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On a Positive Solution for (p, q)-Laplace Equation with Nonlinear Boundary Conditions and Indefinite Weights

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ABSTRACT: In the present paper, we study the existence and non-existence results of a positive solution for the Steklov eigenvalue problem driven by nonhomogeneous operator (p, q)-Laplacian with indefinite weights. We also prove, under appropriate conditions, that the results are completely different from those for the usual Steklov eigenvalue problem involving the p-Laplacian with indefinite weight. Precisely, we show that there exists an interval of principal eigenvalues for our Steklov eigenvalue problem.

Key Words: (p,q)-Laplacian, Nonlinear boundary conditions, Indefinite weight, Mountain pass theorem, Global minimizer.

Contents

1. Introduction

Consider the (p, q)-Laplacian Steklov eigenvalue problem

$$(P_{\lambda,\mu}) \begin{cases} \operatorname{div}[A^{(\mu)}_{p,q}(\nabla u)] = A^{(\mu)}_{p,q}(u) & \text{in } \Omega, \\ \langle A^{(\mu)}_{p,q}(\nabla u), \nu\rangle = \lambda[m_p(x)|u|^{p-2}u + \mu m_q(x)|u|^{q-2}u] & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, $\langle .,. \rangle$ is the scalar product of $\mathbb{R}^N$, $\lambda \in \mathbb{R}$, $\mu \geq 0$ and $1 < q < p < \infty$. Let $r = p, q$ and let $\frac{N-1}{r-1} < s_r < \infty$ if $r < N$ and $s_r \geq 1$ if $r \geq N$. $A^{(\mu)}_{p,q}(s) = |s|^{p-2}s + \mu|s|^{q-2}s$ and the function weight $m_r \in \mathbb{M}_r$ may be unbounded and change sign, where $\mathbb{M}_r := \{m_r \in L^{s_r}(\partial\Omega); m_r^+ \not\equiv 0\}$.

The problem $(P_{\lambda,\mu})$ comes, for example, from a general reaction diffusion system

$$u_t = \operatorname{div}(D(u)\nabla u) + c(x, u), \tag{1.1}$$

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where $D(u) = (|\nabla u|^{p-2} + \mu|\nabla u|^{q-2})$. This system has a wide range of applications in physics and related sciences like chemical reaction design [2], biophysics [5] and plasma physics [14]. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.1) corresponds to the diffusion with a diffusion coefficient $D(u)$; whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x;u)$ has a polynomial form with respect to the concentration.

The nonhomogeneous operator (p,q)-Laplacian have been the topic of many studies (see [6,7,13,17]). However, there are few results one the eigenvalue problems for the (p,q)-Laplacian, we cite [3,9,10,15]. The classical eigenvalue problem for the (p,q)-Laplacian

$$\begin{cases} -\triangle_p u - \mu\triangle_q u &= \lambda[m_p(x)|u|^{p-2}u + \mu m_q(x)|u|^{q-2}u] \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\triangle_r u = \text{ div } (|\nabla u|^{r-2}\nabla u)$ indicate the r-Laplacian, has attracted considerable attention. In [12], the authors study the problem (1.2) for domains with boundary C^2 and bounded weights. They proved, in the case where $\mu > 0$, the existence of an interval of eigenvalues and the existence of positive solutions in nonresonant cases. A non-existence result is also given. In [18], A. Zerouali and B. Karim are proved the same results by assuming the singularities on the domain and the weights. Our purpose in this article is to extend the results of the classical eigenvalue problem involving the (p,q)-Laplacian (see for example [11,12]) and generalize some results knouwn in the classical p-Laplacian Steklov problems (see [4]).

We will write $\|u\|_r := \left(\int_\Omega |u|^r dx\right)^{1/r}$ for the $L^r(\Omega)-$norm and $W^{1,r}(\Omega)$ will denote the usual Sobolev space with usual norm $\|u\|_{W^{1,r}(\Omega)} := (\|\nabla u\|_r^r + \|u\|_r^r)^{1/r}$. We recall that a value $\lambda \in \mathbb{R}$ is an eigenvalue of problem $(P_{\lambda,\mu})$ if and only if there exists $u \in W^{1,p}(\Omega)\backslash\{0\}$ such that

$$\int_\Omega A_{p,q}^{(\mu)}(\nabla u)\nabla\varphi dx + \int_\Omega A_{p,q}^{(\mu)}(u)\varphi dx = \lambda\left[\int_{\partial\Omega} (m_p(x)|u|^{p-2} + \mu m_q(x)|u|^{q-2})u\varphi d\sigma\right] \tag{1.3}$$

for all $\varphi \in W^{1,p}(\Omega)$, where $d\sigma$ is the $N-1$ dimensional Hausdorff measure and u is then called an eigenfunction of λ.

Letting $\mu \to 0^+$, our problem $(P_{\lambda,\mu})$ turns into the $(p-1)$-homogeneous problem known as the usual weighted eigenvalue Steklov problem for the p-Laplacian with indefinite weight m_p:

$$(P_{\lambda,m_p})\begin{cases} \triangle_p u &= |u|^{p-2}u \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} &= \lambda m_p(x)|u|^{p-2}u \quad \text{on } \partial\Omega \end{cases}$$

Moreover, after multiplying our equation $(P_{\lambda,\mu})$ by $1/\mu$ and then letting $\mu \to +\infty$, we obtain the $(q-1)$-homogeneous equations in:

$$(P_{\lambda,m_q})\begin{cases} \triangle_q u &= |u|^{q-2}u \quad \text{in } \Omega, \\ |\nabla u|^{q-2}\frac{\partial u}{\partial\nu} &= \lambda m_q(x)|u|^{q-2}u \quad \text{on } \partial\Omega \end{cases}$$

Nonlinear Steklov eigenvalue problem (P_{λ,m_r}), where $r = p, q$ and with indefinite weight $m_r \in \mathbb{M}_r$ have been studied by several authors, for example (see [4]). These works proved that there exists a first eigenvalue $\lambda_1(r, m_r) > 0$, where

$$\lambda_1(r, m_r) := \inf \left\{ \frac{1}{r} \|u\|^r_{W^{1,r}(\Omega)}; u \in W^{1,r}(\Omega) \text{ and } \frac{1}{r} \int_{\partial\Omega} m_r(x)|u|^r d\sigma = 1 \right\}, \quad (1.4)$$

which is simple in the sense that two eigenfunctions corresponding to it are proportional. Moreover, the corresponding first eigenfunction $\phi_1(r, m_r)$ can be assumed to be positive. It was also shown in [4] that $\lambda_1(r, m_r)$ is isolated and monotone.

