continuous functions. Consider the problem*

$$\begin{cases} \left\{2+\sin\left[\int_\Omega \frac{1}{e^{1+|x|^2}}\left(|\nabla u(x_1,x_2)|^{e^{1+|x|^2}}+\frac{|u(x_1,x_2)|^{e^{1+|x|^2}}}{1+|x|^2}\right)\mathrm{d}x\right]\right\}\times\\ \Big(-\operatorname{div}(|\nabla u(x_1,x_2)|^{e^{1+|x|^2}-2}\nabla u(x_1,x_2))\\ \quad+\frac{1}{1+|x|^2}|u(x_1,x_2)|^{e^{1+|x|^2}-2}u(x_1,x_2)\Big)\\ =\lambda(f_1+f_2)(u(x_1,x_2))+\mu(x_1^2+x_2^2)(1+\sin(u(x_1,x_2))), & \text{in } \Omega,\\ |\nabla u(x_1,x_2)|^{e^{1+|x|^2}-2}\frac{\partial u}{\partial v}=e\lambda|\vartheta u(x_1,x_2)|^{e-2}\vartheta u(x_1,x_2) & \text{on } \partial\Omega \end{cases} \tag{3.16}$$

where $|x|=\sqrt{x_1^2+x_2^2}$*. Thus,* $m_0=1$, $m_1=3$, $p^-=e$, $p^+=e^2$, $\operatorname{meas}(\Omega)=\pi$, $\Gamma(\partial\Omega)=2\pi$, $D=2$,

$$\|\alpha\|_{\mathrm{L}^1(\Omega)}=\int_0^{2\pi}\int_0^1\frac{r}{1+r^2}\mathrm{d}r\mathrm{d}\theta=\pi\ln(2)\quad\text{and}\quad\|\alpha\|_\infty=1,$$

$$k_{p^-}=k_e\le\frac{2}{\sqrt[e]{2}}\max\left\{\frac{1}{\sqrt[e]{\pi\ln 2}},\frac{2(e-1)\pi}{\pi\ln(2)\sqrt[e]{2\pi(e-2)}}\right\}=\frac{2}{\sqrt[e]{2\pi\ln(2)}}$$

and thus, $\varrho \leq \frac{2(1+\pi)}{\sqrt[e]{2\pi \ln 2}}$. *Moreover,*

$$F_1(x_1,x_2,t) = \begin{cases} k(x_1,x_2)t^{10}\big(1+2\sin^2(\ln t)\big) & \text{if}\,(x_1,x_2,t)\in\Omega\times(0,+\infty),\\ 0 & \text{if}\,(x_1,x_2,t)\in\Omega\times(-\infty,0], \end{cases}$$

$$F_2(x_1,x_2,t) = -l(x_1,x_2)\frac{t^2}{1+t^2} \quad \text{for all}\,(x_1,x_2,t)\in\Omega\times\mathbb{R}$$

and

$$H(x_1,x_2,t) = (x_1^2+x_2^2)(1+t^2-\cos t) \quad \text{for all}\,(x_1,x_2,t)\in\Omega\times\mathbb{R}.$$

Put

$$a_n = \begin{cases} n & \text{if } n \text{ is even},\\ e^{-n\pi} & \text{if } n \text{ is odd} \end{cases} \quad \text{and} \quad b_n = e^{n\pi} \quad \text{for every}\, n\in\mathbb{N}.$$

Then

$$\lim_{n\to+\infty}\frac{\iint_\Omega \sup_{|t|\leq a_n} F_1(x_1,x_2,t)+\Gamma(\partial\Omega)G(a_n)}{a_n^e} = \begin{cases} 2\pi & \text{if } n \text{ is odd},\\ +\infty & \text{if } n \text{ is even} \end{cases}$$

and

$$\limsup_{n\to+\infty}\frac{\iint_\Omega F_1(x_1,x_2,b_n)+\Gamma(\partial\Omega)G(b_n)}{b_n^{e^2}} = +\infty.$$

So,

$$\liminf_{\xi\to+\infty}\frac{\iint_\Omega \sup_{|t|\leq\xi} F_1(x_1,x_2,t)+\Gamma(\partial\Omega)G(\xi)}{|\xi|^{p^-}} = 2\pi < +\infty$$

$$\limsup_{\xi\to+\infty}\frac{\iint_\Omega F_1(x_1,x_2,t)+\Gamma(\partial\Omega)G(\xi)}{|\xi|^{p^+}} = +\infty.$$

