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Approximation of the Jensen Type Rational Functional Equation by a Fixed Point Technique

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ABSTRACT: In this study, we propose a new rational functional equation

$$J_r\left(\frac{x+y}{2}\right) = \frac{2J_r(x)J_r(y)}{J_r(x)+J_r(y)}$$

and obtain its solution. We also investigate its various stabilities by a fixed point method.

Key Words: Generalized Ulam-Hyers stability, Jensen functional equation, Reciprocal functional equation.

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1. Introduction

The vital question raised by Ulam [25] forms the basis for the theory of stability of functional equations. Hyers [9] vividly responded to the problem created by Ulam. The result presented by Hyers is called as direct method where an approximate solution near to the exact solution is obtained directly from the given function and this is the most influencing procedure to achieve the stability of various types of functional equations. Using this direct method, Th. M. Rassias [23] took a broad view of Hyers result and established a new outcome to confirm the validity of stability of Cauchy additive functional equation using sum of powers of norms as upper bound. A particular case of Th. M. Rassias theorem regarding the Ulam-Hyers stability of the additive mappings was proved by Aoki [1]. During 1982-1989, J. M. Rassias ([16], [17], [18]) provided a further generalization of the result of Hyers and established a theorem using weaker conditions bounded by product of powers of norms. A different type of the additive functional equations is introduced in [2]. Găvruta [8] generalized the theorem of Hyers by considering a general control function as function of variables. In 2008, Ravi et al. [19] investigated the stability of the new quadratic functional equation

$$Q(ku_1+u_2)+Q(ku_1-u_2) = 2Q(u_1+u_2)+2Q(u_1-u_2)+2\left(k^2-2\right)Q(u_1)-2Q(u_2)$$

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for any arbitrary and fixed real constant k with $k \neq 0$; $k \neq \pm 1$; $k \neq \pm\sqrt{2}$ by using mixed product-sum of powers of norms.

Jung [11] investigated the Hyers-Ulam Stability for Jensen equation on a restricted domain and he applied the result to the study of an interesting asymptotic property of additive mappings. The stability of functional inequalities associated with the Cauchy-Jensen additive functional equalities in non-Archimedean Banach spaces is studied in [13]. Several mathematicians have remarked interesting applications of the Hyers-Ulam-Rassias stability theory to various mathematical problems.

On the other hand, some fixed point methods are other tools to investigate the stability of functional equations. The approximate solution near to the exact solution can be attained from the fixed point alternative. In 1996, Isac and Th.M. Rassias [10] who were the

rst authors to provide applications of stability theorem of functional equations for the proof of new fixed point theorems with applications. Cadariu and Radu applied the fixed point method to investigate the Jensen and Cauchy additive functional equations in [6] and [7]. For more details about the stability of functional equations by using fixed point method, one can refer [3], [4] and [15].

A function $J : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $J(x) = cx + a$ is the most general solution of the Jensen's functional equation

$$J\left(\frac{x+y}{2}\right) = \frac{1}{2}[J(x) + J(y)] \tag{1.1}$$

where c and a are arbitrary constants [24]. Also, when $a = 0$, $J(x) = cx$ is the general solution of the additive functional equation $J(x + y) = J(x) + J(y)$. Ravi and Senthil Kumar [20] investigated the generalized Hyers-Ulam stability for the reciprocal functional equation

$$g(x + y) = \frac{g(x)g(y)}{g(x) + g(y)} \tag{1.2}$$

where $g : X \longrightarrow \mathbb{R}$ is a mapping with X as the space of non-zero real numbers. The reciprocal function $g(x) = \frac{c}{x}$ is a solution of the functional equation (1.2). The functional equation (1.2) holds good for the "Reciprocal formula" of any electric circuit with a couple of parallel resistors [22]. Jung [12] applied fixed point method to investigate the Hyers-Ulam stability for the reciprocal functional equation (1.2). Other forms of the reciprocal type functional equations can be found in [5] and [21].

Inspired by the Jensen's functional equation (1.1) and its general solution, in this paper, we propose a new rational functional equation of the form

$$J_r\left(\frac{x+y}{2}\right) = \frac{2J_r(x)J_r(y)}{J_r(x) + J_r(y)}. \tag{1.3}$$

We obtain the solution of the functional equation (1.3). Using the fixed point alternative, we prove our main result, the generalized Ulam-Hyers stability for the equation (1.3). We also investigate the Ulam-Hyers stability and Hyers-Ulam-Rassias stability for the equation (1.3). It is easy to arrive equation (1.3) from equation (1.2) by simple substitution. Note that the equations (1.3) and (1.2) are not equivalent.