This paper is divided into three sections, organized as follows. In Section 2, we study Rayleigh quotient for our problem $(P_{\lambda,\mu})$. In contrast to homogeneous case, we prove that if $\lambda_1(p, m_p) \neq \lambda_1(q, m_q)$ or $\phi_1(p, m_p) \neq k\phi_1(q, m_q)$ for every $k > 0$, then the infimum in Rayleigh quotient is not attained. We also show non-existence results for positive solutions of the eigenvalue problem $(P_{\lambda,\mu})$ formulated as Theorem 2.5. Our existence results for positive solutions of the eigenvalue problem $(P_{\lambda,\mu})$ are presented in Section 3. We study the non-resonant case (Theorem 3.1) which prove that when $\mu > 0$ there exists an interval of positive eigenvalues for the problem $(P_{\lambda,\mu})$.

2. Rayleigh quotient and non-existence results

This section concerns the Rayleigh quotient and non-existence results for our eigenvalue Steklov problem $(P_{\lambda,\mu})$. It is inspired from [11] and [15].

Remark 2.1. We start by pointing out to find a solution for the problem $(P_{\lambda,\mu})$ is equivalent to seek a solution in the case $\mu = 1$, that is to solve the problem $(P_{\lambda,1})$. Indeed, if u is a solution of $(P_{\lambda,1})$, then multiplying equation $(P_{\lambda,1})$ by s^{p-1} for $s > 0$ we deduce that $v = su$ is a solution for problem $(P_{\lambda,\mu=s^{p-q}})$.
Conversely, let u be a solution of problem $(P_{\lambda,\mu})$. Then it follows that $v = \mu^{1/p-q}u$ is a solution of $(P_{\lambda,1})$.

2.1. Rayleigh quotient for the problem $(P_{\lambda,\mu})$

We introduce now the functionals A and B on $W^{1,p}(\Omega)$ by

$$A(u) := \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} + \frac{1}{q}\|u\|^q_{W^{1,q}(\Omega)} \quad (2.1)$$

$$B(u) := \frac{1}{p}\int_{\partial\Omega} m_p(x)|u|^p d\sigma + \frac{1}{q}\int_{\partial\Omega} m_q(x)|u|^q d\sigma \quad (2.2)$$

for all $u \in W^{1,p}(\Omega)$.

Proposition 2.2. *(i) The functional A is well defined and sequently weakly lower semi-continuous.*
(ii) If $m_p \in \mathbb{M}_p$ and $m_q \in \mathbb{M}_q$, then the functional B is also well defined and weakly continuous.

Proof. (i) The functional A is well defined. Indeed, since Ω bounded and $q < p$, we have $W^{1,p}(\Omega) \subset W^{1,q}(\Omega)$. Then for all $u \in W^{1,p}(\Omega)$, $\frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} < \infty$ and $\frac{1}{q}\|u\|_{W^{1,q}(\Omega)} < \infty$. It follows that $A(u) < \infty$. It is clear that A is sequently weakly lower semi-continuous.
(ii) The functional B is also well defined. Indeed, for $u \in W^{1,p}(\Omega)$, by Hölder's inequality, for $r = p, q$ and $s'_r = s_r/(s_r - 1)$, we obtain

$$\begin{aligned}\frac{1}{r}\int_{\partial\Omega} m_r(x)|u|^r d\sigma &\le \frac{1}{r}\left(\int_{\partial\Omega}|m_r(x)|^{s_r} d\sigma\right)^{1/s_r}\left(\int_{\partial\Omega}|u|^{rs'_r} d\sigma\right)^{1-1/s_r}\\ &= \frac{1}{r}\|m_r\|_{s_r,\partial\Omega}\|u\|^r_{rs'_r,\partial\Omega}\\ &< \infty,\end{aligned}$$

since $m_r \in \mathbb{M}_r$ and the trace embedding $W^{1,r}(\Omega) \longrightarrow L^{rs'_r}(\partial\Omega)$ is compact.
Let us now show that B is weakly continuous. If $u_n \to u$ weakly in $W^{1,r}(\Omega)$, up to a subsequence, $u_n \to u$ strongly in $L^{rs'_r}(\partial\Omega)$ and $|u_n|^r \to |u|^r$ strongly in $L^{s'_r}(\partial\Omega)$ with $r = p, q$. Hence by Hölder's inequality, we have

$$\begin{aligned}|B(u_n) - B(u)| &\le \frac{1}{p}\left|\int_{\partial\Omega} m_p(x)(|u_n|^p - |u|^p)d\sigma\right| + \frac{1}{q}\left|\int_{\partial\Omega} m_q(x)(|u_n|^q - |u|^q)d\sigma\right|\\ &\le \frac{1}{p}\|m_p\|_{s_p,\partial\Omega}\||u_n|^p - |u|^p\|_{s'_p,\partial\Omega} + \frac{1}{q}\|m_q\|_{s_q,\partial\Omega}\||u_n|^q - |u|^q\|_{s'_q,\partial\Omega}\\ &\to 0.\end{aligned}$$

Thus the functional B is weakly continuous. □

Define now the Rayleigh quotient

$$\lambda^* = \inf\left\{\frac{A(u)}{B(u)}; u \in W^{1,p}(\Omega), B(u) > 0\right\}. \tag{2.3}$$

Proposition 2.3. *One assumes that $m_p \in \mathbb{M}_p$ and $m_q \in \mathbb{M}_q$. If $\lambda_1(p, m_p) \neq \lambda_1(q, m_q)$ or $\phi_1(p, m_p) \neq k\phi_1(q, m_q)$, for every $k > 0$. Then the infimum in (2.3) is not attained.*

For the proof of Proposition 2.3, we will need to use the following lemma.