Moreover,

$$\sup_{\xi\in\mathbb{R}} F_2(x_1,x_2,\xi) = 0,$$

$$\liminf_{\xi\to+\infty}\frac{F_2(x_1,x_2,\xi)}{\xi^{p^-}} = \liminf_{\xi\to+\infty}\frac{-l(x_1,x_2)\xi^2}{\xi^e(1+\xi^2)} = 0 > -\infty$$

and

$$h_\infty := \frac{\varrho^{p^-}p^+}{m_0}\lim_{\xi\to+\infty}\frac{\int_\Omega \sup_{|t|\leq\xi} H(x_1,x_2,t)+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}} = 2\pi\frac{\varrho^{p^-}p^+}{m_0}$$
$$\leq \frac{2^e e^2(1+\pi)^e}{\ln 2} < +\infty.$$

Hence, all assumptions of Corollary 3.10 are satisfied. So, for every

$$\lambda \in (0, \frac{\ln 2}{2^e e^2(1+\pi)^e})$$

and

$$\mu \in [0, \frac{\ln 2}{2^e e^2 (1+\pi)^e}(1 - \lambda \frac{2^e e^2(1+\pi)^e}{\ln 2}))$$

the problem (3.16) *has an unbounded sequence of solutions in the space* $\mathrm{W}^{1,e^{1+x_1^2+x_2^2}}(\Omega)$.

Now put

$$\mathcal{B}^0 = \limsup_{\xi \to 0^+} \frac{\int_\Omega F(x,\xi)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}}.$$

Arguing as in the proof of Theorem 3.1, but using conclusion (c) of Theorem 2.1 instead of (b), one establishes the following result.

Theorem 3.12. *Assume that there exist two real sequences* $\{d_n\}$ *and* $\{e_n\}$ *with* $d_n > 1$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to+\infty} e_n = 0$, *such that*

(*A*5) $d_n^{p^+} < \frac{m_0 p^- e_n^{p^-}}{m_1 p^+ \varrho^{p^-} \|\alpha\|_{\mathrm{L}^1(\Omega)}}$;

(*A*6) $\mathcal{A}_0 := \lim_{n\to+\infty} \frac{\int_\Omega \sup_{|t|\le e_n} F(x,t)\mathrm{d}x - \int_\Omega F(x,d_n)\mathrm{d}x + \Gamma(\partial\Omega)(G(e_n)-G(d_n))}{m_0 p^- e_n^{p^-} - m_1 p^+ \varrho^{p^-} \|\alpha\|_{\mathrm{L}^1(\Omega)} d_n^{p^+}}$
$< \frac{m_0 p^-}{m_1 \|\alpha\|_{\mathrm{L}^1(\Omega)} \varrho^{p^-} p^+} \mathcal{B}^0.$

Then, for each $\lambda \in (\lambda_3, \lambda_4)$ *with* $\lambda_3 := \frac{m_1\|\alpha\|_{\mathrm{L}^1(\Omega)}}{p^- \mathcal{B}^0}$ *and* $\lambda_4 := \frac{m_0}{\varrho^{p^-} p^+ \mathcal{A}_0}$, *for every continuous function* $h : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ *whose* $H(x,t) = \int_0^t g(x,\xi)\mathrm{d}\xi$ *for every* $(x,t) \in \Omega \times \mathbb{R}$, *is a nonnegative function satisfying the condition*

$$h_{e_n} := \frac{\varrho^{p^-} p^+}{m_0} \times \lim_{n\to\infty} \frac{\int_\Omega \sup_{|t|\le e_n} H(x,t)\mathrm{d}x - \int_\Omega H(x,d_n)\mathrm{d}x + \Gamma(\partial\Omega)\left(G(e_n) - G(d_n)\right)}{m_0 p^- e_n^{p^-} - m_1 p^+ \varrho^{p^-} \|\alpha\|_{\mathrm{L}^1(\Omega)} d_n^{p^+}} < +\infty \tag{3.17}$$

and for every $\mu \in \left[0, \tilde{\mu}_{h,\lambda}\right)$ *with* $\tilde{\mu}_{h,\lambda} := \frac{m_0 - \lambda p^+ \varrho^{p^-} \mathcal{A}_0}{m_0 h_{e_n}}$, *the problem* ($P_g^{f,h}$) *has a sequence of pairwise distinct solutions which strongly converges to* 0 *in* $\mathrm{W}^{1,p(x)}(\Omega)$.

