Bol. Soc. Paran. Mat. (3s.) **v. 38** 3 (2020): 79–96.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v38i3.36296

Nonresonance Conditions on the Potential for a Nonlinear Nonautonomous Neumann Problem *

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abstract: The aim of this paper is to establish the existence of the principal eigencurve of the $p-$Laplacian operator with the nonconstant weight subject to Neumann boundary conditions. We then study the nonresonce phenomena under the first eigenvalue and under the principal eigencurve, thus we obtain existence results for some nonautonomous Neumann elliptic problems involving the $p-$Laplacian operator.

Key Words: $p-$Laplacian, First eigenvalue, Principal eigencurve, Nonresonance, Neumann problem.

Contents

1. Introduction

In this paper we are concerned with the following class of problems

$$(\mathcal{P}_\alpha) \quad \begin{cases} -\Delta_p u = \alpha m_1(x)|u|^{p-2}u + m_2(x)g(u) + h(x) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $I\!R^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, $-\Delta_p u = -\text{div}\,(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator with $1 < p < \infty$, h is taken

* A. Sanhaji and A. Dakkak
2010 *Mathematics Subject Classification:* 47A16, 47D06, 47D03.
Submitted March 18, 2017. Published October 16, 2017

in $L^\infty(\Omega)$, $\alpha \in I\!R$, ν is the outward unit normal to $\partial\Omega$.
We assume that, m_1 and m_2 are two weight functions belongs to $M^+(\Omega)$ which satisfies the following conditions

$$(\mathbf{A1}) : m_1 \quad \text{changes sign on} \quad \Omega \quad \text{and} \int_\Omega m_1(x)dx < 0,$$

$$(\mathbf{A2}) : ess \inf_\Omega m_2 > 0,$$

where

$$M^+(\Omega) = \{m \in L^\infty(\Omega) : \text{meas}(\{x \in \Omega : m(x) > 0\}) \neq 0\}.$$

Furthermore, we suppose that g is a continuous function on $I\!R$ and satisfying

$$(\mathbf{H1}) : \limsup_{s\to\pm\infty} \frac{g(s)}{|s|^{p-2}s} \leq \beta,$$

$$(\mathbf{H2}) : \liminf_{s\to\pm\infty} \frac{G(s)}{|s|^p} < \beta,$$

where $G(s) = \int_0^s g(t)dt$ for all $s \in I\!R$ and $\beta \in I\!R$.

Since the 80's, several works have been devoted to questions of nonresonance for this kind of problem, in the semilinear and autonomous case ($p = 2$, $m_1 = 0$ and $m_2 = 1$) has been discussed by many authors (see e.g., [8], [9], [12], [13], [14],$\cdots$) in connection with various qualitative assumptions on the function g and its potential G. In the nonautonomous case A. Anane and A. Dakkak considered problem $(\mathcal{P}_\alpha)$ in the following particular case $\alpha = \lambda_1(m_1)$ and $m_2 = 1$, where λ_1 is the first eigenvalue of the p-Laplacian operator with weight and the Neumann boundary condition (see [6]), they showed the existence of the weak solution of the problem $(\mathcal{P}_\alpha)$ with conditions of nonresonance below the first eigenvalue of the $-\Delta_p$.

Motivated by the papers ([6], [7]) and some ideas introduced in ([6]), the goal of this work is to study the existence of solutions in the sense weak for problem $(\mathcal{P}_\alpha)$. In other words, assuming that **(A1)**, **(A2)**, **(H1)** and **(H2)** hold, our purpose is to show that problem $(\mathcal{P}_\alpha)$ has at least one solution that verifies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \int_\Omega (\alpha m_1(x) + m_2 g(u) + h(x))|u|^{p-2}uv dx,$$

for any $v \in W^{1,p}(\Omega)$ and for all $h \in L^\infty(\Omega)$.
The proof of the main result is based on the Leray-Schauder degree method and exploits some techniques introduced in [6].

The remaining part of this paper is organized as follows: In section 2, we recall some results which are necessary in what follows. In section 3, we show (see theorem 3.2) the existence of principal eigencurve of the p-Laplacian operator with

Neumann boundary conditions. In section 4, we show a theorem of nonresonance under the first eigenvalue (see theorem 4.4). Finally the results of sections 3 and 4 are then employed in section 5 in order to obtain the main result the existence of solution for problem $(\mathcal{P}_\alpha)$.

2. Preliminary

Throughout this paper, Ω will be a smooth bonded domain of $\mathbb{R}^N$, $W^{1,p}(\Omega)$ will denote the usual Sobolev space equipped with the norm $\|.\|_{1,p} = (\|.\|_p^p + \|\nabla(.)\|_p^p)^{\frac{1}{p}}$, where $\|.\|_p$ is the $L^p(\Omega)$-norm.
In this preliminary section we collect some results relative to usual nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda m(x)|u|^{p-2}u & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where $m \in M^+(\Omega)$.

Definition 2.1. *Let $u \in W^{1,p}(\Omega)$ and $\lambda \in \mathbb{R}$.*
1) (u, λ) is called a weak solution of problem (2.1) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \lambda \int_\Omega m(x)|u|^{p-2}uv dx, \quad \textit{for all} \quad v \in W^{1,p}(\Omega).$$

2) λ is called an eigenvalue of problem (2.1) if there exists $u \in W^{1,p}(\Omega) \setminus \{0\}$ such that (u, λ) is a solution of problem (2.1). In this case u is called an eigenfunction associated to λ.

The Lusternik-Schnirelman theory asserts that the spectrum of the p-Laplacian operator contains at least an unbounded sequence of positive eigenvalues, say

$$\lambda_1(m) < \lambda_2(m) \leq \lambda_3(m) \leq \ldots \leq \lambda_k(m) \to \infty \quad \text{as} \quad k \to +\infty$$

Unfortunately, to our best knowledge, nothing is known in general about the possible existence of other eigenvalues in $]\lambda_1(m), +\infty[$.

