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Iterated Function Systems: Transitivity and Minimality *

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abstract: We discuss iterated function systems generated by finitely many continuous self-maps on a compact metric space, with a focus on transitivity and minimality properties. More specifically, we are interested in topological transitivity, fiberwise transitivity, minimality and total minimality. A number of results that clarify the relations between topological transitivity and fiberwise transitivity are included. Furthermore, we generalize the notion of regular periodic decomposition for topologically transitive maps, introduced by John Banks [1], to iterated function systems. We will focus on the existence of periodic decompositions for topologically transitive iterated function systems. Finally, we show that each minimal abelian iterated function system consisting of homeomorphisms on a connected compact metric space is totally minimal.

Key Words: Iiterated function systems, Topological transitivity, Minimality, Total transitivity, Periodic decomposition.

Contents

1. Introduction

Transitivity and minimality have recently been the subject of considerable interest in topological dynamics [1,5,6,10,11,13,14]. The concept of transitivity could trace back to Birkhoff [2,3]. After that many articles dealt with such a topic. We continue this investigation in a context of iterated function systems. Iterated function systems are given by a (finite) collection of continuous maps on a metric space, that are composed for iterations. They have been studied extensively because of their role in the study of fractals [7,8]. This paper studies two non equivalent notions of dynamical transitivity and discusses the relations between these notions, in the context of iterated function systems on compact metric spaces.

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The other interesting issue we will be addressed to is the existence of a periodic decomposition. Regular periodic decompositions for topologically transitive maps introduced in [1]. In fact, in a regular periodic decomposition, one can decompose the domain of a topologically transitive map into finitely many regular closed pieces with nowhere dense overlap in such a way that these pieces map into one another in a periodic fashion. Here, we generalize this concept to iterated function systems (IFSs) and provide some relevant results. Then we deal with minimal iterated function systems and minimal sets of IFSs. In particular, we show that each minimal abelian IFS on a compact connected metric space is totally minimal.

In order to state the main results, first we recall some standard definitions about iterated function systems. Next a brief description of the results will be given.

1.1. Preliminaries

We start collecting some basic concepts on iterated function systems. Let X be a compact metric space and $\mathcal{F}$ be a finite family of maps on X. Write $\langle\mathcal{F}\rangle^+$ for the semigroup generated by the collection $\mathcal{F}$. Following [4,9], the action of the semigroup $\langle\mathcal{F}\rangle^+$ is called the *iterated function system* (or IFS) associated to $\mathcal{F}$. In the rest of this paper, IFS$(X;\mathcal{F})$ or IFS$(\mathcal{F})$ will stand for an iterated function system generated by $\mathcal{F}$. An iterated function system can be thought of as a finite collection of maps which can be applied successively in any order. Moreover, iterated function systems (IFSs) are a method of constructing fractals.

Throughout this paper, we assume that $(X;d)$ is a compact metric space with at least two distinct points and without any isolated point. Also, we assume that IFS$(\mathcal{F})$ is an iterated function system generated by a finite family $\mathcal{F}=\{f_1,\dots,f_k\}$ of continuous self-maps on a compact metric space X.

For the semigroup $\langle\mathcal{F}\rangle^+$ and $x\in M$ the *total forward orbit* of x is defined by

$$\mathcal{O}^+_{\mathcal{F}}(x)=\{h(x):h\in\langle\mathcal{F}\rangle^+\}.$$

In a similar way, if the generators f_i, $i=1,\dots,k$, are injective then one can define the *total backward orbit* of x by

$$\mathcal{O}^-_{\mathcal{F}}(x)=\{h^{-1}(x):h\in\langle\mathcal{F}\rangle^+\}.$$

Symbolic dynamic is a way to represent the elements of $\langle\mathcal{F}\rangle^+$. We take $\Sigma_k^+=\{1,\dots,k\}^{\mathbb{N}}$ endowed with the product topology.

For any sequence $\omega=(\omega_1\omega_2\dots\omega_n\dots)\in\Sigma_k^+$, take $f^0_\omega:=Id$ and

$$f^n_\omega:=f_{\omega_n}\circ f^{n-1}_\omega=f_{\omega_n}\circ\dots\circ f_{\omega_1}.$$

If $\omega=(\omega_1\omega_2\dots\omega_n\dots)$ is a sequence in Σ_k^+, then the corresponding ω-*fiberwise orbit* of a point x is a sequence $O^+_\omega(x)=\{f^n_\omega(x):n\in\mathbb{N}\}$.

A subset B is called a *forward invariant* (or *backward invariant*) set if $f(B)\subset B$ (or $f^{-1}(B)\subset B$) for all $f\in\langle\mathcal{F}\rangle^+$. We say that

(1) IFS$(\mathcal{F})$ is *topologically transitive* whenever U and V are two open subsets of X, there exists $h\in\langle\mathcal{F}\rangle^+$ such that $h(U)\cap V\neq\emptyset$;

(2) IFS($\mathcal{F}$) is *fiberwisely transitive* if it admits an fiberwise dense orbit;

(3) IFS($\mathcal{F}$) is *forward minimal* (or *backward minimal*) if any point has a dense total forward orbit (or total backward orbit).