Proof: Fix $\overline{\lambda} \in \left(\lambda_3, \lambda_4\right)$ and let h is the function satisfying the condition (3.17). Since, $\overline{\lambda} < \lambda_4$, one has $\tilde{\mu}_{h,\overline{\lambda}} > 0$. Fix $\overline{\mu} \in]0, \tilde{\mu}_{h,\overline{\lambda}}[$ and set $\nu_3 := \lambda_3$ and $\nu_4 := \frac{\lambda_4}{1+\frac{\overline{\mu}}{\overline{\lambda}}\lambda_4 h_{e_n}}$. If $h_{e_n} = 0$, clearly, $\nu_3 = \lambda_3$, $\nu_4 = \lambda_4$ and $\lambda \in]\nu_3, \nu_4[$. If $h_{e_n} \neq 0$, since $\overline{\mu} < \tilde{\mu}_{h,\overline{\lambda}}$, one has

$$\frac{\overline{\lambda}}{\lambda_4} + \overline{\mu} h_{e_n} < 1,$$

and so

$$\frac{\lambda_4}{1 + \frac{\overline{\mu}}{\overline{\lambda}}\lambda_4 h_{e_n}} > \overline{\lambda},$$

namely, $\overline{\lambda} < \nu_4$. Hence, recalling that $\overline{\lambda} > \lambda_3 = \nu_3$, one has $\overline{\lambda} \in]\nu_3, \nu_4[$.
Now, put $Q(x,t) = F(x,t) + \frac{\overline{\mu}}{\overline{\lambda}} H(x,t)$ for all $t \in \mathbb{R}$ and $x \in \Omega$. Since

$$\frac{\int_\Omega \sup_{|t|\le e_n} Q(x,t)\mathrm{d}x - \int_\Omega Q(x,d_n)\mathrm{d}x}{m_0 p^- e_n^{p^-} - m_1 p^+ \varrho^{p^-} \|\alpha\|_{\mathrm{L}^1(\Omega)} d_n^{p^+}} \le \frac{\int_\Omega \sup_{|t|\le e_n} F(x,t)\mathrm{d}x - \int_\Omega F(x,d_n)\mathrm{d}x}{m_0 p^- e_n^{p^-} - m_1 p^+ \varrho^{p^-} \|\alpha\|_{\mathrm{L}^1(\Omega)} d_n^{p^+}}$$
$$+ \frac{\overline{\mu}}{\overline{\lambda}} \frac{\sum_\Omega \sup_{|t|\le e_n} H(x,t)\mathrm{d}x + \int_\Omega H(x,d_n)\mathrm{d}x}{m_0 p^- e_n^{p^-} - m_1 p^+ \varrho^{p^-} \|\alpha\|_{\mathrm{L}^1(\Omega)} d_n^{p^+}},$$

taking (3.1) into account, one has

$$\begin{aligned} &\frac{\varrho^{p^-} p^+}{m_0} \lim_{n\to+\infty} \frac{\int_\Omega \sup_{|t|\le e_n} Q(x,t)\mathrm{d}x - \int_\Omega Q(x,d_n)\mathrm{d}x}{m_0 p^- e_n^{p^-} - m_1 p^+ \varrho^{p^-} \|\alpha\|_{\mathrm{L}^1(\Omega)} d_n^{p^+}} \\ &\le \frac{\varrho^{p^-} p^+}{m_0} \lim_{n\to+\infty} \frac{\int_\Omega \sup_{|t|\le e_n} F(x,t)\mathrm{d}x - \int_\Omega F(x,d_n)\mathrm{d}x}{m_0 p^- e_n^{p^-} - m_1 p^+ \varrho^{p^-} \|\alpha\|_{\mathrm{L}^1(\Omega)} d_n^{p^+}} + \frac{\overline{\mu}}{\overline{\lambda}} h_{e_n}. \end{aligned} \tag{3.18}$$

Moreover, since H is nonnegative, from Assumption $(A5)$ we have

$$\limsup_{\xi\to 0^+} \frac{\int_\Omega Q(x,\xi)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}} \ge \limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}}. \tag{3.19}$$

Therefore, from (3.18) and (3.19), we obtain

$$\overline{\lambda} \in (\nu_1, \nu_2) \subseteq \left(\frac{m_1 \|\alpha\|_{\mathrm{L}^1(\Omega)}}{p^- \mathcal{B}^0}, \frac{m_0}{\varrho^{p^-} p^+ \mathcal{A}_0} \right) \subseteq (\lambda_3, \lambda_4).$$