Clearly 0 is a principal eigenvalue of problem (2.1), with the constants as eigenfunctions. The search for another principal eigenvalue involves the following quantity:

$$\lambda_1(m) = \inf_{\mathcal{A}} \int_\Omega |\nabla u|^p dx,$$

where $\mathcal{A}=\{u \in W^{1,p}(\Omega); \int_\Omega m(x)|u|^p dx = 1\}$.

Proposition 2.2. *([6], [11]).* ***1)*** *If m changes sign on Ω and $\int_\Omega m dx < 0$. Then $\lambda_1(m) > 0$ and $\lambda_1(m)$ is the unique nonzero principal eigenvalue; this eigenvalue is simple and the corresponding eigenfunction u can be chosen such that $u(x) > 0$ in Ω, moreover $\lambda_1(m)$ is isolated, namely, there exist $b > \lambda_1(m)$ such that*

$\sigma(-\Delta_p)\bigcap]0,b[=\{\lambda_1\}$, where $\sigma(-\Delta_p)$ represents the set of all eigenvalues associated to the problem (2.1).

2) *If $\int_\Omega m dx > 0$. Then $\lambda_1(m) = 0$ and 0 is the unique nonnegative principal eigenvalue.*

3) *If $\int_\Omega m dx = 0$. Then $\lambda_1(m) = 0$ and 0 is the unique principal eigenvalue.*

Proposition 2.3. *([6]). Let $m, m^{'} \in M^+(\Omega)$. If $m \le m^{'}$, then $\lambda_1(m) \ge \lambda_1(m^{'})$. moreover, if $m < m^{'}$, meas$\{x \in \Omega; m < m^{'}\} \ne 0$ and $\int_\Omega m dx < 0$, then $\lambda_1(m) > \lambda_1(m^{'})$.*

Proposition 2.4. *([6]). **1)** $\lambda_1 : m \to \lambda_1(m)$ is continuous in $(M^+(\Omega), \|.\|_\infty)$*
2) *Let $(m_k)_k$ be a sequence in $M^+(\Omega)$ such that $m_k \to m$ in $L^\infty(\Omega)$. then $\lim_{k\to\infty} \lambda_1(m_k) = +\infty$ if and only if $m \le 0$ almost everywhere in Ω.*

3. Existence of the principal eigencurve of the $-\Delta_p$ with weights in the Neumann case

In this section, we study the following problem: Find all the real numbers α and β such that $\lambda_1(\alpha m_1 + \beta m_2) = 1$. Precisely, we seek to find all the pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that the following problem has at least one nontrivial solution $u \in W^{1,p}(\Omega)$

$$\begin{cases} -\Delta_p u = (\alpha m_1(x) + \beta m_2(x))|u|^{p-2}u & \text{in} \quad \Omega \\ \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

the set of pairs (α, β) has the structure of a continuous curve called the principal eigencurve of the $-\Delta_p$ with weights in the Neumann case.

Definition 3.1. *Let $m_1, m_2 \in M^+(\Omega)$. We define the graph of the first eigencurve of the p-Laplacian with weight subject to Neumann boundary conditions by:*

$$\mathcal{C}_1 = \left\{(\alpha, \beta) \in \mathbb{R}^2; \lambda_1(\alpha m_1 + \beta m_2) = 1\right\}.$$

Theorem 3.2. *Let m_1 and m_2 be two weight functions. Assume that $m_1, m_2 \in M^+(\Omega)$ and satisfying assumptions**(A1)** and **(A2)** respectively. Then for all $\alpha \in \mathbb{R}$ there exist a unique real number t_α which satisfies.*

$$\lambda_1(\alpha m_1 + t_\alpha m_2) = 1.$$

Proof: Let $\alpha \in \mathbb{R}$. We consider the function $f_\alpha : t \longmapsto \lambda_1(\alpha m_1 + t m_2)$. In view of propositon 2.3 and propsition 2.4, we affirm that f_α is decreasing continuous. It follows that f_α is injective. In order to show that the equation $f_\alpha(t) = 1$ has a solution we distinguish three cases.

The case $0 \leq \alpha \leq \lambda_1(m_1)$**.**
If $\alpha = 0$ we take $\beta = \lambda_1(m_2)$ and if $\alpha = \lambda_1(m_1)$ we take $\beta = 0$.

For all $0 < \alpha < \lambda_1(m_1)$ we have

$$f_\alpha(0) = \lambda_1(\alpha m_1) = \frac{\lambda_1(m_1)}{\alpha} > 1.$$

Also, since $\frac{\alpha}{t} m_1 + m_2 \to m_2$ in $L^\infty(\Omega)$ as $t \to +\infty$, then

$$\lim_{t\to+\infty} f_\alpha(t) = \lim_{t\to+\infty} \frac{1}{t} \lambda_1(\frac{\alpha}{t} m_1 + m_2) = 0$$

as $1 \in]0, \frac{\lambda_1(m_1)}{\alpha}[$, so there exist a unique real $t_\alpha \in]0, +\infty[$ which verifies $f_\alpha(t_\alpha) = 1$.

The case $\alpha > \lambda_1(m_1)$**.**
In this case $\alpha > 0$ and

$$0 < f_\alpha(0) = \lambda_1(\alpha m_1) = \frac{\lambda_1(m_1)}{\alpha} < 1 \tag{3.2}$$

Let $\gamma_\alpha = \frac{-\alpha \|m_1\|_\infty}{ess \inf_\Omega m_2}$,

$$A_\alpha = \{t < 0; \alpha m_1 + t m_2 \leq 0 \quad \text{a.e.} \quad x \in \Omega\} \quad \text{and} \quad \tau_\alpha = \sup A_\alpha$$

one can easily see that

$$\alpha m_1 + \gamma_\alpha m_2 \leq 0 \quad \text{almost everywher in } \Omega$$