Lemma 2.4. *The infimum in (2.3) verifies*

$$\lambda^* = \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\}$$

Proof. For sufficiently large $k > 0$, using (2.1) and (2.2), we have

$$B(k\phi_1(p, m_p)) = k^q\left(k^{p-q} + \frac{1}{q}\int_{\partial\Omega} m_q(x)\phi_1^q(p, m_p)d\sigma\right) > 0.$$

and

$$A(k\phi_1(p,m_p)) = k^p\left(\lambda_1(p,m_p) + \frac{1}{q}k^{q-p}\|\phi_1(p,m_p)\|^q_{W^{1,q}(\Omega)}\right).$$

By (2.3), we find

$$\begin{aligned}\lambda^* &\le \frac{A(k\phi_1(p,m_p))}{B(k\phi_1(p,m_p))}\\ &= \frac{\lambda_1(p,m_p) + \frac{1}{q}k^{q-p}\|\phi_1(p,m_p)\|^q_{W^{1,q}(\Omega)}}{1 + \frac{1}{q}k^{q-p}\int_{\partial\Omega} m_q(x)\phi_1^q(p,m_p)d\sigma}\\ &\to \lambda_1(p,m_p) \text{ as } k\to+\infty, \text{ because } q<p.\end{aligned}$$

It follows that $\lambda^* \le \lambda_1(p,m_p)$. On the other hand, we also have

$$\begin{aligned}\lambda^* &\le \frac{A(k\phi_1(q,m_q))}{B(k\phi_1(q,m_q))}\\ &= \frac{\lambda_1(q,m_q) + \frac{1}{p}k^{p-q}\|\phi_1(q,m_q)\|^p_{W^{1,p}(\Omega)}}{1 + \frac{1}{p}k^{p-q}\int_{\partial\Omega} m_p(x)\phi_1^p(q,m_q)d\sigma}\\ &\to \lambda_1(q,m_q) \text{ as } k\to 0^+, \text{ because } q<p.\end{aligned}$$

Thus, we obtain $\lambda^* \le \lambda_1(q,m_q)$, which implies that

$$\lambda^* \le \min\{\lambda_1(p,m_p), \lambda_1(q,m_q)\}$$

Conversely, suppose by contradiction that $\lambda^* < \min\{\lambda_1(p,m_p), \lambda_1(q,m_q)\}$. Then, by (2.3), there exists $u \in W^{1,p}(\Omega)$ such that $B(u) > 0$ and

$$\frac{A(u)}{B(u)} < \min\{\lambda_1(p,m_p), \lambda_1(q,m_q)\}.$$

We distinguish three cases.

Case (i): Suppose that $\int_{\partial\Omega} m_p|u|^p d\sigma > 0$ and $\int_{\partial\Omega} m_q|u|^q d\sigma \le 0$. There hold $pB(u) \le \int_{\partial\Omega} m_p|u|^p d\sigma$ and $pA(u) \ge \|u\|^p_{W^{1,p}(\Omega)}$. Using the definition of $\lambda_1(p,m_p)$, we arrive at the contradiction.

$$\min\{\lambda_1(p,m_p), \lambda_1(q,m_q)\} > \frac{A(u)}{B(u)} \ge \frac{\|u\|^p_{W^{1,p}(\Omega)}}{\int_{\partial\Omega} m_p|u|^p d\sigma} \ge \lambda_1(p,m_p). \tag{2.4}$$

Case (ii): Suppose that $\int_{\partial\Omega} m_p|u|^p d\sigma \le 0$ and $\int_{\partial\Omega} m_q|u|^q d\sigma > 0$. Using the definition of $\lambda_1(q,m_q)$, we also arrive at contradiction

$$\min\{\lambda_1(p,m_p), \lambda_1(q,m_q)\} > \frac{A(u)}{B(u)} \ge \frac{\|u\|^q_{W^{1,q}(\Omega)}}{\int_{\partial\Omega} m_q|u|^q d\sigma} \ge \lambda_1(q,m_q). \tag{2.5}$$

Case (iii): Suppose now that $\int_{\partial\Omega} m_p|u|^p d\sigma > 0$ and $\int_{\partial\Omega} m_q|u|^q d\sigma > 0$. It follows from the definition of $\lambda_1(r,m_r)$, where $r = p, q$ that

$$\|u\|^r_{W^{1,r}(\Omega)} \ge \lambda_1(r,m_r)\int_{\partial\Omega} m_r|u|^r d\sigma.$$

Hence we get

$$A(u) \geq \frac{\lambda_1(p, m_p)}{p} \int_{\partial\Omega} m_p |u|^p d\sigma + \frac{\lambda_1(q, m_q)}{q} \int_{\partial\Omega} m_q |u|^q d\sigma \\ \geq \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\} B(u). \tag{2.6}$$

Against the assumption in our reasoning by contradiction.

□

Proof of Proposition 2.3. By contradiction, we suppose that:

$$\text{there exists } u \in W^{1,p}(\Omega) \text{ such that } B(u) > 0 \text{ and } \frac{A(u)}{B(u)} = \lambda^*.$$

Using Lemma 2.4, we give

$$\frac{A(u)}{B(u)} = \lambda^* = \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\}. \tag{2.7}$$

We argue by considering the three cases in the proof of Lemma 2.4.
Case (i): By (2.4), (2.7) and $\int_{\partial\Omega} m_q |u|^q d\sigma \leq 0$, we have

$$\lambda^* = \frac{A(u)}{B(u)} \geq \frac{\|u\|^p_{W^{1,p}(\Omega)} + \frac{p}{q}\|u\|^q_{W^{1,q}(\Omega)}}{\int_{\partial\Omega} m_p |u|^p d\sigma} \geq \frac{\|u\|^p_{W^{1,p}(\Omega)}}{\int_{\partial\Omega} m_p |u|^p d\sigma} \geq \lambda_1(p, m_p) \geq \lambda^*.$$

We deduce that

$$\|u\|^p_{W^{1,p}(\Omega)} = \lambda_1(p, m_p) \int_{\partial\Omega} m_p |u|^p d\sigma \text{ and } \|u\|_{W^{1,q}(\Omega)} = 0.$$

Thus $u = 0$. This contradicts the fact that $u \neq 0$.
Case (ii): similarly, By (2.5), (2.7) and $\int_{\partial\Omega} m_p |u|^p d\sigma \leq 0$, we get

$$\|u\|^q_{W^{1,q}(\Omega)} = \lambda_1(q, m_q) \int_{\partial\Omega} m_q |u|^q d\sigma \text{ and } \|u\|_{W^{1,p}(\Omega)} = 0.$$