Theorem 1.1. *Let* IFS($\mathcal{F}$) *be an iterated function system generated by finitely many homeomorphisms* $\{f_1, \dots, f_k\}$ *defined on a compact metric space* X*. Then the following statements hold:*

1) *If* IFS($\mathcal{F}$) *has a* ω*-fiberwise dense orbit, for some sequence* $\omega \in \Sigma_k^+$*, then the subset*
$$\{x \in X : Cl(\mathcal{O}^-_{\mathcal{F}}(\omega, x)) = Cl(\mathcal{O}^+_{\mathcal{F}}(\omega, x)) = X\}$$
is residual in X*. In particular, fiberwise transitivity implies topological transitivity; the converse is not true, in general.*

2) *If there exists* $h \in \langle \mathcal{F} \rangle^+$ *so that* h *admits a unique attracting fixed point* x *with dense total backward and total forward orbits then* IFS($\mathcal{F}$) *is fiberwisely transitive.*

In [12], the authors proved that if the mappings $f_i, i = 1, \dots, k$, preserve a finite measure with total support, then these two different notions of transitivity, topological transitivity and fiberwise transitivity, are equivalent. While we see that, in the non-conservative case, in general, the two properties are different. Theorem 1.1 above investigates their relationship.

Let $h \in \langle \mathcal{F} \rangle^+$. We say that the length of h is equal to j and denote it by $|h| = j$ if h is a composition of j elements of the generating set $\mathcal{F}$ of IFS($\mathcal{F}$). For each iterated function system IFS($\mathcal{F}$) with generating set $\mathcal{F}$ we denote IFS($\mathcal{F}^n$) an iterated function system generated by $\mathcal{F}^n = \{f_i^n : f_i \in \mathcal{F}\}$.

Definition 1.1. *We say that an iterated function* IFS($\mathcal{F}$) *is totally transitive (or totally minimal) if* IFS($\mathcal{F}^n$) *is transitive (or minimal) for all* $n \in \mathbb{N}$*.*

Let $\mathcal{K}(X)$ denote the set of non-empty closed subsets of X endowed with the Hausdorff metric topology. Then $\mathcal{K}(X)$ is also a complete metric space and it is compact whenever X is compact.
For an iterated function system IFS($\mathcal{F}$) we define the associated *Hutchinson operator* by
$$\mathcal{L} : \mathcal{K}(X) \to \mathcal{K}(X),\ K \mapsto \mathcal{L}(K) = \bigcup_{i=1}^{k} f_i(K). \tag{1.1}$$

Following [4], we say that a compact set A is a *self-similar set* whenever $\mathcal{L}(A) = A$. For each $n \geq 1$, take $\mathcal{L}^n = \mathcal{L} \circ \mathcal{L}^{n-1}$. Then a compact subset A is called *n-self-similar set* if $\mathcal{L}^n(A) = A$.

In the following, we generalize the notion of a regular periodic decomposition for a topologically transitive map, introduced by John Banks [1], to iterated function systems.

Definition 1.2. *A collection* $\mathcal{D} = \{X_0, X_1, \cdots, X_{n-1}\}$ *of closed subsets of* X *such that* $\mathcal{L}(X_i) \subseteq X_{i+1} \pmod{n}$ *for* $0 \leq i < n-1$ *is called a periodic orbit of sets for the Hutchinson operator* $\mathcal{L}$. *Clearly, the union of the* X_i *is a self-similar set.*

Definition 1.3. *We call a periodic orbit* $\mathcal{D} = \{X_0, X_1, \cdots, X_{n-1}\}$ *of closed subsets of* X *is a periodic decomposition of* $\mathrm{IFS}(\mathcal{F})$ *if* $\mathrm{int}(X_i) \cap \mathrm{int}(X_j) = \emptyset$ *whenever* $i \neq j$ *, and the union of the* X_i *is* X. *The number of sets in a decomposition will be called the length of the decomposition. A periodic decomposition is regular if all of its elements are regular closed. This means that for each* $0 \leq i \leq n-1$, $\mathrm{Cl}(\mathrm{int}X_i) = X_i$. *In particular,* $X_i \cap X_j$ *is nowhere dense whenever* $i \neq j$.

Definition 1.4. *Let* $\mathrm{IFS}(\mathcal{F})$ *be an iterated function system on a compact metric space* X *with generators* $\mathcal{F} = \{f_1, \ldots, f_k\}$. *We say that* $\mathrm{IFS}(\mathcal{F})$ *is an abelian IFS if* $f_i \circ f_j = f_j \circ f_i$, *for each* $i, j \in \{1, \ldots, k\}$.

The next theorem provides a necessary and sufficient condition for topological transitivity.

Theorem 1.2. *Let* $\mathcal{D} = \{X_0, X_1, \cdots, X_{n-1}\}$ *be a regular periodic decomposition for the iterated function system* $\mathrm{IFS}(\mathcal{F})$ *generated by a finite family* $\mathcal{F}$ *of open and continuous maps on a compact metric space* X. *Then* $\mathrm{IFS}(\mathcal{F})$ *is transitive if and only if* $\mathrm{IFS}(\mathcal{F}^n)$ *is transitive on each* X_i.

Theorem 1.3. *A minimal abelian iterated function system* $\mathrm{IFS}(\mathcal{F})$ *generated by finitely many homeomorphisms* $\mathcal{F} = \{f_1, \ldots, f_k\}$ *defined on a connected compact metric space* X *is totally minimal and hence it is totally transitive.*

The present paper is organized as follows. In Section 2, we discuss topological transitivity and fiberwise transitivity and clarify the relations between these two distinct notions. Moreover, the proof of Theorem 1.1 is also given in Section 2. In Section 3, we deal with the existence of regular periodic decompositions for topologically transitive iterated function systems, then we prove Theorem 1.2. In Section 4, we deal with minimal iterated function systems and minimal sets of IFSs and finally we prove Theorem 1.3. Finally, Section 5 of this paper outlines motivations and provides a discussion of the paper.

2. Transitivity of IFSs

Topological transitivity is a global characteristic of a dynamical system. This section will concentrate on transitivity properties of iterated function systems.

Throughout this section let $\mathcal{F} = \{f_1, \ldots, f_k\}$ be a finite family of homeomorphisms defined on a compact metric space X and take $\mathrm{IFS}(\mathcal{F})$ the iterated function system generated by $\mathcal{F}$. Assume that X has no any isolated point and write $\langle \mathcal{F} \rangle^+$ for the semigroup generated by $\mathcal{F}$.