Then $A_\alpha \neq \emptyset$, we wile prove that $\tau_\alpha \in A_\alpha$. Indeed, we first show that $\tau_\alpha < 0$. Since $f_\alpha(0) > 0$, and f_α, is a continuous function then there exist $\eta < 0$ such that $f_\alpha(t) > 0$ for all $t \in [\eta, 0]$. So $\lambda_1(\alpha m_1 + t m_2) > 0$ for all $t \in [\eta, 0]$; which gives that

$$\alpha m_1 + t m_2 \in M^+(\Omega) \quad \text{for all} \quad t \in [\eta, 0]$$

hence $\tau_\alpha \leq \eta < 0$. Moreover according to the definition of τ_α we have, for all $n \in I\!N$, there exist $t_n \in A_\alpha$ such that $\tau_\alpha - \frac{1}{n} < t_n$. It follows that

$$\alpha m_1(x) + \tau_\alpha m_2(x) \leq \alpha m_1(x) + t_n m_2(x) + \frac{1}{n} m_2(x) \leq \frac{1}{n} \|m_2\|_\infty \quad \text{a.e.} \quad x \in \Omega$$

and for all $n \in I\!N^*$.
Therefor, by letting n tends to $+\infty$, we conclude that

$$\alpha m_1(x) + \tau_\alpha m_2(x) \leq 0 \quad \text{a.e.} \quad x \in \Omega$$

Hence, $\tau_\alpha \in A_\alpha$. Then proposition 2.4 implies

$$\lim_{t\to\tau_\alpha} f_\alpha(t) = +\infty \tag{3.3}$$

It then follows from (3.2) and (3.3) that there exist a unique real $t_\alpha \in]\tau_\alpha, 0[$ which verifies

$$f_\alpha(t_\alpha) = 1$$

The case $\alpha < 0$.
In this case we have $\int_\Omega \alpha m_1 dx > 0$, thus $f_\alpha(0) = \lambda_1(\alpha m_1) = 0$.
We seek a real θ_α such that $\lim_{t\to\theta_\alpha} f_\alpha(t) = +\infty$.
Indeed, let $\mu_\alpha = \frac{\alpha\|m_1\|_\infty}{ess\inf_\Omega m_2}$,

$$B_\alpha = \{t < 0; \alpha m_1 + t m_2 \leq 0 \quad \text{a.e.} \quad x \in \Omega\} \quad \text{and} \quad \theta_\alpha = \sup B_\alpha$$

Clearly $\mu_\alpha \in B_\alpha$, so $B_\alpha \neq \emptyset$.
The rest of the proof can be carried out in a similar manner to that of the case 2.
□

4. Nonresonance under the first eigenvalue

This section is devoted to the study of the problem $(\mathcal{P}_\alpha)$ in the particular case where $(\alpha = \lambda_1(m_1))$ and the function g and its primitive G satisfying the following conditions

$$(\mathbf{g}^\pm) : \limsup_{s\to\pm\infty} \frac{g(s)}{|s|^{p-2}s} \leq 0,$$

$$(\mathbf{G}^\pm) : \liminf_{s\to\pm\infty} \frac{G(s)}{|s|^p} < 0.$$

That is to say, we show that the following problem admits at least one weak solution

$$(\mathcal{P}_{\lambda_1}) \quad \begin{cases} -\Delta_p u = \lambda_1 m_1(x)|u|^{p-2}u + m_2(x)g(u) + h(x) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where $\lambda_1 = \lambda_1(m_1)$ is the principal eigenvalue of $-\Delta_p$ with weight m_1.
The main result of this section lies in the ([6]). The improvement of this work is due to the insertion of a second weight m_2 in the right side of the problem $(\mathcal{P}_{\lambda_1})$. By using the theory of the Leray-Schauder degree, the hypotheses $(\mathbf{g}^\pm)$ and $(\mathbf{G}^\pm)$ are not sufficient to obtain the result of existence (see theorem4.4), for this we are compelled to treat the following possible cases.

4.1. Possible cases

. In order to study the problem $(\mathcal{P}_{\lambda_1})$ we will suggest four cases.

$$\text{case 1:} \quad (g^+_{-1}) : \liminf_{s\to+\infty} \frac{g(s)}{|s|^{p-2}s} < \delta \quad \text{and} \quad (g^-_{-1}) : \liminf_{s\to-\infty} \frac{g(s)}{|s|^{p-2}s} < \delta,$$

$$\text{case 2:} \quad (g^-_{-1}) : \liminf_{s\to-\infty} \frac{g(s)}{|s|^{p-2}s} < \delta \quad \text{and} \quad (g^+_0) : \liminf_{s\to+\infty} \frac{g(s)}{|s|^{p-2}s} \geq \delta,$$

$$\text{case 3:} \quad (g_{-1}^+): \liminf_{s\to+\infty} \frac{g(s)}{|s|^{p-2}s} < \delta \quad \text{and} \quad (g_0^-): \liminf_{s\to-\infty} \frac{g(s)}{|s|^{p-2}s} \geq \delta,$$

$$\text{case 4:} \quad (g_0^+): \liminf_{s\to+\infty} \frac{g(s)}{|s|^{p-2}s} \geq \delta \quad \text{and} \quad (g_0^-): \liminf_{s\to-\infty} \frac{g(s)}{|s|^{p-2}s} \geq \delta,$$

where $\delta = \frac{-\lambda_1 \|m_1^+\|_\infty}{\gamma}$, $\gamma = ess\inf_\Omega m_2$ and $m_1^+ = max(m_1, 0)$. There are all possible cases and by a classical method of lower and upper solutions (cf. [1], [12]) one can show that if the case 1 holds, then $(\mathcal{P}_{\lambda_1})$ is solvable for every $h \in L^\infty(\Omega)$, so it remains to consider only the other cases.

. Therefore we will keep the hypotheses (g_0^+) and (g_0^-) in order to be used in the next, and (g_{-1}^+), (g_{-1}^-) will be used in the proof of the following proposition and used also in the truncated function f_i which will be defined just after.