Thus $u = 0$. Which contradicts $u \neq 0$.
Case (iii): In this case, using (2.6) and (2.7), we find

$$A(u) = \lambda^* B(u) = \frac{\lambda_1(p, m_p)}{p} \int_{\partial\Omega} m_p |u|^p d\sigma + \frac{\lambda_1(q, m_q)}{q} \int_{\partial\Omega} m_q |u|^q d\sigma.$$

It follows

$$[\lambda_1(p, m_p) - \lambda^*] \int_{\partial\Omega} m_p |u|^p d\sigma + [\lambda_1(q, m_q) - \lambda^*] \int_{\partial\Omega} m_q |u|^q d\sigma = 0.$$

Since $\int_{\partial\Omega} m_p |u|^p d\sigma > 0$, $\int_{\partial\Omega} m_q |u|^q d\sigma > 0$ and $\lambda^* = \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\}$, we have

$$\lambda^* = \lambda_1(p, m_p) = \lambda_1(q, m_q).$$

We deduce that

$$\frac{\|u\|^p_{W^{1,p}(\Omega)}}{\int_{\partial\Omega} m_p |u|^p d\sigma} = \lambda_1(p, m_p) = \lambda_1(q, m_q) = \frac{\|u\|^q_{W^{1,p}(\Omega)}}{\int_{\partial\Omega} m_q |u|^q d\sigma}.$$

Hence, the simplicity of eigenvalue $\lambda_1(r, m_r)$ (for $r = p, q$), guarantees that $u = t\phi_1(p, m_p) = s\phi_1(q, m_q)$ for some $t \neq 0$ and $s \neq 0$. The hypotheses of proposition is thus contradicted. □

2.2. Non-existence results

The following theorem is the main result of this section.

Theorem 2.5. *One assumes that $m_p \in \mathbb{M}_p$ and $m_q \in \mathbb{M}_q$.*

(a) *If $0 < \lambda < \lambda^*$, then the problem $(P_{\lambda,1})$ has no non-trivial solutions.*

(b) *Moreover, if one of the following conditions holds*

(i) $\lambda_1(p, m_p) \neq \lambda_1(q, m_q)$;

(ii) $\phi_1(p, m_p) \neq k\phi_1(q, m_q)$, *for every* $k > 0$,

then the problem $(P_{\lambda,1})$, with $\lambda = \lambda^*$ has no non-trivial solutions.

Remark 2.6. It is easy to see that if $\lambda_1(p, m_p) = \lambda_1(q, m_q)$ and $\phi_1(p, m_p) = k\phi_1(q, m_q)$, for some $k > 0$, then $\phi_1(p, m_p)$ and $\phi_1(q, m_q)$ are positive solutions of problem $(P_{\lambda,1})$, with $\lambda = \lambda_1(p, m_p) = \lambda_1(q, m_q)$.

Proof of Theorem 2.5. Assume by contradiction that there exists a non-trivial solution u of problem $(P_{\lambda,1})$. Then, for every $s > 0$, we have that $v = su$ is a non-trivial solution of problem $(P_{\lambda,s^{p-q}})$ (see Remark 2.1). Choose $s^{p-q} = p/q$ and then act with su as test function on the problem $(P_{\lambda,s^{p-q}})$. We arrive at

$$0 < pA(su) = p\lambda B(su). \tag{2.8}$$

From the estimate (2.8) and according to Lemma 2.4, we obtain

$$\lambda = \frac{A(su)}{B(su)} \geq \lambda^* = \min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\}.$$

This contradiction yields the first assertion of the theorem.
The second part of the Theorem 2.5 follows by Proposition 2.3. □

3. Existence result with non-resonant case

The following theorem is our main existence result for problem $(P_{\lambda,1})$ (or $(P_{\lambda,\mu})$) in the non-resonant case. This result prove that there exists an interval of positive eigenvalues for the problem $(P_{\lambda,1})$ (or $(P_{\lambda,\mu})$, with $\mu > 0$).

Theorem 3.1. *One supposes that $m_p \in \mathbb{M}_p$, $m_q \in \mathbb{M}_q$ and $\lambda_1(p, m_p) \neq \lambda_1(q, m_q)$. If*

$$\min\{\lambda_1(p, m_p), \lambda_1(q, m_q)\} < \lambda < \max\{\lambda_1(p, m_p), \lambda_1(q, m_q)\},$$

then the problem $(P_{\lambda,1})$ has at least one positive solution.

Remark 3.2. The proof of Theorem 3.1 reduces to provide a non-trivial critical point of the functional I_{λ,m_p,m_q} defined for all $u \in W^{1,p}(\Omega)$ by

$$I_{\lambda,m_p,m_q}(u) := A(u) - \lambda B(u^+),$$

where $u^+ = \max\{u, 0\}$ and A, B are the functionals defined by (2.1) and (2.2). This non-trivial critical point u of I_{λ,m_p,m_q} is a non-negative solution of the problem $(P_{\lambda,1})$. Indeed, inserting $-u^- = -\max\{-u, 0\}$ as test function leads to

$$0 = \langle I'_{\lambda,m_p,m_q}(u), -u^-\rangle = \|u^-\|^p_{W^{1,p}(\Omega)} + \|u^-\|^q_{W^{1,q}(\Omega)},$$

thus $u^- = 0$. We can check that $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ (see [1]). Then the maximum principle of Vasquez [16] can be applied to ensure positiveness of u.

The argument will be separately developed in two cases:
(a) $\lambda_1(q, m_q) < \lambda < \lambda_1(p, m_p)$.
(b)$\lambda_1(p, m_p) < \lambda < \lambda_1(q, m_q)$.
In case (a), we apply the minimum principle and in case (b), we use the mountain pass theorem.