Here, we restrict ourselves to topological transitivity and fiberwise transitivity of IFSs. The first problem is, broadly speaking, a question about the relation between these two concepts of transitivity in the context of IFSs.

In what follows, we will prove that fiberwise transitivity implies topological transitivity. However, the next example shows that these two properties are not equivalent.

Example 2.1. *Let X be a a second countable Baire space and $\mathcal{V} = \{V_j : j \in \mathbb{N}\}$ a countable basis of X. Let us take a countable collection $\mathcal{F} = \{f_j : j \in \mathbb{N}\}$ consisting of self continuous maps defined on X, $\langle \mathcal{F} \rangle^+$ the semigroup generated by $\mathcal{F}$ and a point $x \in X$ for which the following properties hold:*

(1) for each j, $f_j(x) \in V_j$;

(2) f_j has an attracting fixed point x_j in V_j, moreover, V_j is contained in the basin of attraction of x_j;

(3) for all $j \neq 1$, f_j has an attracting fixed point in V_1.

Obviously, the total forward orbit of x is dense in X but x has not a fiberwise dense orbit.

Notice that if IFS($\mathcal{F}$) is a topologically transitive IFS on a second countable Baire space X, then the set of points with a dense total forward orbit is a residual set, for instance see [5, Prop. 1]. In the next two results we generalize this observation to the fiberwisely transitive IFSs.

Lemma 2.1. *If* IFS($\mathcal{F}$) *has any point x such that for some $\omega \in \Sigma_k^+$, x has ω-fiberwise dense orbit, then the set of points with ω-fiberwise dense orbit is a residual subset of X.*

Proof: Suppose that there exist a point $x \in X$ and a sequence $\omega \in \Sigma_k^+$ such that the ω-fiberwise orbit of x, $\mathcal{O}^+_{\mathcal{F}}(\omega, x) = \{f^j_\omega(x) : j \in \mathbb{N}\}$, is dense in X. Let us take a countable basis $\mathcal{V} = \{V_j : j \in \mathbb{N}\}$ of X and we set

$$\mathcal{A}_n = \{z \in X : \mathcal{O}^+_{\mathcal{F}}(z) \text{ is } \frac{1}{n}\text{-dense}\}.$$

Since the fiberwise-orbit $\mathcal{O}^+_{\mathcal{F}}(\omega, x)$ of x is dense in X, hence there exist sequences $\{x_{j_l}\} \subset \mathcal{O}^+_{\mathcal{F}}(\omega, x)$ and $\{k_{n,l}\} \subset \mathbb{N}$, for which the following property holds: $x_{j_l} = f^{j_l}_\omega(x) \in V_l$ and the segment orbit $\{f^{j_l}_\omega(x), f^{j_l+1}_\omega(x), \dots f^{j_l+k_{n,l}}_\omega(x)\}$ is $\frac{1}{2n}$-dense. By continuity, there exist $r_{n,l} > 0$ such that $f^i_{\sigma^{j_l-1}\omega}(B_{r_{n,l}}(x_{j_l})) \subset B_{\frac{1}{2n}}(f^i_{\sigma^{j_l-1}\omega}(x_{j_l}))$ for all $0 \le i \le k_{n,l}$. This means that for each $y \in B_{r_{n,l}}(x_{j_l})$, the segment orbit

$$\{y, f_{\sigma^{j_l-1}\omega}(y), \dots, f^{k_{n,l}}_{\sigma^{j_l-1}\omega}(y)\}$$

is $\frac{1}{n}$-dense. Therefore, $\cup_{l\in\mathbb{N}} B_{r_{n,l}}(x_{j_l})$ is an open and dense subset. In particular, this proves that $\cap_{n\in\mathbb{N}}\cup_{l\in\mathbb{N}} B_{r_{n,l}}(x_{j_l})$ is a residual subset contained in $\cap_{n=1}^\infty \mathcal{A}_n$ which consists of the points with ω-fiberwise dense orbit. □

The next result relates forward and backward total orbits.

Proposition 2.2. *If there exists a point $x \in X$ with ω-fiberwise dense orbit, for some $\omega \in \Sigma_k^+$, then $\{y \in X : Cl(\mathcal{O}_{\mathcal{F}}^-(\omega, y)) = Cl(\mathcal{O}_{\mathcal{F}}^+(\omega, y)) = X\}$ is a residual subset of X.*

Proof: From the previous lemma, $\{y \in X : \mathrm{Cl}(\mathcal{O}_{\mathcal{F}}^+(\omega, y)) = X\}$ is a residual subset of X. By assumption, the set $\{x, f_{\omega_0}(x), \ldots, f_\omega^j(x), \ldots\}$ is dense in X. Let us take $x_l = f_\omega^l(x)$ and $\mathcal{A}_n = \{x \in X : \mathcal{O}_{\mathcal{F}}^-(x)$ is $\frac{1}{n}$-dense$\}$. Since $\mathcal{O}_{\mathcal{F}}^+(x, \omega)$ is dense, there exists a sequence $\{k_n\}$ of positive integers such that $\{x, \ldots, f_\omega^{k_n}(x)\}$ is $\frac{1}{2n}$-dense. As $\{x, \ldots, f_\omega^{k_n}(x)\} \subset \mathcal{O}^-(\omega, x_j)$ for all $j \geq k_n$, we get that $x_j \in \mathcal{A}_n$ for each $j \geq k_n$.
By continuity, one can find $r_j > 0$ such that $f_\omega^l(B_{r_j}(x)) \subset B_{\frac{1}{2n}}(f_\omega^l(x))$, for all $0 \leq l \leq k_j$. This proves that $\mathcal{O}_{\mathcal{F}}^-(\omega, y)$ is $\frac{1}{n}$-dense for all $y \in f_\omega^j(B_{r_j}(x))$ and $j \geq k_n$. As $f_\omega^j(B_{r_j}(x))$ is an open neighborhood of x_j and $\{x_j : j \geq k_n\}$ is dense, then $\cup_{j \geq k_n} f_\omega^j(B_{r_j}(x)) \subset \mathcal{A}_n$ is an open and dense subset of X. Therefore $\cap_{n\geq 1} \cup_{j \geq k_n} f_\omega^j(B_{r_j}(x)) \subset \cap_{n \geq 1}\mathcal{A}_n$ is a residual subset of X. Since $\cap_{n\geq 1}\mathcal{A}_n$ consists of points with dense ω-backward orbit, then