Proposition 4.1. ***i)*** *If g satisfy the hypothesis (g_{-1}^-) then, for any given $h \in L^\infty(\Omega)$ there exists $A = A_h < 0$ such that:*

$$\lambda_1 m_1(x)|A|^{p-2}A + m_2(x)g(A) + h(x) > 0 \quad a.e. \ in \quad \Omega. \tag{4.1}$$

ii) *If g satisfy the hypothesis (g_{-1}^+) then, for any given $h \in L^\infty(\Omega)$ there exists $B = B_h > 0$ such that:*

$$\lambda_1 m_1(x)|B|^{p-2}B + m_2(x)g(B) + h(x) < 0 \quad a.e. \ in \quad \Omega. \tag{4.2}$$

Proof: We only show the first assertion since the proof of the second one proceeds in the same way. According to the hypothesis (g_{-1}^-), let us fix $\varepsilon > 0$, such that

$$\liminf_{s\to-\infty} \frac{g(s)}{|s|^{p-2}s} < \frac{-\lambda_1\|m_1^+\|_\infty}{\gamma} - \varepsilon$$

Let $A_h < 0$ such that

$$-\varepsilon\gamma|A_h|^{p-2}A_h \geq \|h\|_\infty$$

then

$$\varepsilon\gamma|A_h|^{p-1} \geq \|h\|_\infty,$$

thus,

$$\varepsilon\gamma|A_h|^{p-1} + h(x) \geq 0 \quad \text{a.e. in} \quad \Omega$$

for this A_h, there exists $A < A_h$ such that

$$\varepsilon\gamma|A|^{p-1} + h(x) > \varepsilon\gamma|A_h|^{p-1} + h(x) \geq 0 \quad \text{a.e. in} \quad \Omega \tag{4.3}$$

and

$$\frac{g(A)}{|A|^{p-2}A} < \frac{-\lambda_1\|m_1^+\|_\infty}{\gamma} - \varepsilon. \tag{4.4}$$

Indeed, assume by contradiction, that for all $s < A_h$, we have

$$\frac{g(s)}{|s|^{p-2}s} \geq \frac{-\lambda_1\|m_1^+\|_\infty}{\gamma} - \varepsilon$$

then

$$\liminf_{s\to-\infty} \frac{g(s)}{|s|^{p-2}s} \geq \frac{-\lambda_1\|m_1^+\|_\infty}{\gamma} - \varepsilon$$

which gives a contradiction.
According to (4.4), we have

$$-m_2(x)\frac{g(A)}{|A|^{p-2}A} > \frac{m_2(x)}{\gamma}\lambda_1\|m_1^+\|_\infty + \varepsilon m_2(x) \quad \text{a.e. in} \quad \Omega$$

since $\gamma = ess\inf_\Omega m_2$, we get

$$(-\lambda_1 m_1(x) - m_2(x)\frac{g(A)}{|A|^{p-2}A})|A|^{p-1} + h(x) \quad > \quad (-\lambda_1 m_(x) + \lambda_1\|m_1^+\|_\infty)|A|^{p-1}$$
$$+\varepsilon\gamma|A|^{p-1} + h(x) \quad \text{a.e. in} \quad \Omega$$

It is easy to see that

$$-\lambda_1 m_(x) + \lambda_1\|m_1^+\|_\infty \geq 0 \quad \text{a.e. in} \quad \Omega$$

Therefore, using (4.3) we conclude that

$$\lambda_1\, m_1(x)|A|^{p-2}A + m_2(x)\, g(A) + h(x) > 0 \quad \text{a.e. in} \quad \Omega$$

This concludes the proof of (4.1).

□

4.2. Homotopic Problems.

Let $\theta < 0$ be fixed, and let $\mu \in [0,1]$ and consider for $i \in \{2,3,4\}$ the following problem

$$(\mathcal{P}_{i,\mu}) \quad \begin{cases} -\Delta_p u = (1-\mu)\theta|u|^{p-2}u + \mu f_i(x,u) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $f_i(.,.)$ is defined for every $s \in I\!R$ and a.e $x \in \Omega$ by

$$f_2(x,s) = f_4(x, T_A^+(s)), \tag{4.5}$$

$$f_3(x,s) = f_4(x, T_B^-(s)),$$

and

$$f_4(x,s) = \lambda_1 m_1(x)|s|^{p-2}s + m_2(x)g(s) + h(x),$$

where $T_A^+(s) = max(s,A)$, $T_B^-(s) = min(s,B)$, A and B comes from the Proposition 4.1.

Proposition 4.2. *If u is a solution of $(\mathcal{P}_{i,\mu})$ for $i = 2$ or $i = 3$, then we have*
1) *If $i = 2$, $u(x) \geq A$ a.e. $x \in \Omega$ and u is also a solution of $(\mathcal{P}_{4,\mu})$*
2) *If $i = 3$, $u(x) \leq B$ a.e. $x \in \Omega$ and u is also a solution of $(\mathcal{P}_{4,\mu})$,*
where A and B comes from the proposition 4.1.

Proof: 1) Since u is a solution of $(\mathcal{P}_{2,\mu})$, then

$$\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla(u-A)^- dx = \int_{\Omega^-} [(1-\mu)\theta|u|^{p-2}u + \mu f_2(x,u)](u-A)^- dx,$$

where $\Omega^- = \{x \in \Omega, u(x) \leq A\}$.
Using the fact that $u - A = (u-A)^+ - (u-A)^-$, we get

$$-\int_\Omega |\nabla(u-A)^-|^p dx = \int_{\Omega^-} [(1-\mu)\theta|u|^{p-2}u + \mu f_2(x,A)](u-A)^- dx, \tag{4.6}$$

since $A < 0$ and $\theta < 0$, it is easy to see that

$$(1-\mu)\theta|u|^{p-2}u(u-A)^- \geq 0 \quad \text{a.e.} \quad x \in \Omega. \tag{4.7}$$

By using (4.6), (4.7) and proposition 4.1, we obtain

$$-\int_\Omega |\nabla(u-A)^-|^p dx \geq 0,$$

thus, $\nabla(u-A)^- = 0$ which means $(u-A)^- = C$, where C is a constant real. If $C \neq 0$, then C is positive and $u(x) = A - C$ a.e. $x \in \Omega$, according to the fact that u is a solution of $(\mathcal{P}_{2,\mu})$ we get

$$0 = \Delta_p u = (1-\mu)\theta|A-C|^{p-2}(A-C) + \mu f_2(x,A) > 0,$$

which gives a contradiction, so $(u-A)^- \equiv 0$. This completes the proof.
For **2)**, using similar arguments as in the proof of **1)**. □

Corollary 4.3. ***1)*** *If g satisfy (g^-_{-1}) and u is a solution of $(\mathcal{P}_{2,\mu})$, then u is also a solution of $(\mathcal{P}_{\lambda_1})$.*
2) *If g satisfy (g^+_{-1}) and u is a solution of $(\mathcal{P}_{3,\mu})$, then u is also a solution of $(\mathcal{P}_{\lambda_1})$.*

We are now in position to give the following result.