We take X, Φ, Ψ and $I_{\overline{\lambda}}$ as in the proof of Theorem 3.1. We prove that $\delta < +\infty$. For this, put $r_n = \frac{m_0}{\varrho^{p^-} p^+} b_n^{p^-}$ for all $n \in \mathbb{N}$. Let us show that the functional $I_{\overline{\lambda}}$ has not a local minimum at zero. For this, let $\{c_n\}$ be a sequence of positive numbers and $\tau > 0$ such that $c_n \to 0^+$ as $n \to \infty$ and

$$\frac{1}{\overline{\lambda}} < \tau < \frac{p^-}{m_1 \|\alpha\|_{\mathrm{L}^1(\Omega)}} \frac{\int_\Omega F(x,c_n)\mathrm{d}x + \Gamma(\partial\Omega)G(\xi)}{c_n^{p^+}} \tag{3.20}$$

for each $n \in \mathbb{N}$ large enough. Let $\{y_n\}$ be a sequence in $\mathrm{W}^{1,p(x)}(\Omega)$ defined by (3.8). So, owing to (3.9), (3.10) and (3.20) we obtain

$$\begin{aligned} I_{\overline{\lambda}}(y_n) = \Phi(y_n) - \overline{\lambda}\Psi(y_n) &\le \frac{m_1 \|\alpha\|_{\mathrm{L}^1(\Omega)}}{p^-} c_n^{p^+} - \overline{\lambda} \int_\Omega F(x,c_n)\mathrm{d}x \\ &< \frac{m_1 \|\alpha\|_{\mathrm{L}^1(\Omega)}(1 - \overline{\lambda}\tau)}{p^-} c_n^{p^+} < 0, \end{aligned}$$

for every $n \in \mathbb{N}$ large enough. Since $I_{\overline{\lambda}}(0) = 0$, that means that 0 is not a local minimum of the functional $I_{\overline{\lambda}}$. Hence, the part (c) of Theorem 2.1 ensures that there exists a sequence $\{u_n\}$ in $\mathrm{W}^{1,p(x)}(\Omega)$ of pairwise distinct critical points of $I_{\overline{\lambda}}$ such that $\|u_n\| \to 0$ as $n \to \infty$, and the proof is complete. □

Theorem 3.13. *Assume that*

(A7) $\liminf_{\xi\to 0^+} \frac{\int_\Omega \sup_{|t|\le\xi} F(x,t)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}$
$< \frac{m_0 p^-}{m_1\|\alpha\|_{\mathrm{L}^1(\Omega)}\varrho^{p^-}p^+} \limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}}$.

Then, for each

$$\lambda \in \left(\frac{m_1\|\alpha\|_{\mathrm{L}^1(\Omega)}}{p^- \limsup_{\xi\to 0^+} \frac{\int_\Omega F(x,\xi)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}}}, \frac{m_0}{\varrho^{p^-}p^+ \liminf_{\xi\to 0^+} \frac{\int_\Omega \sup_{|t|\le\xi} F(x,t)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}} \right),$$

for every continuous function $h:\Omega\times\mathbb{R}\to\mathbb{R}$ whose $H(x,t)=\int_0^t h(x,\xi)\mathrm{d}\xi$ for every $(x,t)\in\Omega\times\mathbb{R}$, is a nonnegative function satisfying the condition

$$h_0 := \frac{\varrho^{p^-}p^+}{m_0} \lim_{\xi\to 0^+} \frac{\int_\Omega \sup_{|t|\le\xi} H(k,t)+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}} < \infty$$

and for every $\mu\in[0,\tilde{\mu}'_{h,\lambda})$ where

$$\tilde{\mu}'_{h,\lambda} := \frac{1}{h_0}\left(1-\frac{\lambda\varrho^{p^-}p^+}{m_0}\liminf_{\xi\to 0^+}\frac{\int_\Omega \sup_{|t|\le\xi} F(x,t)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}\right),$$

the problem $(P_g^{f,h})$ has a sequence of pairwise distinct solutions which strongly converges to 0 in $\mathrm{W}^{1,p(x)}(\Omega)$.

Proof: We choose the sequence $\{e_n\}$ of positive numbers such that goes to zero and

$$\lim_{n\to+\infty}\frac{\int_\Omega \sup_{|t|\le e_n} F(x,t)\mathrm{d}x+\Gamma(\partial\Omega)G(e_n)}{e_n^{p^-}} = \liminf_{\xi\to+\infty}\frac{\int_\Omega \sup_{|t|\le\xi} F(x,t)\mathrm{d}x+\Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}}.$$

Now, since $\Phi(0)=\Psi(0)=0$ we can taking $d_n=0$, from Theorem 3.12 the conclusion follows. □

Remark 3.14. *Applying Theorem 3.12, results similar to Remark 3.2 and Corollaries 3.6, 3.7, 3.9 and 3.10 can be obtained.*

We end this paper by giving the following example as an application of Theorem 3.13.