$$\{y \in X : \mathrm{Cl}(\mathcal{O}_{\mathcal{F}}^-(\omega, y)) = X\} \cap \{y \in X : \mathrm{Cl}(\mathcal{O}_{\mathcal{F}}^+(\omega, y)) = X\}$$

is also residual. $\square$

In particular, the next result can be followed immediately.

Corollary 2.3. *If* IFS($\mathcal{F}$) *is fiberwisely transitive, then it is topologically transitive.*

Lemma 2.4. *If* IFS($\mathcal{F}$) *is minimal, then it is fiberwisely transitive.*

Proof: Suppose that $\mathcal{V} = \{V_j : j \in \mathbb{N}\}$ is a countable basis of X and IFS($\mathcal{F}$) acts minimally on X. It is not hard to see that for each $x \in X$ and every open subset $V \subset X$, there exists $h \in \langle \mathcal{F} \rangle^+$ so that $h(x) \in V$. Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a finite word so that $h = f_\alpha = f_{\alpha_\ell} \circ \ldots \circ f_{\alpha_1}$. If we apply this conclusion for the countable basis $\mathcal{V}$, we will provide a sequence $h_i \in \langle \mathcal{F} \rangle^+$ so that $h_1(x) = x_1$, $h_i(x_i) = x_{i+1}$ and $x_i \in V_i$, for each $i \in \mathbb{N}$. Assume $\alpha^{(i)}$ is the finite word which satisfies $h_i = f_{\alpha^{(i)}}$. Then for sequence ω, that is the concatenation of $\alpha^{(i)}$s, ω-fiberwise orbit of x is dense in X. This fact ensures that every point x has a dense fiberwise orbit and hence IFS($\mathcal{F}$) is fiberwisely transitive. $\square$

As a consequence of the above observations we get the next result.

Corollary 2.5. *We have the following implications:*

minimality $\Rightarrow$ fiberwise transitivity $\Rightarrow$ topological transitivity.

The next result provides a sufficient condition for fiberwise transitivity.

Proposition 2.6. *If there exists $h \in \langle \mathcal{F} \rangle^+$ with a unique attracting fixed point x so that the total backward and total forward orbits of x are dense in X, then there exists a sequence $\omega \in \Sigma_k^+$ so that the ω-fiberwise orbit of x is dense in X. In particular,* IFS($\mathcal{F}$) *is fiberwisely transitive.*

Proof: Suppose that U and V are two open subsets of X. Since the total forward orbit of x is dense in X, there exist two mappings $k_1, g_1 \in \langle \mathcal{F} \rangle^+$ such that $g_1(x) \in U$ and $k_1(x) \in V$.
By density of $\mathcal{O}^-_{\mathcal{F}}(x)$, we can choose two sequences $\{z_i\} \subset U$ and $\{h_i\} \subset \langle \mathcal{F} \rangle^+$ so that $h_i(z_i) = x$ and z_i converges to $g_1(x)$ whenever i tends to infinity. On the other hand, by continuity of k_1, one can find a small real $r > 0$ so that $k_1(B_r(x)) \subset V$, where $B_r(x)$ is the ball with radius r and center x. Let B be an open neighborhood of x so that $h_i(g_1(x))$ is contained in B for some large enough i. Since x is an attracting fixed point of h, one can find $n \in \mathbb{N}$ such that $x \in h^n(B) \subset B_r(x)$, and hence $k_1(h^n(h_i(g_1(x)))) \in V$. Let us take $g_1(x) := x_1$ and $g_2 := k_1 \circ h^n \circ h_i$, then $g_1(x) \in U$ and $g_2(x_1) \in V$. We apply the above argument for a countable basis $\mathcal{V} = \{V_i : i \in \mathbb{N}\}$ of X. Then we will provide a sequence of mappings $g_i \in \langle \mathcal{F} \rangle^+$ enjoying the following properties: $g_1(x) = x_1$, $g_i(x_i) = x_{i+1}$ and $x_i \in V_i$, for each $i \in \mathbb{N}$. Let $\alpha_i = (\alpha_{1,i}, \ldots, \alpha_{n_i,i})$ be a finite word of the alphabets $\{1, \ldots, k\}$ so that $g_i = f_{\alpha_i} = f_{\alpha_{n_i,i}} \circ \ldots \circ f_{\alpha_{1,i}}$. If we take $\omega = (\alpha_1, \alpha_2, \ldots) \in \Sigma_k^+$ the concatenation of α_is then clearly the ω-fiberwise orbit of x is dense in X. $\square$

Now, Theorem 1.1 is a consequence of Propositions 2.2 and 2.6.

3. Periodic decompositions of IFSs

This section will concentrate on dynamical properties relative to a regular periodic decomposition in the context of iterated function systems. We follow the approach introduced by Banks in [1] to our setting. We will prove that each iterated function system IFS($\mathcal{F}$) with a regular periodic decomposition $\mathcal{D} = \{X_0, \ldots, X_{n-1}\}$ of length n is transitive if and only if IFS($\mathcal{F}^n$) is transitive on each X_i which establishes Theorem 1.2.