4.3. Main result

Theorem 4.4. *Let $m_1, m_2 \in M^+(\Omega)$. Assume that the weight m_1 and m_2 satisfy $(\boldsymbol{A1})$ and $(\boldsymbol{A2})$ respectively and the assumptions $(\boldsymbol{g}^\pm)$ and $(\boldsymbol{G}^\pm)$ hold. Then the problem $(\mathcal{P}_{\lambda_1})$ has at least one nontrivial weak solution $u \in W^{1,p}(\Omega)$ for any given $h \in L^\infty(\Omega)$.*

The proof needs some technical lemmas, the two next lemmas concern an a-priori estimates on the possible solutions of the homotopic problem $(\mathcal{P}_{i,\mu})$.

Lemma 4.5. *We assume that g satisfy ($\boldsymbol{g}^{\pm}$) and the hypotheses of the case i, where $i \in \{2,3,4\}$. Let be (u_n, μ_n) be a sequence of solutions of $(\mathcal{P}_{i,\mu_n})$, then we have*
1) *$(u_n)_n$ is a sequence of $L^\infty(\Omega)$.*
2) *If $\|u_n\|_\infty \to +\infty$ when $n \to \infty$. Then, for a subsequence $v_n = \frac{u_n}{\|u_n\|_\infty} \to v$ strongly in $\mathcal{C}^1(\overline{\Omega})$, where $v = +\varphi$ if $i = 2$, $v = -\varphi$ if $i = 3$, $v = \pm\varphi$ if $i = 4$ and φ is a normed positive eigenfunction associated to the first eigenvalue λ_1 of $-\Delta_p$ with weight m_1. Moreover, we have*

$$\int_\Omega \frac{|g(u_n)|}{\|u_n\|_\infty^{p-1}} dx \to 0 \quad \textit{when} \quad n \to +\infty. \tag{4.8}$$

Proof: 1) From the Anane's L^∞-estimation [2] and the Tolksdorf's regularity [15] we can see that $(u_n)_n \subset \mathcal{C}^{1,\alpha}(\overline{\Omega})$, since the embedding $\mathcal{C}^{1,\alpha}(\overline{\Omega}) \hookrightarrow L^\infty(\Omega)$ is continuous for some $\alpha \in]0,1[$ independent on n, furthermore $v_n = \frac{u_n}{\|u_n\|_\infty}$ remains a bounded sequence in $\mathcal{C}^{1,\alpha}(\overline{\Omega})$.
2) By using the following compact embedding $\mathcal{C}^{1,\alpha}(\overline{\Omega}) \hookrightarrow\hookrightarrow \mathcal{C}^1(\overline{\Omega})$, then there exists a subsequence still denoted $(v_n)_n$ such that

$$v_n \to v \quad \text{stongly in} \quad \mathcal{C}^1(\overline{\Omega}) \quad \text{and} \quad \|v\|_\infty = 1. \tag{4.9}$$

Let use assume for instance that the case $i = 2$ holds, so by combining (g_{-1}^-) and (g_1^+), we can write

$$g(s) = q(s)|s|^{p-2}s + r(s) \quad \text{for every} \quad s \in [A, +\infty[, \tag{4.10}$$

where A comes from proposition 4.1, q and r are two continuous functions on $[A, +\infty[$ satisfying

$$-\lambda_1 \|m_1^+\|_\infty \le q(s) \le 0 \quad \text{for every} \quad s \in [A, +\infty[,$$

and

$$\frac{r(s)}{|s|^{p-2}s} \to 0 \quad \text{uniformly, when} \quad s \to +\infty. \tag{4.11}$$

Since u_n is a solution of $(\mathcal{P}_{2,\mu_n})$, we get

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla w dx = \int_\Omega [(1-\mu_n)\theta |u_n|^{p-2}u_n + \mu_n f_2(x, u_n)] w dx \tag{4.12}$$

for all $w \in W^{1,p}(\Omega)$.
According to the proposition 4.2, we have $u_n \ge A$ a.e. in Ω, so by using (4.4) and (4.10) it is easy to see that

$$f_2(x, u_n) = (\lambda_1 m_1(x) + m_2(x) q(u_n)) |u_n|^{p-2} u_n + m_2(x) r(u_n) + h(x). \tag{4.13}$$

On the other hand, since $(u_n)_n \subset L^\infty(\Omega)$ and q is a continuous function, it follows that $q(u_n)$ is bonded in $L^\infty(\Omega)$, then for a subsequence we get

$$q(u_n) \rightharpoonup q_0 \quad \text{in} \quad L^\infty(\Omega) \quad \text{weak} - *,$$

where $-\lambda_1\|m_1^+\|_\infty \leq q_0(x) \leq 0$ a.e. in Ω
and from (4.11), we have for a subsequence

$$\frac{|r(u_n)|}{\|u_n\|_\infty^{p-1}} \to 0 \quad \text{strongly in} \quad L^\infty(\Omega).$$

We can also suppose that for a further subsequence, $\mu_n \to \mu \in [0,1]$.
By combining (4.12), (4.13) and dividing by $\|u_n\|_\infty^{p-1}$ and passing to the limit as $n \to \infty$, we get