Example 3.15. *Let* $N = 3$, $\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^2;\ x_1^2 + x_2^2 + x_3^2 < 1\} \subset \mathbb{R}^3$, $M(t) = 2+\tanh t$ *for all* $t \in \mathbb{R}$, $p(x_1, x_2, x_3) = 4+x_1^2+x_2^2+x_3^2$ *for all* $(x_1, x_2, x_3) \in \overline{\Omega}$, $\alpha(x_1, x_2, x_3) = x_1^2+x_2^2+x_3^2$ *for all* $(x_1, x_2, x_3) \in \overline{\Omega}$, $f_1 : \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ *be the function defined by*

$$\begin{aligned} f_1(t) =& 2t^5 \left(3\ln(\ln(\frac{1}{t^2})) - \ln^{-1}(\frac{1}{t^2})\right) \sin^2(\ln(\ln(\ln(\frac{1}{t^2})))) \\ & - 4t^5 \ln^{-1}(\frac{1}{t^2}) \sin(\ln(\ln(\ln(\frac{1}{t^2})))) \cos(\ln(\ln(\ln(\frac{1}{t^2})))) \\ & + 2t^3 \ln^{-2}(\frac{1}{t^2})(1 + 2\ln(\frac{1}{t^2})), \end{aligned}$$

$$f(x_1, x_2, x_3, t) = \begin{cases} e^{x_1^2+x_2^2+x_3^2} f_1(t) & \textit{if } (x,t) \in \Omega \times (\mathbb{R} \setminus \{0\}), \\ 0 & \textit{if } (x,t) \in \Omega \times \{0\}, \end{cases}$$

$h(x_1, x_2, x_3, t) = 6t^5 \ln(x_1^2+x_2^2+x_3^2+1)$ *for all* $(x_1, x_2, x_3, t) \in \Omega\times\mathbb{R}$ *and* $g(t) = 5t^4$ *for every* $t \in \mathbb{R}$. *Direct calculations give* $m_0 = 1$, $m_1 = 3$, $p^- = 4$, $p^+ = 5$, $\mathrm{meas}(\Omega) = \frac{4\pi}{3}$, $D = 2$,

$$\|\alpha\|_{L^1(\Omega)} = \int_0^{2\pi} \int_0^{\pi} \int_0^{1} r^4 \sin\phi \mathrm{d}r\mathrm{d}\phi\mathrm{d}\theta = \frac{4\pi}{5} \quad \text{and} \quad \|\alpha\|_\infty = 1,$$

$$k_{p^-} = k_4 \le \frac{2}{\sqrt[4]{2}} \max\left\{\sqrt[4]{\frac{5}{4\pi}}, \frac{5}{\sqrt[4]{4\pi}}\right\} = \frac{10}{\sqrt[4]{8\pi}}$$

and thus, $\varrho \le \frac{10}{\sqrt[4]{8\pi}}(1 + \frac{4\pi}{3})$. *Also* $\Gamma(\partial\Omega) = 4\pi$,

$$F(x_1, x_2, x_3, t) = \begin{cases} e^{x_1^2+x_2^2+x_3^2} t^6 \ln(\ln(\frac{1}{t^2})) \sin^2(\ln(\ln(\ln(\frac{1}{t^2})))) + t^4 \ln^{-1}(\frac{1}{t^2}), \\ \qquad\qquad \textit{if } (x_1, x_2, x_3, t) \in \Omega \times (\mathbb{R} \setminus \{0\}), \\ 0 \qquad\qquad \textit{if } (x_1, x_2, x_3, t) \in \Omega \times \{0\}, \end{cases}$$

$$H(x_1, x_2, x_3, t) = t^6 \ln(x_1^2 + x_2^2 + x_3^2 + 1) \quad \text{for all}\,(x_1, x_2, x_3, t) \in \Omega \times \mathbb{R}$$

and

$$G(t) = t^5 \quad \text{for all}\, t \in \mathbb{R}.$$

Thus

$$\begin{aligned} &\liminf_{\xi\to0^+} \frac{\iiint_\Omega \sup_{|t|\le\xi} F(x_1, x_2, x_3, t)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^-}} \\ &= \liminf_{\xi\to0^+} \frac{\iiint_\Omega \sup_{|t|\le\xi} F(x_1, x_2, x_3, t)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 + 4\pi\xi^5}{\xi^4} = 0 \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} &\limsup_{\xi\longrightarrow0^+} \frac{\iiint_\Omega F(x_1, x_2, x_3, \xi)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}} \\ &= \limsup_{\xi\to0^+} \frac{\iiint_\Omega F(x_1, x_2, x_3, \xi)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 + 4\pi\xi^5}{\xi^5} = +\infty. \end{aligned} \tag{3.22}$$