Proof of case (a). By Proposition 2.2, A is sequently weakly lower semi-continuous and B is weakly continuous. It follows that I_{λ,m_p,m_q} is sequently weakly lower semi-continuous. Moreover I_{λ,m_p,m_q} is bounded from below. Indeed for all $u \in W^{1,p}(\Omega)$, we have

$$I_{\lambda,m_p,m_q}(u) \geq -\lambda B(u^+) > -\infty. \tag{3.1}$$

It is remains to show that I_{λ,m_p,m_q} is coercive in $W^{1,p}(\Omega)$.
Fix $\varepsilon > 0$ such that

$$(1-\varepsilon)\lambda_1(p, m_p) > \lambda \tag{3.2}$$

which is possible due to the assumption in case (a). For every $u \in W^{1,p}(\Omega)$ with $\int_{\partial\Omega} m_p(u^+)^p d\sigma \leq 0$, through Holder's inequality we obtain

$$\begin{aligned} I_{\lambda,m_p,m_q}(u) &\geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} + \frac{1}{q}\|u\|^q_{W^{1,q}(\Omega)} - \frac{\lambda}{q}\int_{\partial\Omega} m_q(u^+)^q d\sigma \\ &\geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} + \frac{1}{q}\|u\|^q_{W^{1,q}(\Omega)} - \frac{\lambda}{q}\|m_q\|_{\mathrm{L}^{s_q}(\partial\Omega)}\|(u^+)^q\|_{\mathrm{L}^{s'_q}(\partial\Omega)} \\ &\geq \frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - \frac{\lambda C}{q}\|m_q\|_{\mathrm{L}^{s_q}(\partial\Omega)}\|u\|^q_{W^{1,p}(\Omega)} \end{aligned} \tag{3.3}$$

for $u \in W^{1,p}(\Omega)$ with $\int_{\partial\Omega} m_p(u^+)^p d\sigma > 0$, by (1.4) we have

$$\|u^+\|^p_{W^{1,p}(\Omega)} \geq \lambda_1(p, m_p)\int_{\partial\Omega} m_p(u^+)^p d\sigma.$$

Then, taking into account (3.2), we derive

$$\begin{aligned} I_{\lambda,m_p,m_q}(u) &\geq \frac{\varepsilon}{p}\|u\|^p_{W^{1,p}(\Omega)} + \frac{(1-\varepsilon)\lambda_1(p,m_p)-\lambda}{p}\int_{\partial\Omega} m_p(u^+)^p d\sigma \\ &\quad - \frac{\lambda}{q}\|m_q\|_{\mathrm{L}^{s_q}(\partial\Omega)}\|u^+\|^q_{\mathrm{L}^{s'_q}(\partial\Omega)} \\ &\geq \frac{\varepsilon}{p}\|u\|^p_{W^{1,p}(\Omega)} - \frac{C\lambda}{q}\|m_q\|_{\mathrm{L}^{s_q}(\partial\Omega)}\|u\|^q_{W^{1,p}(\Omega)} \end{aligned} \tag{3.4}$$

Since $q < p$, it follows from (3.3) and (3.4) that I_{λ,m_p,m_q} is coercive and bounded from below. Consequently, by a standard result(see, e.g., [[8],Theorem 1.1]), there exists a global minimizer u_0 of I_λ. In order to have $u_0 \neq 0$ it suffices to show that $I_\lambda(u_0) = \min_{W^{1,p}(\Omega)} I_\lambda < 0$. Let ψ_1 be the eigenfunction corresponding to $\lambda_1(q,m_q)$ that satisfies $\int_{\partial\Omega} m_q\psi_1^q d\sigma = 1$. Because $\lambda > \lambda_1(q,m_q)$, for sufficiently small $t > 0$ it holds

$I_{\lambda,m_p,m_q}(t\psi_1) = t^q\left(\frac{t^{p-q}}{p}\|\psi_1\|^p_{W^{1,p}(\Omega)} - \frac{\lambda t^{p-q}}{p}\int_{\partial\Omega} m_p\psi_1^p d\sigma + \frac{\lambda_1(q,m_q)-\lambda}{q}\right) < 0,$

which completes the proof.

Proof of case (b). We organize the proof of this case in several lemmas. In the sequel, we design by $o(1)$ a quantity tending to 0 as $n \longrightarrow \infty$.

Lemma 3.3. *Let $m_p \in M_p$, $m_q \in M_q$. If $0 \leq \lambda \neq \lambda_1(p,m_p)$, Then the functional I_{λ,m_p,m_q} satisfies the Palais-Smale condition.*

Proof. Let $(u_n) \subset W^{1,p}(\Omega)$ be a sequence such that $I_{\lambda,m_p,m_q}(u_n) \longrightarrow c$ for $c \in \mathbb{R}$ and $I'_{\lambda,m_p,m_q}(u_n) \longrightarrow 0$ in $(W^{1,p}(\Omega))^*$ as $n \longrightarrow \infty$. Let us first show that the sequence (u_n) is bounded in $W^{1,p}(\Omega)$. It is sufficient only to prove the boundedness of $\|u_n\|_{ps'_p}$, because using the Hölder's inequality and the continuous embedding $W^{1,p}(\Omega) \subset L^{qs'_q}(\partial\Omega)$, we have

$$\|u_n\|^p_{W^{1,p}(\Omega)} \leq pc + c'\lambda\|m_p\|_{s_p}\|u_n\|^p_{ps'_p} + c''\frac{p\lambda}{q}\|m_q\|_{s_q}\|u_n\|^q_{W^{1,p}(\Omega)} \tag{3.5}$$

where c' and c'' are the positive constants.

Suppose by contradiction that $\|u_n\|_{ps'_p} \to +\infty$ and let $v_n := \frac{u_n}{\|u_n\|_{ps'_p}}$. We claim that the sequence v_n bounded in $W^{1,p}(\Omega)$.

Indeed, dividing (3.5) by $\|u_n\|^p_{ps'_p}$ we have

$$\|v_n\|^p_{W^{1,p}(\Omega)} \leq \frac{pc}{\|u_n\|^p_{ps'_p}} + C_1 + C_2\|v_n\|^q_{W^{1,p}(\Omega)}, \tag{3.6}$$

where the positive contants C_1 and C_2 are defined by $C_1 = c'\lambda\|m_p\|_{s_p}$ and $C_2 = c''\frac{p\lambda}{q}\|m_q\|_{s_q}$. Since $1 < q < p$, the inequality (3.6) implies the boundedness of v_n in $W^{1,p}(\Omega)$. for a subsequence, $v_n \rightharpoonup v$ (weakly) in in $W^{1,p}(\Omega)$. By the compact embedding. $W^{1,r}(\Omega) \subset L^{rs'_r}(\partial\Omega), (r = p,q)$ we have $v_n \to v$ strongly in $L^{rs'_r}(\partial\Omega)$ $(r = p,q)$. First we observe that $v^- \equiv 0$ in Ω. In fact, acting with $-u_n^-$ as test function, we have