Throughout this section take $\mathcal{F} = \{f_1, \ldots, f_k\}$ a finite family of continuous and open self maps defined on a compact metric space X and let IFS($\mathcal{F}$) be the iterated function system generated by $\mathcal{F}$. Assume that X has no any isolated point and write $\langle \mathcal{F} \rangle^+$ for the semigroup generated by $\mathcal{F}$.

Lemma 3.1. *For the iterated function system* IFS($\mathcal{F}$) *let* $\mathcal{D} = \{X_0, X_1, \cdots, X_{n-1}\}$ *be a periodic orbit of subsets of* X *for the Hutchinson operator* $\mathcal{L}$. *Then for each* $i = 0, \ldots, n-1$, *and each* $h \in \langle \mathcal{F} \rangle^+$ *with* $|h| = j$, *one has* $h(X_i) \subseteq X_{i+j} \pmod n$ *for all* $j \geq 0$. *In particular,* X_i *is an* $\mathcal{F}^n$*-invariant.*

Proof: By induction for each $j \geq 0$, $\mathcal{L}^j(X_i) \subseteq X_{i+j}$, where $\mathcal{L}^j = \mathcal{L} \circ \mathcal{L}^{j-1}$. Let $h \in \langle \mathcal{F} \rangle^+$ with $|h| = j$. Then $h(X_i) \subseteq \mathcal{L}^j(X_i) \subseteq X_{i+j} \pmod n$. In particular, $\mathcal{L}^n(X_i) \subseteq X_i$. This finishes the proof of the lemma. $\square$

Remark 3.2. *Let* $\mathcal{L}$ *be the Hutchinson operator of an iterated function system* IFS($\mathcal{F}$). *Inductively, for each* $n \in \mathbb{N}$ *take* $\mathcal{L}^n = \mathcal{L} \circ \mathcal{L}^{n-1}$. *Then it is not hard to see that for each compact subset* $A \subseteq X$ *one has*

$$\mathcal{L}^n(A) = \bigcup_{h \in \langle \mathcal{F} \rangle^+, |h| = n} h(A).$$

In particular, a subset $A \subset X$ is $\mathcal{F}^n$-invariant iff it is $\mathcal{L}^n$-invariant.

Lemma 3.3. *Let* IFS($\mathcal{F}$) *be a transitive iterated function system generated by finitely many continuous and open self-maps $\mathcal{F} = \{f_1, \ldots, f_k\}$ on a compact metric space X. If $\mathcal{D} = \{X_0, X_1, \cdots, X_{n-1}\}$ is a regular periodic decomposition for* IFS($\mathcal{F}$), *then the following properties hold:*

1. *$\mathcal{L}^j(X_i) = X_{i+j} \pmod{n}$ for all $0 \le i \le n-1$ and $j \ge 1$, in particular, X_i is n-self-similar set;*
2. *for each $h \in \langle \mathcal{F} \rangle^+$ with $|h| = j$, $h(X_i) \subseteq X_i$ if and only if $j = 0 \pmod{n}$;*
3. *$h^{-1}(\mathrm{int}(X_i)) \subseteq \mathrm{int}(X_{i-j}) \pmod{n}$ for all $0 \le i \le n-1$, $h \in \langle \mathcal{F} \rangle^+$ with $|h| = j$ and $j \ge 0$;*
4. *$Y = \bigcup_{i \ne j} X_i \cap X_j$ is $\mathcal{F}$-invariant and nowhere dense in X.*

Proof: (i) Since $X_{i+j} \pmod{n}$ is a closed set and by the previous lemma, one has that $\mathrm{Cl}(\mathcal{L}^j(X_i)) \subseteq X_{i+j} \pmod{n}$. Suppose $U = X_{i+j} \pmod{n} \setminus \mathrm{Cl}(\mathcal{L}^j(X_i))$ is non-empty. Then U is open in the regular closed set $X_{i+j} \pmod{n}$ and hence $V = U \cap \mathrm{int}(X_{i+j} \pmod{n})$ is open and non-empty in X. Let us take $W = \mathrm{int}(X_i)$. Since X_i is a regular set so W is non-empty and open in X. By transitivity of IFS($\mathcal{F}$), there is a mapping $g \in \langle \mathcal{F} \rangle^+$ so that $g(\mathrm{int}(X_i)) \cap V = g(W) \cap V \ne \emptyset$.

By definition, $\mathrm{int}(X_i) \cap \mathrm{int}(X_j) = \emptyset$ whenever $i \ne j$. Also by Lemma 3.1, for each $h \in \langle \mathcal{F} \rangle^+$ with $|h| = j$, one has $h(\mathrm{int}(X_i)) \subseteq h(X_i) \subseteq X_{i+j} \pmod{n}$ for all $j \ge 0$. Now, these observations ensure that $|g| = j \pmod{n}$. Therefore,

$$g(W) = g(\mathrm{int}(X_i)) \subseteq g(X_i) \subseteq \mathcal{L}^j(X_i) \subseteq \mathrm{Cl}(\mathcal{L}^j(X_i).$$

Since $U \cap \mathrm{Cl}(\mathcal{L}^j(X_i) = \emptyset$, hence $g(W) \cap U = \emptyset$ and therefore $g(W) \cap V = \emptyset$ contracting the fact that IFS($\mathcal{F}$) is transitive.