Hence, using Theorem 3.13, since

$$h_0 := \frac{\varrho^{p^-}p^+}{m_0}\lim_{\xi\to 0^+}\frac{\int_\Omega \sup_{|t|\le\xi} H(x_1,x_2,x_3,t)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 + \Gamma(\partial\Omega)G(\xi)}{\xi^{p^+}}$$
$$= \frac{25\times 10^3}{3^4}(3+4\pi)^4,$$

the problem

$$\begin{cases} \left\{2+\tanh\left[\int_\Omega \frac{|\nabla u(x_1,x_2,x_3)|^{4+|x|^2}+|x|^2|u(x_1,x_2,x_3)|^{4+|x|^2}}{4+|x|^2}\mathrm{d}x\right]\right\}\times \\ \Big(-\operatorname{div}(|\nabla u(x_1,x_2,x_3)|^{2+|x|^2}\nabla u(x_1,x_2,x_3)) \\ \quad +|x|^2|u(x_1,x_2,x_3)|^{2+|x|^2}u(x_1,x_2,x_3)\Big) \\ =\lambda f(x,u(x_1,x_2,x_3))+6\mu u^5(x_1,x_2,x_3)\ln(x_1^2+x_2^2+x_3^2), & \text{in } \Omega, \\ |\nabla u(x_1,x_2,x_3)|^{2+|x|^2}\frac{\partial u}{\partial v} = 5\lambda(\vartheta u(x_1,x_2,x_3))^4 & \text{on } \partial\Omega \end{cases}$$

where $|x|=\sqrt{x_1^2+x_2^2+x_3^2}$, *for every* $(\lambda,\mu)\in(0,+\infty)\times\left[\frac{3^4}{25\times 10^3(3+4\pi)^4},\infty\right)$ *has a sequence of pairwise distinct solutions which strongly converges to* 0 *in the space* $\mathrm{W}_0^{1,4+x_1^2+x_2^2+x_3^2}(\Omega)$.

References

1. Afrouzi, G.A., Hadjian, A., Heidarkhani, S., *Steklov problem involving the $p(x)$-Laplacian,* Electronic J. Differ. Equ. Vol. **2014**, No. 134, pp. 1–11, (2014).

2. Alves, C.O., Corrêa, F.S.J.A. , Ma, T.F., *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl. **49**, 85–93, (2005).

3. Arosio, A., Panizzi, S., *On the well-posedness of the Kirchhoff string*, Trans. Am. Math. Soc. **348**, 305–330, (1996).

4. Autuori, G., Colasuonno, F., Pucci, P., *Blow up at infinity of solutions of polyharmonic Kirchhoff systems,* Complex Var. Elliptic Equ. **57**, 379–395, (2012).

5. Autuori, G., Colasuonno, F., Pucci, P., *Lifespan estimates for solutions of polyharmonic Kirchhoff systems,* Math. Mod. Meth. Appl. Sci. **22**, 1150009 [36 pages], (2012).

6. Autuori, G., Colasuonno, F., Pucci, P., *On the existence of stationary solutions for higher-order p-Kirchhoff problems,* Commun. Contemp. Math. **16**, 1450002 [43 pages], (2014).

7. Bonanno, G., Candito, P., *Infinitely many solutions for a class of discrete non-linear boundary value problems,* Appl. Anal. **88**, 605–616, (2009).

8. Bonanno, G., Candito, P., *Three solutions to a Neumann problem for elliptic equations involving the p-Laplacian,* Arch. Math. (Basel) **80**, 424–429, (2003).

9. Bonanno, G., Chinnì, A., *Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent,* J. Math. Anal. Appl. **418**, 812-827, (2014).

10. Bonanno, G., Chinnì, A., *Multiple solutions for elliptic problems involving the $p(x)$-Laplacian,* Le Matematiche, Vol. **LXVI-Fasc. I**, 105-113, (2011).

11. Bonanno,, G., Di Bella, B., *Infinitely many solutions for a fourth-order elastic beam equation,* Nonlinear Differ. Equ. Appl. NoDEA **18**, 357–368, (2011).

12. Bonanno, G., Molica Bisci, G., *A remark on perturbed elliptic Neumann problems,* Studia Univ. "Babeş-Bolyai", Mathematica, Volume **LV**, Number 4, December 2010.