$$o(1)\|u_n^-\|_{ps'_p} = \langle I'_{\lambda,m_p,m_q}(u_n), -u_n^-\rangle = \frac{1}{p}\|u_n^-\|^p_{W^{1,p}(\Omega)} + \frac{1}{q}\|u_n^-\|^q_{W^{1,q}(\Omega)} \geq \|u_n^-\|^p_{W^{1,p}(\Omega)} \tag{3.7}$$

the inequality(3.7) quarantines the boundedness of $\|v_n^-\|_{W^{1,p}(\Omega)}$ and so $\|v_n^-\|_{W^{1,p}(\Omega)} = \frac{\|u_n^-\|_{W^{1,p}(\Omega)}}{\|u_n^-\|_{ps'_p}} \to 0$, thus $v^- \equiv 0$, holds, hence $v \geq 0$ in Ω.
Now, by taking $(v_n - v)/\|u_n^-\|_{ps'_p}^{p-1}$ as test function, we have

$$
\begin{aligned}
o(1) &= \langle I'_{\lambda,m_p,m_q}(u_n), \frac{(v_n - v)}{\|u_n^-\|_{ps'_p}^{p-1}} \rangle \\
&= \frac{1}{p}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)dx + \frac{1}{q\|u_n\|_{ps'_p}^{p-q}}\int_\Omega |\nabla v_n|^{q-2}\nabla v_n\nabla(v_n - v)dx \\
&+ \frac{1}{p}\int_\Omega |v_n|^{p-2}v_n(v_n - v)dx + \frac{1}{q\|u_n\|_{ps'_p}^{p-q}}\int_\Omega |v_n|^{q-2}v_n(v_n - v)dx \\
&- \frac{\lambda}{p}\int_{\partial\Omega} m_p(v_n^+)^{p-2}v_n^+(v_n - v)d\sigma \\
&- \frac{\lambda}{q\|u_n\|_{ps'_p}^{p-q}}\int_{\partial\Omega} m_q(v_n^+)^{q-2}v_n^+(v_n - v)d\sigma \\
&= \frac{1}{p}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n\nabla(v_n - v)dx + \frac{1}{p}\int_\Omega |v_n|^{p-2}v_n(v_n - v)dx \\
&- \frac{\lambda}{p}\int_{\partial\Omega} m_p(v_n^+)^{p-2}v_n^+(v_n - v)d\sigma + o(1)
\end{aligned}
\tag{3.8}
$$

because $q < p$, $\|u_n^-\|_{ps'_p} \to +\infty$, v_n is bounded in $W^{1,p}(\Omega)$ and converge to v strongly in $L^{ps'_p}(\partial\Omega)$. Thus by (3.8) and (S_+) property of $-\Delta_p u + u^{p-2}u$ in $W^{1,p}(\Omega)$, we deduce that $v_n \to v$ strongly in $W^{1,p}(\Omega)$. For any $\varphi \in W^{1,p}(\Omega)$, by taking $\frac{\varphi}{\|u_n^-\|_{ps'_p}^{p-1}}$ as test function, we obtain

$$
\begin{aligned}
o(1) &= \langle I'_{\lambda,m_p,m_q}(u_n), \frac{\varphi}{\|u_n\|_{ps'_p}^{p-1}} \rangle \\
&= \frac{1}{p}\int_\Omega |\nabla v_n|^{p-2}\nabla v_n\nabla\varphi dx + \frac{1}{q\|u_n\|_{ps'_p}^{p-q}}\int_\Omega |\nabla v_n|^{q-2}\nabla v_n\nabla\varphi dx \\
&+ \frac{1}{p}\int_\Omega |v_n|^{p-2}v_n\varphi dx + \frac{1}{q\|u_n\|_{ps'_p}^{p-q}}\int_\Omega |v_n|^{q-2}v_n\varphi dx \\
&- \frac{\lambda}{p}\int_{\partial\Omega} m_p(v_n^+)^{p-2}v_n^+\varphi d\sigma - \frac{\lambda}{q\|u_n\|_{ps'_p}^{p-q}}\int_{\partial\Omega} m_q(v_n^+)^{q-2}v_n^+\varphi d\sigma
\end{aligned}
\tag{3.9}
$$

Passing to the limit in (3.9), we see that v is a non-negative and non-trivial solution of problem (P_{λ,m_p}) (note $v \geq 0$ and $\|v\|_{W^{1,p}(\Omega)} = 1$). The eigenfunction v is $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$ (see [1]). According to maximum principle of Vasquez, we have $v > 0$ in $W^{1,p}(\Omega)$. This implies that $\lambda = \lambda_1(p, m_p)$ because any positive eigenvalue other than $\lambda_1(p, m_p)$ has no positive eigenfunction. Therefore, we obtain a contradiction since we assumed $\lambda \neq \lambda_1(p, m_p)$. Hence u_n is bounded in $W^{1,p}(\Omega)$. For a subsequence, $u_n \rightharpoonup u$ (weakly) in $W^{1,p}(\Omega)$ and $u_n \to u$ (strongly) in $L^{ps'_p}(\partial\Omega)$. We claim now that $u_n \to u$ in $W^{1,p}(\Omega)$. As $W^{1,p}(\Omega)$ is reflexive and uniformly convex, it suffices to prove that $\|u_n\|_{W^{1,p}(\Omega)} \to \|u\|_{W^{1,p}(\Omega)}$. It is clear that

$$\begin{aligned} o(1) &= \left\langle I'_{\lambda,m_p,m_q}(u_n), u_n - u \right\rangle \\ &= \frac{1}{p}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)dx + \frac{1}{p}\int_\Omega |u_n|^{p-2}u_n(u_n - u)dx \\ &+ \frac{1}{q}\int_\Omega |\nabla u_n|^{q-2}\nabla u_n \nabla(u_n - u)dx + \frac{1}{q}\int_\Omega |u_n|^{q-2}u_n(u_n - u)dx + o(1). \end{aligned} \tag{3.10}$$