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla w dx = \int_\Omega m_{\mu,\theta,q_0}(x)|v|^{p-2}vwdx \quad \text{for all} \quad w \in W^{1,p}(\Omega), \quad (4.14)$$

where v is given by (4.9) and $m_{\mu,\theta,q_0}(x) = (1-\mu)\theta + \mu(\lambda_1 m_1(x) + m_2(x)q_0(x))$.
First of all we remark that $\mu \neq 0$, because if $\mu = 0$, testing (4.14) against $w = v$, we get $\int_\Omega |\nabla v|^p dx = \theta \int_\Omega |v|^p dx < 0$, which gives a contradiction.
Now let us prove

$$meas(\{x \in \Omega, m_{\mu,\theta,q_0}(x) > 0\}) \neq 0. \quad (4.15)$$

Indeed, arguing by contradiction, taking v as test function in (4.14), we get

$$\int_\Omega |\nabla v|^p dx = \int_\Omega m_{\mu,\theta,q_0}(x)|v|^p dx.$$

Since $m_{\mu,\theta,q_0}(x) \leq 0$ a.e. in Ω, then $\int_\Omega |\nabla v|^p dx = 0$ this implies that

$$m_{\mu,\theta,q_0}(x)|v|^p = 0,$$

and the fact that $v \neq 0$, assures that $m_{\mu,\theta,q_0} = 0$.
On the other hand, it is easy to see that $m_{\mu,\theta,q_0}(x) \leq \mu\lambda_1 m_1(x) \leq \lambda_1 m_1(x)$.
Then

$$0 = \int_\Omega m_{\mu,\theta,q_0}(x)dx \leq \lambda_1 \int_\Omega m_1(x)dx.$$

consequently $\int_\Omega m_1(x)dx \geq 0$, which contradicts $\int_\Omega m_1(x)dx < 0$.

According to (4.14) and (4.15) we can see that 1 is an eigenvalue of $p-$Laplacain with weight m_{μ,θ,q_0}, then

$$\frac{1}{\mu} = \lambda_1(\mu\lambda_1 m_1) \leq \lambda_1(m_{\mu,\theta,q_0}) \leq 1. \quad (4.16)$$

Since $\mu \in [0,1]$, then from (4.16) necessarily $\mu = 1$. By using the strict monotony property of λ_1 with respect to the weight and the definition of q_0, we deduce

$$q_0 \equiv 0 \quad \text{and} \quad m_{\mu,\theta,q_0} \equiv \lambda_1 m_1.$$

Thus, by virtue of (4.14) and the simplicity of λ_1, we get $v = \pm\varphi$. Finally, since $u_n \geq A$, then it is clear to see that $v \geq 0$ and consequently $v = \varphi$.

The other cases follows directly by the same proceedings.
According to (4.10) and (4.11), we get

$$\int_{\Omega}\frac{|m_2(x)g(u_n)|}{\|u_n\|_\infty^{p-1}}dx \leq \|m_2\|_\infty \int_{\Omega} -q(u_n)|v_n|^{p-1} + \|m_2\|_\infty \int_{\Omega}\frac{|r(u_n)|}{\|u_n\|_\infty^{p-1}}dx$$

by passage to the limit in the above inequality, we find (4.8). □

Lemma 4.6. *Let u_n be a solution of $(\mathcal{P}_{i,\mu_n})$ for some $i \in \{2,3,4\}$ and for all n, such that $\|u_n\|_\infty \to \infty$ when $n \to +\infty$, and let us fix $a \in \Omega$ and $\eta > 0$ such that $B(a,\eta) \subset \Omega$. So, if g satisfy $(\boldsymbol{g}^{\pm})$ and the case i holds, then by putting $\sigma_x(t) = a + t(x-a)$, we have*

$$\lim_{n\to\infty}\int_0^1 \frac{|g(u_n(\sigma_x(t))||\nabla u_n(\sigma_x(t))||x-a|}{\| u_n \|_\infty{}^p}dt = 0 \quad a.e.\ x \in \partial B(a,\eta), \tag{4.17}$$

where $B(a,\eta)$ is the ball of a center and radius η.

Proof: For simplicity of the task we prove the lemma only in the case $i = 2$ and other cases can be treated in a similar way.
Using relation (4.8), we deduce that

$$\int_{B(a,\eta)}\frac{|m_2(x)g(u_n)|}{\|u_n\|_\infty^{p-1}}dx \leq \int_{\Omega}\frac{|m_2(x)g(u_n)|}{\|u_n\|_\infty^{p-1}}dx \to 0 \quad \text{when} \quad n\to\infty.$$

By using the spherical coordinates, we obtain

$$\lim_{n\to\infty}\int_{[0,\pi]^n}\int_0^{2\pi}\int_0^{\eta} t^{N-1}\frac{|g(u_n(a+t\omega))|}{\| u_n \|_\infty{}^{p-1}}\prod_{j=1}^{N-2}(\sin\theta_j)^{N-1-j}d\theta_j d\theta_{N-1}dt = 0 \tag{4.18}$$

where, $\omega = \frac{x-a}{\eta} \in \partial B(0,1)$.
The above equality imply

$$\frac{|g(u_n(\sigma_x(\tau)))|}{\|u_n\|_\infty^{p-1}} \to 0 \quad \text{when} \quad n\to\infty \quad \text{a.e.} \quad x\in\partial B(a,\eta) \quad \text{and a.e.} \quad \tau\in[0,1].$$

By using $(\mathbf{g}^{\pm})$ we can see that g satisfy the following growth condition:

$$|g(s)| \leq a|s|^{p-1} + b, \quad \text{for some positive reals} \quad a,b \quad \text{and for all} \quad s\in[A,+\infty[.$$

According to the proposition 4.2, we obtain $u_n \geq A$, Consequently $\left(\frac{|g(u_n(\sigma_x(.)))|}{\|u_n\|_\infty^{p-1}}\right)_n$ and $\left(\frac{|\nabla(u_n(\sigma_x(.)))|}{\|u_n\|_\infty}\right)_n$ are bounded in $L^\infty([0,1])$.
By using the Lebesgue dominated convergence theorem, we conclude this proof. □

Lemma 4.7. *Let $r \in]0,1[$, and assume that there exists $d < 0$ such that*

$$d < \liminf_{s\to+\infty} \frac{G(s)}{|s|^p} \leq \limsup_{s\to+\infty} \frac{G(s)}{|s|^p} \leq 0, \tag{4.19}$$

then there is an equivalence between $(\boldsymbol{G}^+)$ and $(\boldsymbol{G}_r^+)$, where

$$(\boldsymbol{G}^+) : \liminf_{s\to+\infty} \frac{G(s)}{|s|^p} < 0 \quad ; \quad (\boldsymbol{G}_r^+) : \liminf_{s\to+\infty} \frac{G(s) - G(rs)}{|s|^p} < 0.$$

The same conclusion holds if we replace the sign $+$ by the sign $-$.