13. Bonanno, G., Molica Bisci, G., *Infinitely many solutions for a boundary value problem with discontinuous nonlinearities,* Bound. Value Probl. **2009**, 1–20, (2009).

14. Cammaroto, F., Chinnì, A., Di Bella, B., *Multiple solutions for a Neumann problem involving the $p(x)$-Laplacian,* Nonlinear Anal. TMA **71**, 4486–4492, (2009).

15. Cammaroto, F., Vilasi, L., *Existence of three solutions for a degenerate Kirchhoff-type transmission problem,* Num. Func. Anal. Opt. **35**, 911–931, (2014).

16. Cammaroto, F., Vilasi, L., *Multiple solutions for a Kirchhoff-type problem involving the $p(x)$-Laplacian operator,* Nonlinear Anal. TMA **74**, 1841–1852, (2011).

17. Chipot, M., Lovat, B., *Some remarks on non local elliptic and parabolic problems,* Nonlinear Anal. **30**, 4619–4627, (1997).

18. Colasuonno, F., Pucci, P., *Multiplicity of solutions for $p(x)$-polyharmonic elliptic Kirchhoff equations,* Nonlinear Anal. TMA **74**, 5962–5974, (2011).

19. D'Aguì, G., Sciammetta, A., *Infinitely many solutions to elliptic problems with variable exponent and nonhomogeneous Neumann conditions,* Nonlinear Anal. TMA **75**, 5612-5619, (2012).

20. Dai, G., Hao, R., *Existence of solutions for a $p(x)$-Kirchhoff-type equation,* J. Math. Anal. Appl. **359**, 275–284, (2009).

21. Dai, G., Wei, J., *Infinitely many non-negative solutions for a $p(x)$-Kirchhoff-type problem with Dirichlet boundary condition,* Nonlinear Anal. TMA **73**, 3420–3430, (2010).

22. De Araujo, A.L.A., *Infinitely many solutions for the Dirichlet problem involving the p-Laplacian in annulus*, Submitted for publication.

23. De Araujo, A.L.A., Heidarkhani, S., Afrouzi, G.A., Moradi, S., *A variational approach for nonlocal problems with variable exponent and nonhomogeneous Neumann conditions,* preprint.

24. De Araujo, A.L.A., Heidarkhani, S., Caristi, G., Salari, A., *Multiplicity results for nonlocal problems with variable exponent and nonhomogeneous Neumann conditions,* preprint.

25. Deng, S.G., *A local mountain pass theorem and applications to a double perturbed $p(x)$–Laplacian equations,* Appl. Math. Comput. **211**, 234–241, (2009).

26. Diening, L., Harjulehto, P., Hästö, P., Ružička, M., *Lebesgue and Sobolev Spaces with Variable Exponents,* Lecture Notes in Math., vol. 2017, Springer-Verlag, Heidelberg, 2011.

27. Fan, X., *Boundary trace embedding theorems for variable exponent Sobolev space,* J. Math. Anal. Appl. J. Math. Anal. Appl. **339**, 1395–1412, (2008).

28. Fan, X.L., Zhang, Q.H., Zhao, Y.Z., *A strong maximum principle for $p(x)$-Laplace equations,* Chinese. J. Contemp. Math. **24**, 277–282, (2003).

29. Fan, X., Zhao, D., *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, J. Math. Anal. Appl. **263**, 424–446, (2001).

30. Graef, J.R., Heidarkhani, S., Kong, L., *A variational approach to a Kirchhoff-type problem involving two parameters*, Results Math. **63**, 877–889, (2013).

31. Halsey, T.C., *Electrorheological fluids,* Science **258**, 761–766, (1992).

32. He, X., Zou, W., *Infinitely many positive solutions for Kirchhoff-type problems*, Nonlinear Anal. TMA **70**, 1407–1414, (2009).

33. Heidarkhani, S., *Infinitely many solutions for systems of n two-point Kirchhoff-type boundary value problems,* Ann. Polon. Math. **107(2)**, 133–152, (2013).

34. Heidarkhani, S., De Araujo, A.L.A., Afrouzi, G.A., Moradi, S., *Existence of three weak solutions for Kirchhoff-type problems with variable exponent and nonhomogeneous Neumann conditions,* preprint.

35. Heidarkhani, S., De Araujo, A.L.A., Afrouzi, G.A., Moradi, S., *Multiple solutions for Kirchhoff-type problems with variable exponent and nonhomogeneous Neumann conditions,* Math. Nachr., v. 291, 326–342, (2018).