2. Stability of the Jensen type rational functional equation

In this section, induced by the solution of the Jensen functional equation (1.1) in [24], we firstly attain the solution of functional equation (1.3). Then, we investigate the stability of functional equation (1.3) pertinent to Găvruta by using the fixed point alternative. We also extend the results associated with Hyers and Th. M. Rassias stabilities.

Theorem 2.1. *The function $J_r(x) = \frac{1}{cx+k}$ is a solution of the equation (1.3).*

Proof: Equation (1.3) can be written as

$$\frac{1}{J_r\left(\frac{x+y}{2}\right)} = \frac{1}{2}\left[\frac{1}{J_r(x)} + \frac{1}{J_r(y)}\right]. \tag{2.1}$$

Putting $y = 0$ in (2.1), we find

$$\frac{1}{J_r\left(\frac{x}{2}\right)} = \frac{1}{2}\left[\frac{1}{J_r(x)} + k\right] \tag{2.2}$$

where $J_r(0) = \frac{1}{k}$ with $k \neq 0$. Hence, the equation (2.2) can be written as

$$J_r\left(\frac{x}{2}\right) = \frac{2}{\frac{1}{J_r(x)} + k}. \tag{2.3}$$

Using (2.3) in (2.1), we have

$$\frac{J_r(x)J_y(y)}{J_r(x)+J_r(y)} = \frac{1}{\frac{1}{J_r(x+y)} + k}$$

or equivalently

$$\frac{1}{J_r(x+y)} = \frac{1}{J_r(x)} + \frac{1}{J_r(y)} - k. \tag{2.4}$$

Let $A(x) = \frac{1}{J_r(x)} - k$. Then

$$\begin{aligned} A(x+y) &= \frac{1}{J_r(x+y)} - k = \left[\frac{1}{J_r(x)} + \frac{1}{J_r(y)} - k\right] - k \\ &= \left[\frac{1}{J_r(x)} - k\right] + \left[\frac{1}{J_r(y)} - k\right] = A(x) + A(y) \end{aligned} \tag{2.5}$$

which is Cauchy additive functional equation, whose solution is $A(x) = cx$. Therefore, $J_r(x) = \frac{1}{cx+k}$. □

From the above theorem, we have the following definition:

Definition 2.1. *A mapping J_r is said to be Jensen type rational mapping if it satisfies the equation (1.3). Hence, the equation (1.3) is said to be Jensen type rational functional equation.*

The following theorem is very useful for proving our main results which is due to Margolis and Diaz [14].

Theorem 2.2. *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping (that is $d(J(x), J(y)) \leq Ld(y, x)$, for all $x, y \in X$ and a Lipschitz constant $L < 1$). Then for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(i) *$d\left(J^n x, J^{n+1} x\right) < \infty$ for all $n \geq n_0$;*

(ii) *the sequence $\{J^n x\}$ converges to a fixed point $y^\star$ of J;*

(iii) *$y^\star$ is the unique fixed point of J in the set*

$$Y = \{y \in X | d(J^{n_0} x, y) < \infty\};$$

(iv) *$d(y, y^\star) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In the following theorems, we assume that $y \neq -x$, for all $(x, y) \in \mathbb{R}^2$.

Theorem 2.3. *Suppose that the mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the condition $f(x)$ approches ∞ as $x \to 0$ and the inequality*

$$\left| \frac{1}{2} f\left(\frac{x+y}{2}\right) - \frac{1}{\frac{1}{f(x)} + \frac{1}{f(y)}} \right| \leq \phi(x, y) \tag{2.6}$$

for all $x, y \in \mathbb{R}$, where $\phi : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is a given function. If there exists $L < 1$ such that the mapping $x \mapsto \psi(x) = \phi(x, 0)$ has the property $\psi\left(\frac{x}{2}\right) \leq 2L\psi(x)$ for all $x \in \mathbb{R}$ and the mapping ϕ has the property

$$\lim_{n\to\infty} 2^{-n} \phi\left(2^{-n}x, 2^{-n}y\right) = 0, \quad \text{for all } x, y \in \mathbb{R}, \tag{2.7}$$

then there exists a unique Jensen type rational mapping $r : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$|r(x) - f(x)| \leq \frac{1}{1-L} \psi(x) \tag{2.8}$$

for all $x \in \mathbb{R}$.