(ii) Let $h \in \langle \mathcal{F} \rangle^+$ with $|h| = j$. Since the generators f_i, $i = 1, \ldots, k$, are open mappings hence h is also open. If $h(X_i) \subseteq X_i$ but $m = i + j \pmod{n} \ne i$, then since $h(X_i) \subseteq X_{i+j} \pmod{n} = X_m$, we have $h(X_i) \subseteq X_i \cap X_m$. But this implies that $h(X_i)$ is nowhere dense contracting the facts that X_i is a regular set and h is an open mapping. The other implication follows from the definition and the previous lemma.

(iii) We assume that $i \ge j$. The proof for $i < j$ is similar. Let $h \in \langle \mathcal{F} \rangle^+$ with $|h| = j$. If $h^{-1}(\mathrm{int}(X_i)) \cap X_m \ne \emptyset$ for some $m \ne i - j$, we would have a point of X_m that maps into $\mathrm{int}(X_i)$ under h. But $h(X_m) \cap \mathrm{int}(X_i) = \emptyset$ because $h(X_m) \ne X_i$. This contradiction establishes that $h^{-1}(\mathrm{int}(X_i)) \cap X_m = \emptyset$ for $m \ne i - j$ and hence $h^{-1}(\mathrm{int}(X_i)) \subseteq X_{i-j}$. Since $h^{-1}(\mathrm{int}(X_i))$ is open we have $h^{-1}(\mathrm{int}(X_i)) \subseteq \mathrm{int}(X_{i-j})$.

(iv) It suffices to show that Y is $\mathcal{F}$-invariant. If $i \ne j \pmod{n}$, then $i + 1 \ne j + 1 \pmod{n}$, hence $f_\ell(Y) \subseteq \bigcup_{i \ne j} f_\ell(X_i) \cap f_\ell(X_j) \subseteq \bigcup_{i \ne j} (X_{i+1} \pmod{n}) \cap (X_{j+1} \pmod{n})$ which ensures that Y is $\mathcal{F}$-invariant. $\square$

Now, we will complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Assume IFS($\mathcal{F}$) is transitive and let U and V be open sets intersecting X_i. Since X_i is regular closed, hence $\mathrm{int}(X_i) \cap U$ and $\mathrm{int}(X_i) \cap V$ are non-empty open subsets of X. By transitivity of IFS($\mathcal{F}$), there exists $h \in \langle \mathcal{F} \rangle^+$ such that $h(\mathrm{int}(X_i) \cap U) \cap \mathrm{int}(X_i) \cap V \neq \emptyset$, hence $h(\mathrm{int}(X_i)) \cap \mathrm{int}(X_i) \neq \emptyset$ and so $h^{-1}(\mathrm{int}(X_i)) \cap \mathrm{int}(X_i) \neq \emptyset$. Let $|h| = j$. By part *(iii)* of Lemma 3.3, $h^{-1}(\mathrm{int}(X_i)) \subseteq X_{i-j} \pmod n$. Since the interiors of X_i, $i = 0, \dots, n-1$, are disjoint, $i = i - j \pmod n$, hence j is a multiple of n. This establishes the transitivity of IFS($\mathcal{F}^n$) on X_i.

Conversely, assume IFS($\mathcal{F}^n$) is transitive on each X_i. Let U and V be open sets intersecting $\mathrm{int}(X_i)$ and $\mathrm{int}(X_j)$, respectively. Clearly, there exists $h \in \langle \mathcal{F} \rangle^+$ with $|h| = m$, for some $0 \leq m \leq n-1$, so that $W = h^{-1}(V \cap \mathrm{int}(X_j))$ is a non-empty open subset of X_i and hence there is $g \in \langle \mathcal{F}^n \rangle^+$ so that $g(U) \cap W \neq \emptyset$ by transitivity of IFS($\mathcal{F}^n$) on X_i. Therefore, $h \circ g(U) \cap V \neq \emptyset$. Thus, the proof of Theorem 2 is completed.

The next result is a consequence of Theorem 1.2.

Corollary 3.4. *If an iterated function system* IFS($\mathcal{F}$) *is totally transitive, then it has no regular periodic decompositions.*

The following result is evident.

Lemma 3.5. *Let* IFS($\mathcal{F}$) *be an iterated function system generated by finitely many open and continuous self maps $\mathcal{F} = \{f_1, \dots, f_k\}$ on a compact metric space X. Then, the followings are equivalent:*

1. IFS($\mathcal{F}$) *is topologically transitive;*
2. *any $\mathcal{F}$-invariant subset of X is either dense or nowhere dense;*
3. *every proper closed invariant subset of X is nowhere dense;*
4. *every backward invariant subset of X with non-empty interior is dense.*

The next result provides sufficient conditions for the existence of a regular periodic decompositions for transitive IFSs.

Proposition 3.6. *Let* IFS($\mathcal{F}$) *be an iterated function system generated by a finite family $\mathcal{F} = \{f_1, \dots, f_k\}$ of continuous and open self maps on a compact metric space X. Assume* IFS($\mathcal{F}$) *is a transitive IFS and G a non-empty, non-dense, open subset in X and forward $\mathcal{F}^n$-invariant. If $G_0 = G$, $\mathcal{L}(G_i) = G_{i+1}$, $0 \leq i \leq n-1$, and $\{G_i : 0 \leq i \leq n-1\}$ are pairwise disjoint, then $\{\mathrm{Cl}(G_0), \mathrm{Cl}(G_1), \dots, \mathrm{Cl}(G_{n-1})\}$ is a regular periodic decomposition for* IFS($\mathcal{F}$).