36. Heidarkhani, S., Ge, B., *Critical points approaches to elliptic problems Dervin by a $p(x)$-Laplacian,* Ukrainian Math. J. **66**, 1883–1903, (2015).

37. Hssini, M., Massar, M., Tsouli, N., *Existence and multiplicity of solutions for a $p(x)$-Kirchhoff type problems,* Bol. Soc. Paran. Mat. **33**, 201–215, (2015).

38. Kirchhoff, G., *Vorlesungen über mathematische Physik*, Mechanik. Teubner, Leipzig (1883).

39. Kováčik, O., Rákosník, J., *On the spaces* $L^{p(x)}(\Omega)$ *and* $W^{1,p(x)}(\Omega)$, Czechoslovak Math. **41**, 592–618, (1991).

40. Lions, J.L., *On some questions in boundary value problems of mathematical physics,* North-Holland Mathematics Studies, **30**, 284–346, (1978).

41. Mao, A., Zhang, Z., *Sign-changing and multiple solutions of Kirchhoff-type problems without the P.S. condition*, Nonlinear Anal. TMA **70**, 1275–1287, (2009).

42. Mihăilescu, M., *Existence and multiplicity of solutions for a Neumann problem involving the $p(x)$-Laplacian operator,* Nonlinear Anal. TMA **67**, 1419–1425, (2007).

43. Molica Bisci, G., Pizzimenti, P., *Sequences of weak solutions for non-local elliptic problems with Dirichlet boundary condition,* Proc. Edinb. Math. Soc. **257**, 779–809, (2014).

44. Molica Bisci, G., Rădulescu, V., *Applications of local linking to nonlocal Neumann problems,* Commun. Contemp. Math. **17**, 1450001 [17 pages], (2014).

45. Molica Bisci, G., Rădulescu, V., *Mountain pass solutions for nonlocal equations,* Annales AcademiæScientiarum FennicæMathematica **39**, 579-59, (2014).

46. Molica Bisci, G., Rădulescu, V., Servadei, R., *Variational Methods for Nonlocal Fractional Problems, Encyclopedia of Mathematics and its Applications,* vol. 162, Cambridge University Press, Cambridge, (2016).

47. Moschetto, D.S., *A quasilinear Neumann problem involving the $p(x)$-Laplacian,* Nonlinear Anal. **71**, 2739–2743, (2009).

48. Pfeiffer, C., Mavroidis, C., Bar-Cohen, Y., Dolgin, B., *Electrorheological fluid based force feedback device, in Proceedings of the 1999 SPIE Telemanipulator and Telepresence Technologies VI Conference (Boston, MA),* **3840**, 88–99, (1999).

49. Qian, C., Shen, Z., Yang, M., *Existence of solutions for $p(x)$-Laplacian nonhomogeneous Neumann problems with indefinite weight,* Nonlinear Anal. RWA **11**, 446–458, (2010).

50. Rădulescu, V., *Nonlinear elliptic equations with variable exponent: old and new,* Nonlinear Anal. TMA **121**, 336–369, (2015).

51. Rădulescu, V., Repovš, D., *Partial Differential Equations with Variable Exponents, Variational Methods and Qualitative Analysis, Monographs and Research Notes in Mathematics,* CRC Press, Boca Raton, FL, (2015).

52. Repovš, D., *Stationary waves of Schrödinger-type equations with variable exponent,* Anal. Appl. **13**, 645–661, (2015).

53. Ricceri, B., *A general variational principle and some of its applications*, J. Comput. Appl. Math. **113**, 401–410, (2000).

54. Ricceri, B., *On an elliptic Kirchhoff-type problem depending on two parameters*, J. Global Optim. **46**, 543–549, (2010).

55. Ružička, M., *Electro-rheological Fluids: Modeling and Mathematical Theory* Lecture Notes in Math., 1784, Springer, Berlin, (2000).

56. Vilasi, L., *Eigenvalue estimates for stationary $p(x)$-Kirchhoff problems* Electron. J. Diff. Equ., Vol. **2016**, No. 186, 1–9, (2016).

57. Yao, J., *Solutions for Neumann boundary value problems involving the $p(x)$-Laplacian operators,* Nonlinear Anal. TMA **68**, 1271–1283, (2008).

58. Yin, H., *Existence of three solutions for a Neumann problem involving the $p(x)$-Laplace operator,* Math. Meth. Appl. Sci. **35**, 307–313, (2012).

59. Zhikov, V.V., Kozlov, S.M., Oleinik, O.A., *Homogenization of Differential Operators and Integral Functionals,* Springer-Verlag, Berlin, (1994).

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