Proof: Define a set S by $S = \{h : \mathbb{R} \longrightarrow \mathbb{R} | h \text{ is a function}\}$ and introduce the generalized metric d on S as follows:

$$d(g, h) = d_\psi(g, h) = \inf\{C \in \mathbb{R}^+ : |g(x) - h(x)| \leq C\psi(x), \text{ for all } x \in \mathbb{R}\}. \tag{2.9}$$

It is easy to see that (S, d) is complete. Define a mapping $\sigma : S \longrightarrow S$ by

$$\sigma h(x) = \frac{1}{2} h\left(\frac{x}{2}\right) \qquad (x \in X) \tag{2.10}$$

for all $h \in S$. We claim that σ is strictly contractive on S. Given $g, h \in S$, let $C_{gh} \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq C_{gh}$. Hence

$$\begin{aligned} d(g,h) < C_{gh} &\Longrightarrow |g(x) - h(x)| \leq C_{gh}\psi(x), \quad (\forall x \in \mathbb{R}) \\ &\Longrightarrow \left|\frac{1}{2}g\left(\frac{x}{2}\right) - \frac{1}{2}h\left(\frac{x}{2}\right)\right| \leq \frac{1}{2}C_{gh}\psi\left(\frac{x}{2}\right), \quad (\forall x \in \mathbb{R}) \\ &\Longrightarrow \left|\frac{1}{2}g\left(\frac{x}{2}\right) - \frac{1}{2}h\left(\frac{x}{2}\right)\right| \leq LC_{gh}\psi(x), \quad (\forall x \in \mathbb{R}) \\ &\Longrightarrow d(\sigma g, \sigma h) \leq LC_{gh}. \end{aligned}$$

Therefore, we see that$d(\sigma g, \sigma h) \leq Ld(g, h)$ for all $g, h \in S$, that is, σ is strictly contractive mapping of S, with the Lipschitz constant L. Now, replacing (x, y) by $(x, 0)$ in (2.6), we get

$$\left|\frac{1}{2}f\left(\frac{x}{2}\right) - f(x)\right| \leq \phi(x, 0) = \psi(x)$$

for all $x \in \mathbb{R}$. Hence (2.9) implies that $d(\sigma f, f) \leq 1$. So, by applying the fixed point alternative Theorem 2.2, there exists a function $r : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the following:

(I) r is a fixed point of σ, that is

$$r\left(\frac{x}{2}\right) = 2r(x) \tag{2.11}$$

for all $x \in \mathbb{R}$. The mapping r is the unique fixed point of σ in the set

$$\mu = \{g \in S : d(f, g) < \infty\}.$$

This implies that r is the unique mapping satisfying (2.11) such that there exists $C \in (0, \infty)$ satisfying

$$|r(x) - f(x)| \leq C\psi(x), \ \forall x \in \mathbb{R}.$$

(II) $d\left(\sigma^n f, r\right) \to 0$ as $n \to \infty$. Thus, we have

$$\lim_{n\to\infty} 2^{-n} f\left(2^{-n}x\right) = r(x) \tag{2.12}$$

for all $x \in \mathbb{R}$.

(III) $d(r, f) \leq \frac{1}{1-L} d(r, \sigma f)$, which implies $d(r, f) \leq \frac{1}{1-L}$.

So, the inequality (2.8) holds. On the other hand, from (2.6), (2.7) and (2.12), we have

$$\begin{aligned}\left|\frac{1}{2}r\left(\frac{x+y}{2}\right)-\frac{1}{\frac{1}{r(x)}+\frac{1}{r(y)}}\right| &= \lim_{n\to\infty} 2^{-n}\left|\frac{1}{2}f\left(\frac{2^{-n}x+2^{-n}y}{2}\right)-\frac{1}{\frac{1}{f(2^{-n}x)}+\frac{1}{f(2^{-n}y)}}\right| \\ &\leq \lim_{n\to\infty} 2^{-n}\phi\left(2^{-n}x,2^{-n}y\right)=0\end{aligned}$$