Proof: Assume that $(\mathbf{G}^+)$ hold, then there exists a sequence $(s_n)_n$ with

$$\lim_{n\to\infty} s_n = +\infty,$$

such that

$$\lim_{n\to\infty} \frac{G(s_n)}{|s_n|^p} = \liminf_{s\to+\infty} \frac{G(s)}{|s|^p} = l < 0.$$

According to (4.19), we get $d < l < 0$ and $(\frac{G(rs_n)}{|rs_n|^p})_n$ is bounded, so there exists $k \in [l, 0]$ such that $\lim_{n\to\infty} \frac{G(rs_n)}{|rs_n|^p} = k$ for some subsequence, thus we get

$$\liminf_{s\to+\infty} \frac{G(s) - G(rs)}{|s|^p} \leq \lim_{n\to\infty} \frac{G(s_n) - G(rs_n)}{|s_n|^p} = l - r^p k \leq l(1 - r^p) < 0.$$

Reciprocally, let us assume that (G_r^+), then there exists a sequence $(\widetilde{s}_n)_n$ with

$$\lim_{n\to\infty} \widetilde{s}_n = +\infty$$

such that

$$\lim_{n\to\infty} \frac{G(\widetilde{s}_n) - G(r\widetilde{s}_n)}{|\widetilde{s}_n|^p} = \liminf_{s\to+\infty} \frac{G(s) - G(rs)}{|s|^p} = l_r < 0,$$

it is easy to see that $(\frac{G(\widetilde{s}_n)}{\widetilde{s}_n^p})_n$ and $(\frac{G(r\widetilde{s}_n)}{r\widetilde{s}_n^p})_n$ are bounded, and there exists $k, k' \in]d, 0]$ such that $\lim_{n\to\infty} \frac{G(\widetilde{s}_n)}{|\widetilde{s}_n|^p} = k$ and $\lim_{n\to\infty} \frac{G(r\widetilde{s}_n)}{r|\widetilde{s}_n|^p} = k'$ for some subsequence, then we obtain

$$k - k' r^p = l_r < 0,$$

so, we have $k \neq 0$ or $k' \neq 0$, and consequently

$$\liminf_{s\to+\infty} \frac{G(s)}{|s|^p} \leq \min\{k, k'\}.$$

□

4.4. Proof of main result

Let us fix $h \in L^\infty(\Omega)$, so we will distinguish three cases (i.e. 2, 3 and 4), thus for any fixed $i \in \{2,3,4\}$ we will assume that the case i holds and we will show the existence of solutions of $(\mathcal{P}_{i,\mu})$, so according to the corollary 4.3, we can deduce the existence of solution of $(\mathcal{P}_{\lambda_1})$. Our proof consists in building in $\mathcal{C}(\overline{\Omega})$ an open bounded set $\mathcal{O}$, with $0 \in \mathcal{O}$ such that no solution of $(\mathcal{P}_{i,\mu})$ with $\mu \in [0,1[$ occurs on the boundary $\partial\mathcal{O}$. Homotopy invariance of the degree then yields the conclusion of the Theorem 4.4. This open set $\mathcal{O}$ will have the form

$$\mathcal{O} = \mathcal{O}_{S,T} = \{u \in \mathcal{C}(\overline{\Omega}); \quad T < u < S\}, \tag{4.20}$$

where, S and T satisfy $T < 0 < S$.
In order to simplify this proof we will assume that the case 2 holds, since the proof with the other cases is hardly the same. We take $T = 2A$, where $A < 0$ with A coming from proposition 4.1. By using the hypothesis $(\mathbf{G}^+)$ and lemma 4.7, we can get the existence of a sequence of positive real numbers $(s_n)_n$ which satisfy

$$\lim_{n\to\infty} s_n = +\infty$$

and,

$$\lim_{n\to\infty} \frac{G(s_n) - G(rs_n)}{s_n^p} = \liminf_{s\to+\infty} \frac{G(s) - G(rs)}{|s|^p} < 0, \tag{4.21}$$

where, $r = \frac{min\varphi}{max\varphi}$ and φ coming from lemma 4.5.
The proof is carried out by contradiction. precisely, one assumes the existence of a sequence $(u_n)_n$ of solutions to $(\mathcal{P}_{i,\mu_n})$, with $\mu_n \in [0,1]$ and $u_n \in \partial\mathcal{O}_{S_n,T}$.
then by proposition 4.2, we get $u_n \geq A > 2A$.
It follows that

$$\max(u_n) = s_n.$$

Let $x_n, y_n \in \overline{\Omega}$ such that $\max_{\overline{\Omega}}(u_n) = u_n(x_n)$ and $\min_{\overline{\Omega}}(u_n) = u_n(y_n)$, we clearly can suppose that $x_n \to x_0$ in $\overline{\Omega}$ and $y_n \to y_0$ in $\overline{\Omega}$, where x_0 and y_0 are two points where φ attains its maximum and minimum respectively.
By character $\mathcal{C}^1$ of Ω, we obtain the existence of a sequence $(z_k)_{k=1,\ldots,m} \subset \Omega$ such that

$$\bigcup_{k=1}^{m} B(z_k, |z_k - z_{k-1}|) \subset \Omega \quad \text{where,} \quad z_0 = x_0 \quad \text{and} \quad z_{m+1} = y_0.$$