Using Hölder's inequality and for $(r = p, q)$, we have

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{r-2}\nabla u_n \nabla(u_n - u)dx + \int_\Omega |u_n|^{r-2}u_n(u_n - u)dx \\ &= \int_\Omega |\nabla u_n|^r dx + \int_\Omega |\nabla u|^r dx - \int_\Omega |\nabla u_n|^{r-2}\nabla u_n u dx - \int_\Omega |\nabla u|^{r-2}\nabla u \nabla u_n dx \\ &= \int_\Omega |u_n|^r dx + \int_\Omega |u|^r dx - \int_\Omega |u_n|^{r-2}u_n u dx - \int_\Omega |u|^{r-2}u u_n dx \\ &\geq \int_\Omega |\nabla u_n|^r dx + \int_\Omega |\nabla u|^r dx - \left(\int_\Omega |\nabla u_n|^r dx\right)^{(r-1)/r}\left(\int_\Omega |\nabla u|^r dx\right)^{1/r} \\ &+ \int_\Omega |u_n|^r dx + \int_\Omega |u|^r dx - \left(\int_\Omega |u_n|^r dx\right)^{(r-1)/r}\left(\int_\Omega |u|^r dx\right)^{1/r} \\ &= \left(\|u_n\|^{r-1}_{W^{1,r}(\Omega)} - \|u\|^{r-1}_{W^{1,r}(\Omega)}\right)\left(\|u_n\|_{W^{1,r}(\Omega)} - \|u\|_{W^{1,r}(\Omega)}\right) \\ &\geq 0 \end{aligned} \tag{3.11}$$

Moreover, (3.10) and (3.11) imply that $\|u_n\|_{W^{1,p}(\Omega)} \to \|u\|_{W^{1,p}(\Omega)}$. Thus $u_n \to u$ strongly in $W^{1,p}(\Omega)$. □

Lemma 3.4. *Let $m_p \in M_p$, $m_q \in M_q$. If $\lambda < \lambda_1(q, m_q)$, then there exist $\delta > 0$ and $\rho > 0$ such that*

$$I_\lambda(u) \geq \delta \text{ whenever } \|u\|_{L^{qs'_q}(\partial\Omega)} = \rho. \tag{3.12}$$

To prove the Lemma 3.4, we need the following lemma.

Lemma 3.5.

$$X(d) := \left\{u \in W^{1,p}(\Omega);\ \|u\|^p_{W^{1,p}(\Omega)} \leq d\|u\|^p_{L^{ps'_p}(\partial\Omega)}\right\} \tag{3.13}$$

for $d > 0$. Then there exists $C = C(d) > 0$ such that

$$\|u\|_{W^{1,p}(\Omega)} \leq C\|u\|_{L^{qs'_q}}(\partial\Omega) \text{ for all } u \in X(d).$$

Proof. By way contradiction, we assume that

$$\forall n \in \mathbb{N}, \exists u_n \in X(d),\ \frac{1}{n}\|u_n\|_{W^{1,p}(\Omega)} > \|u_n\|_{L^{qs'_q}(\partial\Omega)} \tag{3.14}$$

Set $v_n = \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}}$; hence v_n bounded in $W^{1,p}(\Omega)$, then there exists a subsequence that we still denote v_n such that $v_n \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$. By the compact embedding $W^{1,r}(\Omega) \subset L^{rs'_r}(\partial\Omega)(r = p, q)$ we have $v_n \to v$ strongly in $L^{rs'_r}(\partial\Omega)$. By (3.14), we have $\|v_n\|_{L^{qs'_q}(\partial\Omega)} < \frac{1}{n}$, thus $v_n \to 0$ in $L^{qs'_q}(\partial\Omega)$. By uniqueness of the limit we have $v = 0$, hence $v_n \to 0$ in $L^{ps'_p}(\partial\Omega)$. As $u_n \in X(d)$ we have

$$\frac{1}{d} \le \frac{\|u_n\|^p_{L^{ps'_p}(\partial\Omega)}}{\|u_n\|^p_{W^{1,p}(\Omega)}} = \|v_n\|^p_{L^{ps'_p}(\partial\Omega)}.$$

Passing to the limit, we obtain a contradiction. □

Proof of Lemma 3.4. Let C_r the constant from embedding $W^{1,r}(\Omega) \subset L^{rs'_r}(\partial\Omega)$, where $r = p, q$. According to Lemma 3.5, there exists $C(d) > 0$ such that

$$\|u\|_{W^{1,p}(\Omega)} \le C(d)\|u\|_{L^{qs'q}(\partial\Omega)} \ \textit{for all}\ u \in X(d), \tag{3.15}$$

where d such that

$$d > max\left\{1, C_p\lambda\|m_p\|_{L^{s_p}(\partial\Omega)}, \lambda\|m_p\|_{L^{s_p}(\partial\Omega)}\right\}. \tag{3.16}$$

For any $u \in X(d)$ satisfying $\int_{\partial\Omega} m_q u_+^q d\sigma \le 0$ by (3.16) and (3.15) we have

$$\begin{aligned}
I_\lambda(u) \ge& \frac{1-d}{p}\|u\|^p_{W^{1,p}(\Omega)} + \frac{1}{qC_q}\|u\|^q_{\mathrm{L}^{qs'_q}(\partial\Omega)} + \frac{d}{p}\|u\|^p_{W^{1,p}(\Omega)}\\
&- \frac{\lambda}{p}\|m_p\|_{\mathrm{L}^{s_p}(\partial\Omega)}\|u\|^p_{\mathrm{L}^{ps'_p}(\partial\Omega)}\\
\ge& \frac{1-d}{p}\|u\|^p_{W^{1,p}(\Omega)} + \frac{1}{qC_q}\|u\|^q_{\mathrm{L}^{qs'_q}(\partial\Omega)} + \frac{d}{p}\|u\|^p_{W^{1,p}(\Omega)}\\
&- \frac{C_p\lambda}{p}\|m_p\|_{\mathrm{L}^{s_p}(\partial\Omega)}\|u\|^p_{W^{1,p}(\Omega)}\\
\ge& \frac{1-d}{p}\|u\|^p_{W^{1,p}(\Omega)} + \frac{1}{qC_q}\|u\|^q_{\mathrm{L}^{qs'_q}(\partial\Omega)} + \frac{(d - C_p\lambda\|m_p\|_{\mathrm{L}^{s_p}(\partial\Omega)})}{p}\|u\|^p_{W^{1,p}(\Omega)}\\
\ge& \frac{(1-d)C^p(d)}{p}\|u\|^p_{\mathrm{L}^{qs'_q}(\partial\Omega)} + \frac{1}{qC_q}\|u\|^q_{\mathrm{L}^{qs'_q}(\partial\Omega)}
\end{aligned} \tag{3.17}$$