for all $x, y \in \mathbb{R}$. Therefore, r is a solution of the functional equation (1.3) and hence $r : \mathbb{R} \longrightarrow \mathbb{R}$ is a reciprocal mapping. Next, we show that r is the unique reciprocal mapping satisfying (1.3) and (2.8). Suppose, let $r' : \mathbb{R} \longrightarrow \mathbb{R}$ be another reciprocal function satisfying (1.3) and (2.8). Since r' is a fixed point of σ and $d(f, r') < \infty$, we have $r' \in S^\star = \{g \in S | d(f, g) < \infty\}$. From Theorem 2.2 (iii) and since both r and r' are fixed points of σ, we have $r = r'$. Therefore, r is unique which completes the proof of Theorem. □

In analogy with Theorem 2.3, we have the upcoming result. Since the proof is similar, we omit it.

Theorem 2.4. *Suppose that the mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the condition $f(0) = \infty$ and the inequality (2.6) for all $x, y \in \mathbb{R}$, where $\phi : \mathbb{R} \times \mathbb{R} \longrightarrow [0, \infty)$ is a given function. If there exists $L < 1$ such that the mapping $x \mapsto \psi(x) = 2\phi(2x, 0)$ has the property $\psi(2x) \leq \frac{1}{2}L\psi(x)$, for all $x \in \mathbb{R}$ and the mapping ϕ has the property*

$$\lim_{n\to\infty} 2^n\phi\left(2^n x, 2^n y\right) = 0, \quad \textit{for all } x, y \in \mathbb{R}, \tag{2.13}$$

then there exists a unique Jensen type rational mapping $r : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$|r(x) - f(x)| \leq \frac{1}{1-L}\psi(x) \tag{2.14}$$

for all $x \in \mathbb{R}$.

Here, we bring some corollaries regarding the stability of functional equation (1.3) which are direct consequences of Theorems 2.3 and 2.4.

Corollary 2.5. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a mapping for which there exists a constant ϵ (independent of x, y)> 0 such that the functional inequality*

$$\left|\frac{1}{2}f\left(\frac{x+y}{2}\right)-\frac{1}{\frac{1}{f(x)}+\frac{1}{f(y)}}\right| \leq \epsilon$$

holds for all $x, y \in \mathbb{R}$. Then there exists a unique Jensen type rational mapping $r : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the equation (1.3) and

$$|r(x) - f(x)| \leq 2\epsilon$$

for all $x \in \mathbb{R}$.

Proof: Taking $\phi(x,y) = \epsilon$, for all $x, y \in \mathbb{R}$ and choosing $L = \frac{1}{2}$ in Theorem 2.3, we obtain the desired result. $\square$

Corollary 2.6. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a mapping and let there exist real numbers $p \neq -1$ and $c_1 > 0$ such that*

$$\left| \frac{1}{2} f\left(\frac{x+y}{2}\right) - \frac{1}{\frac{1}{f(x)} + \frac{1}{f(y)}} \right| \leq c_1 \left(|x|^p + |y|^p\right)$$

for all $x, y \in \mathbb{R}$. Then, there exists a unique Jensen type rational mapping $r : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the equation (1.3) and

$$|r(x) - f(x)| \leq \begin{cases} \frac{2^{p+1}c_1}{2^{p+1}-1}|x|^p, & p > -1 \\ \frac{2^{p+1}c_1}{1-2^{p+1}}|x|^p, & p < -1 \end{cases}$$

for all $x \in \mathbb{R}$.

Proof: The result follows from Theorems 2.3 and 2.4 by taking

$$\phi(x,y) = c_1 \left(|x|^p + |y|^p\right)$$

for all $x, y \in \mathbb{R}$ and $L = 2^{-p-1}$ and $L = 2^{p+1}$ respectively. $\square$

Remark 2.7. If upper bound is taken as $\phi(x,y) = c_2|x|^p|y|^q$ and then letting $y = 0$ in the above corollary, one can find that the upper bound becomes 0 and it is trivial that the mapping f satisfies the Jensen type rational functional equation (1.3). Similarly, if upper bound is assumed as $\phi(x,y) = c_3\left(|x|^p| + y|^p + \left(|x|^{\frac{p}{2}}|y|^{\frac{p}{2}}\right)\right)$ in the above corollary, then the result is akin to Corollary 2.6. Hence we omit the above two results in this study.

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