Proof: Consider an open set $G \subseteq X$ satisfies the hypothesis of the proposition. Take $G_0 = G$, $\mathcal{L}(G_i) = G_{i+1}$, $0 \leq i \leq n-1$, hence $\{G_0, G_1, \dots, G_{n-1}\}$ are pairwise disjoint. Let us take $X_0 = \mathrm{Cl}(G)$ and $X_i = \mathcal{L}^i(X_0)$ for $1 \leq i \leq n$. Clearly, X_0 is a closed regular set. Moreover, it is not hard to see that if f is a continuous open map on a compact metric space X and $A \subset X$ a closed regular set, then $f(A)$ is also a closed regular set. Also finite unions of regular closed sets are regular closed. These observations ensure that X_i, $0 \leq i \leq n$, are closed regular sets. Also, by hypothesis for each $h \in \langle \mathcal{F} \rangle^+$ with $|h| = n$, one has $h(G_0) \subseteq G_0$ which implies that $X_n \subseteq X_0$. It is enough to show that $X = X_0 \cup X_1 \cup \ldots \cup X_{n-1}$. Clearly $X_0 \cup X_1 \cup \ldots \cup X_{n-1}$ is a closed regular $\mathcal{F}$-invariant set with non-empty interior. Hence by part (ii) of Lemma 3.5 the subset $X_0 \cup X_1 \cup \ldots \cup X_{n-1}$ is dense in X and so it is equal to X. □

In the following we give an example of a transitive iterated function system which admits a regular periodic decomposition. However, it is neither total transitive nor minimal.

Example 3.1. *Following [1], consider the piecewise linear map $f_1 : [0,1] \to [0,1]$ defined by*

$$f_1(x) := \begin{cases} 2x + \frac{1}{2} & \text{if } x \in [0, \frac{1}{4}], \\ \frac{3}{2} - 2x & \text{if } x \in [\frac{1}{4}, \frac{3}{4}], \\ 2x - \frac{3}{2} & \text{if } x \in [\frac{3}{4}, 1] \end{cases} \tag{3.1}$$

and let $f_2 : [0,1] \to [0,1]$ be a linear map defined by $f_2(x) = 1-x$. Clearly, f_1 and f_2 are continuous and open mappings. Take $\mathcal{F} = \{f_1, f_2\}$ and let IFS($\mathcal{F}$) *be the iterated function system generated by $\mathcal{F}$. It is easy to see that $\mathcal{D} = \{[0, \frac{1}{2}], [\frac{1}{2}, 1]\}$ forms a regular periodic decomposition for* IFS($\mathcal{F}$)*, and hence, by Theorem 1.2,* IFS($\mathcal{F}^2$) *cannot be transitive on $[0,1]$. It is not hard to check that f_1 is topologically mixing [1] and hence, totally transitive on each element of $\mathcal{D}$. In particular, it is transitive on $[0,1]$. Notice that each iterated function system containing a transitive map is also transitive, thus* IFS($\mathcal{F}$) *is transitive. By these facts and Theorem 2,* IFS($\mathcal{F}^2$) *is transitive on each element of $\mathcal{D}$. Furthermore, since $x = \frac{1}{2}$ is a common fixed point for f_1 and f_2, hence* IFS($\mathcal{F}$) *is not minimal.*

The next example shows that total transitivity does not imply minimality.

Example 3.2. *Take the doubling map $f_1 : S^1 \to S^1$, $f_1(x) = 2x \ (mod\ 1)$, with the fixed point p. Clearly, f_1 is an expanding map, therefore it is topologically mixing and hence it is totally transitive. Also, take a map $f_2 : S^1 \to S^1$ with a fixed point at p. Consider the iterated function system* IFS($\mathcal{F}$) *generated by $\mathcal{F} = \{f_1, f_2\}$. Total transitivity of the mapping f_1 implies the total transitivity of* IFS($\mathcal{F}$)*. Since f_1 and f_2 have a common fixed point p hence* IFS($\mathcal{F}$) *is not minimal.*

4. Minimal properties of IFSs

In this section, we deal with minimal iterated function systems and minimal sets of IFSs. The following lemma is evident.

Lemma 4.1. *For an iterated function system* IFS($\mathcal{F}$) *on a compact metric space X the following conditions are equivalent.*

1. IFS($\mathcal{F}$) *is forward minimal.*

2. *If A is a closed forward invariant set, then either $A = \emptyset$; or $A = X$.*

3. *If $U \neq \emptyset$ is an open set, then $X = \bigcup_{h \in \langle \mathcal{F} \rangle^+} h^{-1}(U)$.*

Usually a dynamical system is not minimal. However, the basic and well-known fact due to Birkhoff is that any compact dynamical system has minimal subsystems. The next lemma establishes the same result for iterated function systems. The proof is straightforward. It is enough to apply Zorn's lemma to every totally ordered subset (or chain) of forward $\mathcal{F}$-invariant sets of X. Now the result can be followed by part (ii) of Lemma 4.1.

Lemma 4.2. *Let* IFS($\mathcal{F}$) *be an iterated function system on a compact metric space X. There exists a non-empty closed forward invariant set $Y \subset X$ so that $\langle \mathcal{F} \rangle^+$ acts minimally on Y.*

Notice that the problem of finding all minimal sets of the system is central in topological dynamics.

Theorem 4.3. *Let* IFS($\mathcal{F}$) *be an abelian iterated function system on a compact metric space X with generators $\mathcal{F} = \{f_1, \ldots, f_k\}$. If* IFS($\mathcal{F}$) *is forward minimal but* IFS($\mathcal{F}^n$) *is not, then there are pairwise disjoint compact subsets $X_i \subset X$, $i = 0, 1, \ldots, \ell - 1$, with $X = X_0 \cup X_1 \cup \ldots \cup X_{\ell-1}$ such that $\ell \geq 2$ is a divisor of n, $\mathcal{L}(X_i) = X_{i+1} \pmod{\ell}$, and $\langle \mathcal{F}^n \rangle^+$ acts forward minimally on X_i for each i, where $\mathcal{L}$ is the Hutchinson operator of* IFS($\mathcal{F}$).

