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A Note on Super Integral Rings

Rajat Kanti Nath

ABSTRACT: Let R be a finite non-commutative ring with center $Z(R)$. The commuting graph of R, denoted by Γ_R, is a simple undirected graph whose vertex set is $R \setminus Z(R)$ and two distinct vertices x and y are adjacent if and only if $xy = yx$. Let $\text{Spec}(\Gamma_R)$, $\text{Ł} - spec(\Gamma_R)$ and $\text{Q-Spec}(\Gamma_R)$ denote the spectrum, Laplacian spectrum and signless Laplacian spectrum of Γ_R respectively. A finite non-commutative ring R is called super integral if $\text{Spec}(\Gamma_R)$, $\text{Ł} - spec(\Gamma_R)$ and $\text{Q-Spec}(\Gamma_R)$ contain only integers. In this paper, we obtain several classes of super integral rings.

Key Words: Integral graph, Commuting graph, Spectrum of graph.

Contents

1. Introduction

Throughout the paper R denotes a finite non-commutative ring with center $Z(R)$ and $\frac{R}{Z(R)}$ denotes the additive quotient group. The commuting graph of R, denoted by Γ_R, is a simple undirected graph whose vertex set is $R \setminus Z(R)$ and two vertices x, y are adjacent if and only if $xy = yx$. Many mathematicians have considered commuting graphs of several classes of finite non-commutative rings and studied various graph theoretic aspects (see [1,3,4,16,18,19,20]) in recent years. Some generalizations of Γ_R are also considered in [2,10]. Let $A(\Gamma_R)$ and $D(\Gamma_R)$ denote the adjacency matrix and degree matrix of Γ_R respectively. Then the Laplacian matrix and signless Laplacian matrix of Γ_R are given by $L(\Gamma_R) = D(\Gamma_R) - A(\Gamma_R)$ and $Q(\Gamma_R) = D(\Gamma_R) + A(\Gamma_R)$ respectively. We write $\text{Spec}(\Gamma_R)$, $\text{Ł} - spec(\Gamma_R)$ and $\text{Q-Spec}(\Gamma_R)$ to denote the spectrum, Laplacian spectrum and Signless Laplacian spectrum of Γ_R respectively. Then $\text{Spec}(\Gamma_R) = \{\alpha_1^{a_1}, \alpha_2^{a_2}, \dots, \alpha_l^{a_l}\}$, $\text{Ł} - spec(\Gamma_R) = \{\beta_1^{b_1}, \beta_2^{b_2}, \dots, \beta_m^{b_m}\}$ and $\text{Q-Spec}(\Gamma_R) = \{\gamma_1^{c_1}, \gamma_2^{c_2}, \dots, \gamma_n^{c_n}\}$ where $\alpha_1, \alpha_2, \dots, \alpha_l$ are the eigenvalues of $A(\Gamma_R)$ with multiplicities $a_1, a_2, \dots, a_l$; $\beta_1, \beta_2, \dots, \beta_m$ are the eigenvalues of $L(\Gamma_R)$ with multiplicities $b_1, b_2, \dots, b_m$ and $\gamma_1, \gamma_2, \dots, \gamma_n$ are the eigenvalues of $Q(\Gamma_R)$ with multiplicities $c_1, c_2, \dots, c_n$ respectively. A finite non-commutative ring R is said to be *super integral* if $\text{Spec}(\Gamma_R)$, $\text{Ł} - spec(\Gamma_R)$ and $\text{Q-Spec}(\Gamma_R)$ contain only integers. In this paper, we obtain several classes of super

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integral rings. It may be mentioned here that several classes of finite groups having integral commuting graphs are determined in [12,13].

Let x be an element of R. Then centralizer of x in R denoted by $C_R(x)$ is the set given by $\{y \in R : xy = yx\}$. Let $\mathrm{Cent}(R) = \{C_R(x) : x \in R\}$. A ring R is called an n-centralizer ring if $|\mathrm{Cent}(R)| = n$. The study of n-centralizer rings was initiated by Dutta et al. in [7]. The readers may conf. [7,9,11] for various results on n-centralizer rings. As an application of our results obtained in Section 2, we determine some positive integers n such that R is super integral if $|\mathrm{Cent}(R)| = n$.