For any $u \notin X(d)$ satisfying $\int_{\partial\Omega} m_q u_+^q d\sigma \le 0$ thanks to (3.13) and (3.16) we find

$$\begin{aligned}
I_\lambda(u) &\ge \frac{d - \lambda\|m_p\|_{\mathrm{L}^{s_p}(\partial\Omega)}}{p}\|u\|^p_{\mathrm{L}^{ps'_p}(\partial\Omega)} + \frac{1}{qC_q}\|u\|^q_{\mathrm{L}^{qs'_q}(\partial\Omega)}\\
&\ge \frac{1}{qC_q}\|u\|^q_{\mathrm{L}^{qs'_q}(\partial\Omega)}
\end{aligned} \tag{3.18}$$

If $u \in W^{1,p}(\Omega)$ fulfills $\int_{\partial\Omega} m_q u_+^q d\sigma > 0$, we get

$$\|u\|_{W^{1,q}(\Omega)}^q \geq \|u_+\|_{W^{1,q}(\Omega)}^q \geq \lambda_1(q, m_q) \int_{\partial\Omega} m_q u_+^q d\sigma \tag{3.19}$$

Our assumption on λ enables us to fix $(1 > \varepsilon > 0)$ with

$$(1-\varepsilon)\lambda_1(q, m_q) > \lambda. \tag{3.20}$$

If in addition $u \notin X(d)$ then due to (3.16) and (3.20) we have the estimate

$$\begin{aligned} I_\lambda(u) \geq \frac{d - \lambda\|m_p\|_{\mathrm{L}^{s_p}(\partial\Omega)}}{p}\|u\|_{\mathrm{L}^{ps'_p}(\partial\Omega)}^p + \frac{(1-\varepsilon)\lambda_1(q, m_q) - \lambda}{q}\int_{\partial\Omega} m_q u_+^q d\sigma \\ + \frac{\varepsilon}{q}\|u\|_{W^{1,q}(\Omega)}^q \geq \frac{\varepsilon}{q}\|u\|_{W^{1,q}(\Omega)}^q \geq \frac{\varepsilon}{qC_q}\|u\|_{\mathrm{L}^{qs'_q}(\partial\Omega)}^q \end{aligned} \tag{3.21}$$

Finally, if $u \in X(d)$ and $\int_{\partial\Omega} m_q u_+^q d\sigma > 0$, then (3.16), (3.19), (3.20) imply

$$\begin{aligned} I_\lambda(u) &\geq \frac{1-d}{p}\|u\|_{W^{1,p}(\Omega)}^p + \frac{\varepsilon}{q}\|u\|_{W^{1,q}(\Omega)}^q + \frac{(1-\varepsilon)\lambda_1(q, m_q) - \lambda}{q}\int_{\partial\Omega} m_q u_+^q d\sigma \\ &+ \frac{d}{p}\|u\|_{W^{1,p}(\Omega)}^p \frac{\lambda\|m_p\|_{\mathrm{L}^{s_p}(\partial\Omega)}}{p}\|u\|_{\mathrm{L}^{ps'_p}(\partial\Omega)}^p \\ &\geq \frac{(1-d)C^p(d)}{p}\|u\|_{\mathrm{L}^{qs'_q}(\partial\Omega)}^p + \frac{\varepsilon}{q}\|u\|_{W^{1,q}(\Omega)}^q + \frac{d}{p}\|u\|_{W^{1,p}(\Omega)}^p \\ &- \frac{C_p\lambda\|m_p\|_{\mathrm{L}^{s_p}(\partial\Omega)}}{p}\|u\|_{W^{1,p}(\Omega)}^p \\ &\geq \frac{(1-d)C^p(d)}{p}\|u\|_{\mathrm{L}^{qs'_q}(\partial\Omega)}^p + \frac{\varepsilon}{qC_q}\|u\|_{\mathrm{L}^{qs'_q}(\partial\Omega)}^q \\ &+ \frac{d - C_p\lambda\|m_p\|_{\mathrm{L}^{s_p}(\partial\Omega)}}{p}\|u\|_{W^{1,p}(\Omega)}^p \\ &\geq \frac{(1-d)C^p(d)}{p}\|u\|_{\mathrm{L}^{qs'_q}(\partial\Omega)}^p + \frac{\varepsilon}{qC_q}\|u\|_{\mathrm{L}^{qs'_q}(\partial\Omega)}^q \end{aligned} \tag{3.22}$$

Using that $q < p$, the claim in (3.12) follows from (3.17), (3.18), (3.21), and (3.22). □

Lemma 3.6. *Let $m_p \in M_p$, $m_q \in M_q$. If $\lambda_1(p, m_p) < \lambda$, then there exist $R > 0$ such that*

$$\|R\varphi_1\|_{\mathrm{L}^{qs'}(\partial\Omega)} > \rho \text{ and } I_\lambda(R\varphi_1) < 0, \tag{3.23}$$

where $\rho > 0$ is the constant in (3.12) and φ_1 is the positive eigenfunction corresponding to $\lambda_1(p, m_p)$ satisfying $\int_{\partial\Omega} m_p \varphi_1^p d\sigma = 1$.

Proof. Taking into account that $\lambda > \lambda_1(p, m_p)$ and $p > q$, we claim that (3.23) is true because for a sufficiently large $R > 0$ we have

$$\frac{I_\lambda(R\varphi_1)}{R^p} = \frac{\lambda_1(p, m_p) - \lambda}{1} + \frac{1}{qR^{p-q}}\left(\|\varphi_1\|^q_{W^{1,q}(\Omega)} - \int_{\partial\Omega} m_q \varphi_1^q d\sigma\right) < 0.$$

□

Recalling that I_{λ,m_p,m_q} satisfies the Palais-Smale condition by virtue of Lemma 3.3, the properties pointed out in (3.12) and (3.23) allow us to apply the mountain pass theorem, which guarantees the existence of a critical value $c \geq \delta$ of I_λ, with $\delta > 0$ in (3.12), namely

$$c := \inf_{\gamma\in\Sigma} \max_{t\in[0,1]} I_\lambda(\gamma(t)),$$

$$\Sigma := \left\{\gamma \in C([0,1], W^{1,p}(\Omega)); \gamma(0) = 0, \gamma(1) = R\varphi_1\right\}.$$

This completes the proof of Theorem 3.1.

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