Proof: Since IFS($\mathcal{F}$) is abelian hence $f_i \circ f_j = f_j \circ f_i$, for each $i, j \in \{1, \ldots, k\}$. By Lemma 4.2 IFS($\mathcal{F}^n$) admits a forward minimal set $X_0 \subset X$ and since IFS($\mathcal{F}^n$) is not minimal so X_0 is a proper subset of X. For each $i \in \mathbb{N}$ we set $\mathcal{L}^i(X_0) = X_i$, where $\mathcal{L}^i(X_0)$ is defined inductively by $\mathcal{L}^0(X_0) = X_0$, $\mathcal{L}^i(X_0) = \mathcal{L}(\mathcal{L}^{i-1}(X_0))$ and $\mathcal{L}(A) = \cup_{j=1}^k f_j(A)$ for each compact set $A \subset X$. Clearly, all the sets X_i are closed and non-empty. We prove by induction $f_\ell^n(X_i) \subset X_i$, for each $i \in \mathbb{N}$ and $\ell = 1, \ldots, k$. First, since X_0 is forward minimal for IFS($\mathcal{F}^n$) so it is forward $\mathcal{F}^n$-invariant and hence $f_\ell^n(X_0) \subset X_0$ for each $\ell = 1, \ldots, k$. Indeed, let $f_\ell^n(X_{i-1}) \subset X_{i-1}$. Then,

$$
\begin{aligned}
f_\ell^n(X_i) &= f_\ell^n(\mathcal{L}(X_{i-1})) = f_\ell^n(\bigcup_{j=1}^k f_j(X_{i-1})) \subset \bigcup_{j=1}^k f_\ell^n \circ f_j(X_{i-1}) \\
&= \bigcup_{j=1}^k f_j \circ f_\ell^n(X_{i-1}) \subset \bigcup_{j=1}^k f_j(X_{i-1}) = \mathcal{L}(X_{i-1}) = X_i.
\end{aligned}
$$

Hence, the subsets X_i, $i = 0, \ldots, n-1$, are forward $\mathcal{F}^n$-invariant. We claim that they are also forward minimal subsets of IFS($\mathcal{F}^n$).

Suppose that for some i, the subset X_i is not a minimal set of IFS($\mathcal{F}^n$). Since every closed forward $\mathcal{F}^n$-invariant non-empty subset of X contains a minimal set of IFS($\mathcal{F}^n$), hence there exists a subset Y of X_i, not equal to X_i, such that it is a forward minimal set of IFS($\mathcal{F}^n$). For some $j \in \mathbb{N}$, the number $i+j$ is divisible by n. We claim that $\mathcal{L}^j(Y) \subset X_0$ is non-empty, closed and forward $\mathcal{F}^n$-invariant. Indeed, by the above argument, Y is forward $\mathcal{F}^n$-invariant. Also, $Y \subset X_i$ which implies that $\mathcal{L}^j(Y) \subset \mathcal{L}^j(X_i) = X_{i+j} = X_0$. Now, since X_0 is a forward minimal set of IFS($\mathcal{F}^n$), hence $\mathcal{L}^j(Y) = X_0$. Since n divides $i+j$, we get $X_i = \mathcal{L}^i(X_0) = \mathcal{L}^{i+j}(Y) \subset Y$, which is a contradiction and the claim holds.

Notice that any two distinct forward minimal sets of an IFS are disjoint. Since X_n is not disjoint from X_0, we get that there exists $\ell \in \mathbb{N}$ such that $X_\ell = X_0$ and $X_i \cap X_0 = \emptyset$ for all $0 < i < \ell$. This ℓ has to divide n, since otherwise, we would have $X_\ell \subset X_i$ for some $0 < i < \ell$, hence X_n would be disjoint from X_0.

Now, the set $Z = X_0 \cup X_1 \cup \ldots \cup X_{\ell-1}$ is a closed non-empty subset of X. We claim the it is forward $\mathcal{F}$-invariant. In fact, for each $i = 0, \ldots, \ell-1$, and $m = 1, \ldots, k$, $f_m(X_i) \subset \cup_{j=1}^k f_j(X_i) = X_{i+1}$ which implies that $f_m(Z) \subset Z$. Thus, by minimality of IFS($\mathcal{F}$), the set Z is equal to X. This completes the proof. □

Since distinct minimal sets are pairwise disjoint, we get the next result by the previous theorem.

Corollary 4.4. *Let* IFS($\mathcal{F}$) *be an abelian iterated function system on a compact metric space X. If X is connected and* IFS($\mathcal{F}$) *is forward minimal then it is totally minimal.*

Now, the proof of Theorem 1.3 is established.

5. Conclusion and discussion

Transitivity forms part of a popular definition of chaos in discrete dynamical systems. Two different transitivity properties, topological transitivity and fiberwise transitivity, for iterated function systems are investigated. The relation between these transitivities is studied. Summing up the main results obtained in Section 2, we have the following implifications:

minimality ⇒ fiberwise transitivity ⇒ topological transitivity.

In general, topological transitivity is a weaker condition than fiberwise transitivity. Several conditions on spaces for topological transitivity to imply fiberwise transitivity are given.

It is known that the transitive systems are dynamically indecomposable. But they often admit a particular kind of decomposability in which their domains decompose into finitely many topologically non-trivial closed pieces which map into each other in a periodic fashion. The results of Section 3 give some information about regular periodic decompositions for transitive IFSs. As a consequence, it is shown that totally transitive iterated function systems do not have any regular periodic decomposition. Moreover, sufficient conditions for existence of a regular periodic decompositions for transitive IFSs are given.

In Section 4, minimal IFSs are studied. It is investigated that every minimal IFS on a compact and connected metric space is totally minimal. In this context, a natural question arises: *Under which conditions a non-abelian minimal IFS on a compact and connected metric space is totaly minimal*?

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