The commuting probability of R denoted by $\Pr(R)$ is the probability that a randomly chosen pair of elements of R commute. Clearly, $\Pr(R) = 1$ if and only if R is commutative. MacHale [17] initiated the study of $\Pr(R)$ in the year 1976. Various results on $\Pr(R)$ and its generalizations can be found in [5,6,8,15,17]. Using our results obtained in Section 2, we also determine some positive rationals r such that R is super integral if $\Pr(R) = r$. We conclude the paper by computing various energies of a class of super integral rings.

2. Main results

It is well-known that the spectrum, Laplacian spectrum and signless Laplacian spectrum of the complete graph K_n on n vertices are given by $\{(-1)^{n-1}, (n-1)^1\}, \{0^1, n^{n-1}\}$ and $\{(2n-2)^1, (n-2)^{n-1}\}$ respectively. Further, we have the following theorem which will be used in the next results.

Theorem 2.1. *Let $\mathcal{G} = l_1K_{m_1} \sqcup l_2K_{m_2} \sqcup \cdots \sqcup l_kK_{m_k}$, where $l_iK_{m_i}$ denotes the disjoint union of l_i copies of K_{m_i} for $1 \le i \le k$. Then*

(a) *the Laplacian spectrum of $\mathcal{G}$ is*

$$\left\{0^{\sum_{i=1}^{k} l_i}, m_1^{l_1(m_1-1)}, m_2^{l_2(m_2-1)}, \ldots, m_k^{l_k(m_k-1)}\right\}.$$

(b) *the signless Laplacian spectrum of $\mathcal{G}$ is*

$$\{(2m_1-2)^{l_1}, (m_1-2)^{l_1(m_1-1)}, (2m_2-2)^{l_2}, (m_2-2)^{l_2(m_2-1)}, \ldots, (2m_k-2)^{l_k}, (m_k-2)^{l_k(m_k-1)}\}.$$

The following theorem shows that R is super integral if $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$, where p is a prime.

Theorem 2.2. *Let R be a finite ring and p be a prime. If $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then*

$$\begin{aligned}
\mathrm{Spec}(\Gamma_R) =& \{(-1)^{(p^2-1)|Z(R)|-p-1}, ((p-1)|Z(R)|-1)^{p+1}\},\\
Ł-spec(\Gamma_G) =& \{0^{p+1}, ((p-1)|Z(G)|)^{(p^2-1)|Z(G)|-p-1}\} \text{ and}\\
\text{Q-Spec}(\Gamma_G) =& \{(2(p-1)|Z(G)|-2)^{p+1}, ((p-1)|Z(G)|-2)^{(p^2-1)|Z(G)|-p-1}\}.
\end{aligned}$$

Proof. By [14, Theorem 2.3], we have

$$\mathrm{Spec}(\Gamma_R) = \{(-1)^{(p^2-1)|Z(R)|-p-1}, ((p-1)|Z(R)|-1)^{p+1}\}.$$

Also, in the proof of [14, Theorem 2.3], it was shown that $\Gamma_R = (p+1)K_{(p-1)|Z(R)|}$. Hence the result follows from Theorem 2.1. □

Note that if $\frac{R}{Z(R)}$ is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$ then all the centralizers of non-central elements of R are commutative. A non-commutative ring R is called a CC-ring if all the centralizers of its non-central elements are commutative. In [16], Erfanian et al. have initiated the study of CC-rings. In the following theorem we compute various spectra of Γ_R for a finite CC-ring R.

Theorem 2.3. *If R is a finite CC-ring such that* $\mathrm{Cent}(R) = \{R, S_1, S_2, \dots, S_n\}$ *then*

$$\mathrm{Spec}(\Gamma_R) = \{(-1)^{\sum_{i=1}^{n}|S_i|-n(|Z(R)|+1)}, (|S_1|-|Z(R)|-1)^1, \dots, (|S_n|-|Z(R)|-1)^1\},$$

$$L-spec(\Gamma_R) = \{0^n, (|S_1|-|Z(R)|)^{|S_1|-|Z(R)|-1}, \dots, (|S_n|-|Z(R)|)^{|S_n|-|Z(R)|-1}\}$$

and

$$\text{Q-Spec}(\Gamma_R) = \{(2(|S_1|-|Z(R)|)-2)^1, (|X_1|-|Z(R)|-2)^{|S_1|-|Z(R)|-1}, \dots, (2(|S_n|-|Z(R)|)-2)^1, (|S_n|-|Z(R)|-2)^{|S_n|-|Z(R)|-1}\}.$$

Proof. By [14, Theorem 2.1], we have

$$\mathrm{Spec}(\Gamma_R) = \{(-1)^{\sum_{i=1}^{n}|S_i|-n(|Z(R)|+1)}, (|S_1|-|Z(R)|-1)^1, \dots, (|S_n|-|Z(R)|-1)^1\}.$$

Also, in the proof of [14, Theorem 2.1], it was shown that

$$\Gamma_R = K_{|S_1|-|Z(R)|} \sqcup K_{|S_2|-|Z(R)|} \sqcup \dots \sqcup K_{|S_n|-|Z(R)|}.$$

Hence, the result follows from Theorem 2.1. □

Corollary 2.4. *Let R be a finite CC-ring and* $\mathrm{Cent}(R) = \{R, S_1, S_2, \dots, S_n\}$. *If A is any finite commutative ring then*

$$\mathrm{Spec}(\Gamma_{R\times A}) = \{(-1)^{|A|(\sum_{i=1}^{n}|S_i|-|Z(R)|)-n}, (|A|(|S_1|-|Z(R)|)-1))^1, \dots, (|A|(|S_n|-|Z(R)|)-1))^1\},$$

$$L-spec(\Gamma_{R\times A}) = \{0^n, (|A|(|S_1|-|Z(H)|))^{|A|(|S_1|-|Z(R)|)-1}, \dots, (|A|(|S_n|-|Z(R)|))^{|A|(|S_n|-|Z(R)|)-1}\}$$ *and*

$$\text{Q-Spec}(\Gamma_{R\times A}) = \{(2|A|(|S_1|-|Z(R)|)-2)^1, (|A|(|S_1|-|Z(R)|)-2)^{|A|(|S_1|-|Z(R)|)-1}, \dots, (2|A|(|S_n|-|Z(R)|)-2)^1, (|A|(|S_n|-|Z(R)|)-2)^{|A|(|S_n|-|Z(R)|)-1}\}.$$

Proof. Note that $Z(R \times A) = Z(R) \times A$ and $\text{Cent}(R \times A) = \{R \times A, S_1 \times A, S_2 \times A, \dots, S_n \times A\}$. Therefore, if R is a CC-ring then $R \times A$ is also a CC-ring. Hence, the result follows from Theorem 2.3. □

By Theorem 2.3, it follows that a finite CC-ring is super integral. Further, if R is a finite CC-ring and A is any finite commutative ring then, by Corollary 2.4, $R \times A$ is also super integral. It may be interesting to characterize all super integral rings.

3. Some consequences

In this section, we obtain several consequences of the results obtained in Section 2. We begin with the following result.

Proposition 3.1. *For any prime p, a non-commutative ring of order p^2 is super integral.*

Proof. Let R be a non-commutative ring of order p^2. Note that $|Z(R)| = 1$ and $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. So, by Theorem 2.2, we have

$$\text{Spec}(\Gamma_R) = \{(-1)^{p^2-p-2}, (p-2)^{p+1}\}, \text{Ł} - spec(\Gamma_R) = \{0^{p+1}, (p-1)^{p^2-p-2}\}$$

and

$$\text{Q-Spec}(\Gamma_R) = \{(2p-4)^{p+1}, (p-3)^{p^2-p-2}\}.$$

Hence, R is super integral. □

Proposition 3.2. *For any prime p, a non-commutative ring with unity having order p^3 is super integral.*

Proof. Let R be a ring with unity having order p^3. Then $|Z(R)| = p$ and $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. So, by Theorem 2.2, we have

$$\text{Spec}(\Gamma_R) = \{(-1)^{p^3-2p-1}, (p^2-p-1)^{p+1}\}, \text{Ł} - spec(\Gamma_R) = \{0^{p+1}, (p^2-p)^{(p^3-2p-1}\}$$

and

$$\text{Q-Spec}(\Gamma_R) = \{(2p^2-2p-2)^{p+1}, (p^2-p-2)^{p^3-2p-1}\}.$$

These show that R is super integral. □

Proposition 3.3. *A finite 4-centralizer ring is super integral.*

Proof. If R is a finite 4-centralizer ring then, by [7, Theorem 3.2], we have $\frac{R}{Z(R)} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, by Theorem 2.2, we have

$$\text{Spec}(\Gamma_R) = \{(-1)^{3|Z(R)|-3}, (|Z(R)|-1)^3\}, \text{Ł} - spec(\Gamma_R) = \{0^3, (|Z(R)|)^{3|Z(R)|-3}\}$$

and

$$\text{Q-Spec}(\Gamma_R) = \{(2|Z(R)|-2)^3, (|Z(R)|-2)^{3|Z(R)|-3}\}.$$

Hence, R is super integral. □

Proposition 3.4. *A finite* 5*-centralizer ring is super integral.*

Proof. If R is a finite 5-centralizer ring then, by [7, Theorem 4.3], we have $\frac{R}{Z(R)} \cong \mathbb{Z}_3 \times \mathbb{Z}_3$. Therefore, by Theorem 2.2, we have

$$\text{Spec}(\Gamma_R) = \{(-1)^{9|Z(R)|-4}, (2|Z(R)|-1)^4\}, \text{Ł}-spec(\Gamma_R) = \{0^4, (2|Z(R)|)^{8|Z(R)|-4}\}$$

and

$$\text{Q-Spec}(\Gamma_R) = \{(4|Z(R)|-2)^4, (2|Z(R)|-2)^{8|Z(R)|-4}\}.$$

Hence, R is super integral. □

We also have the following result.

Proposition 3.5. *For any prime p, a $(p+2)$-centralizer p-ring is super integral.*

Proof. If R is a finite $(p+2)$-centralizer p-ring then, by [7, Theorem 2.12], we have $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, the result follows from Theorem 2.2. □

Proposition 3.6. *Let R be a finite ring and p the smallest prime divisor of $|R|$. Then R is super integral if* $\Pr(R) = \frac{p^2+p-1}{p^3}$.

Proof. If $\Pr(R) = \frac{p^2+p-1}{p^3}$ then, by [17, Theorem 2], we have $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Hence, the result follows from Theorem 2.2. □

As a corollary to Proposition 3.6 we have the following result.

Corollary 3.7. *A finite ring R is super integral if* $\Pr(R) = \frac{5}{8}$.

We conclude this paper by computing various energies of the commuting graphs of a class of super integral rings.

Theorem 3.8. *Let p be a prime and R a finite ring. If $\frac{R}{Z(R)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then the energy, Laplacian energy and signless Laplacian energy of Γ_R are all equal to* $2(p^2-1)|Z(R)| - 2(p+1)$.

Proof. The energy $E(\Gamma_R)$, Laplacian energy $LE(\Gamma_R)$ and signless Laplacian energy $LE^+(\Gamma_R)$ of Γ_R are given by

$$E(\Gamma_R) = \sum_{\lambda \in \text{Spec}(\Gamma_R)} |\lambda|, \quad LE(\Gamma_R) = \sum_{\mu \in \text{Ł}-spec(\Gamma_R)} \left|\mu - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right|$$

and

$$LE^+(\Gamma_R) = \sum_{\nu \in \text{Q-Spec}(\Gamma_R)} \left|\nu - \frac{2|e(\Gamma_R)|}{|v(\Gamma_R)|}\right|,$$

where $v(\Gamma_R)$ and $e(\Gamma_R)$ denotes the set of vertices and edges of Γ_R respectively. Hence, the result follows from Theorem 2.2 noting that $|v(\Gamma_R)| = (p^2-1)|Z(R)|$ and $2|e(\Gamma_R)| = (p^2-1)|Z(R)|((p-1)|Z(R)|-1)$. □

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Rajat Kanti Nath,
Department of Mathematical Sciences,
Tezpur University,
Napaam-784028, Sonitpur, Assam, India.
E-mail address: `rajatkantinath@yahoo.com`