We write

$$\sigma_k = [z_k, z_{k+1}], \quad \text{for} \quad k = 0, ..., m$$

where, $\bigcup_{k=1}^{m} \sigma_k$ is a smashed line.
Join x_n to x_0 by a C^1 path $\delta_{0,n}$ having range in $\overline{\Omega}$, and join y_0 to y_n by a C^1 path $\gamma_{n,0}$ having range in $\overline{\Omega}$.
Then, $\mathcal{C}_n{=}\delta_{0,n} \bigcup (\bigcup_{k=1}^{m} \sigma_k) \bigcup \gamma_{n,0}$ is a C^1 with morsels line which connects the

extremity x_n and y_n.
By lemma 4.6, we can rectify the sequence $(z_k)_{k=1,...,m}$ such that

$$\lim_{n\to\infty}\int_0^1 \frac{|g(u_n(\sigma_k(t))||\nabla u_n(\sigma_k(t))||\sigma_k^{'}(t)|}{\| u_n \|_\infty{}^p}dt = 0 \text{ for } \quad k=1,...,m-1. \tag{4.22}$$

Put $r_n = \frac{min(u_n)}{max(u_n)}$, by using lemma 4.5, it is clear to see that

$$\lim_{n\to+\infty} r_n = \frac{min(\varphi)}{max(\varphi)} \in]0,1[.$$

The proof is achieved if we obtain a contradiction from the formula (4.21), thus we will proceed in two claims:

Claim 1. We will show that $\lim_{n\to\infty}\frac{|G(s_n)-G(r_n s_n)|}{|s_n|^p} = 0$.
We write

$$\frac{G(s_n)-G(r_n s_n)}{|s_n|^p} = \frac{G(u_n(x_n))-G(u_n(y_n))}{\|u_n\|_\infty^p} = \frac{1}{\|u_n\|_\infty^p}\int_{\mathcal{C}_n} d(G\circ u_n), \tag{4.23}$$

where, $\int_{\mathcal{C}_n} = \int_{\delta_{0,n}} + \sum_{k=0}^m \int_{\sigma_k} + \int_{\gamma_{n,0}}$ and $d(G\circ u_n)(\sigma_k) = g(u_n(\sigma_k))\nabla u_n(\sigma_k).\sigma_k^{'}$, with $\int_{\mathcal{C}_n}$ denotes a line integral.
So by using lemma 4.6, we obtain

$$\lim_{n\to+\infty}\frac{\sum_{k=1}^{m-1}\int_{\sigma_k} d(G\circ u_n)}{\|u_n\|_\infty^p} = 0. \tag{4.24}$$

Furthermore, we have for all $\varepsilon > 0$, there exists $x_\varepsilon \in \partial B(z_1, |z_1 - x_0|)$ such that

$$|x_\varepsilon - x_0| < 0 \quad \text{and} \quad \lim_{n\to+\infty}\frac{\sum_{k=1}^{m-1}\int_{\sigma_{0,\varepsilon}} d(G\circ u_n)}{\|u_n\|_\infty^p} = 0, \tag{4.25}$$

where, $\sigma_{0,\varepsilon} = [z_1, x_\varepsilon]$.
Similarly, for all $\varepsilon > 0$, there exists $y_\varepsilon \in \partial B(z_m, |z_m - y_0|)$ such that

$$|y_\varepsilon - y_0| < 0 \quad \text{and} \quad \lim_{n\to+\infty}\frac{\sum_{k=1}^{m-1}\int_{\sigma_{m,\varepsilon}} d(G\circ u_n)}{\|u_n\|_\infty^p} = 0, \tag{4.26}$$

where, $\sigma_{m,\varepsilon} = [z_1, x_\varepsilon]$.
By the C^1 character of $\partial\Omega$, $\delta_{0,n}$ and $\gamma_{n,0}$ can be taken such that

$$\ell(\delta_{0,n}) \to 0 \quad \text{and} \quad \ell(\gamma_{n,0}) \to 0 \quad \text{when} \quad n\to\infty,$$

where, $\ell(.)$ denotes the length of the corresponding path.
By combining (4.24), (4.25) and (4.26), we deduce that there exists $c > 0$ such that for all $\varepsilon > 0$ and for all n sufficiently large, we get

$$\frac{|\int_{\mathcal{C}_n} d(G\circ u_n)|}{\|u_n\|_\infty^p} < \varepsilon c.$$

So, this concludes the proof of the first claim by replacing in (4.23).

Claim 2. We will show that $\lim_{n\to\infty} \frac{|G(s_n)-G(rs_n)|}{|s_n|^p} = 0$.
It is easy to see that for all $n \in \mathbb{N}^*$, there exists c_n which lives between $r_n s_n$ and rs_n such that

$$\frac{|G(r_n s_n) - G(rs_n)|}{s_n^p} = \frac{(r_n - r)g(c_n)}{s_n^{p-1}}.$$

Since g satisfy the following growth condition

$$|g(s)| \le a|s|^{p-1} + b, \quad \text{for some positive reals} \quad a, b \quad \text{and for all} \quad s \in [A, +\infty[$$

then, the sequence $(\frac{g(c_n)}{s_n^{p-1}})_n$ is bounded.
On the other hand, we have

$$\frac{|G(s_n) - G(rs_n)|}{|s_n|^p} \le \frac{|G(s_n) - G(r_n s_n)|}{|s_n|^p} + \frac{|G(r_n s_n) - G(rs_n)|}{|s_n|^p}.$$

According to the first claim and the fact $\lim_{n\to\infty} r_n = r$, we conclude the proof of the second claim.
Finally, we get a contradiction from (4.21). This concludes the proof of theorem 4.4.

5. Nonresonance under the principal eigencurve

In this section, we turn to the problem $(\mathcal{P}_\alpha)$ and we show that it has at least weak solution. For this purpose, we will apply the main results the two of sections previous.

Theorem 5.1. *Let $(\alpha, \beta) \in \mathcal{C}_1$ and let $m_1, m_2 \in M^+(\Omega)$. Assume that the weight m_1 and m_2 satisfy (**A1**) and (**A2**) respectively and the assumptions (**H1**) and (**H2**) hold. Then the problem $(\mathcal{P}_\alpha)$ has at least one nontrivial weak solution $u \in W^{1,p}(\Omega)$ for any given $h \in L^\infty(\Omega)$.*

Proof: Clearly, problem $(\mathcal{P}_\alpha)$ can be written in the following equivalent form

$$(\mathcal{P}_{\alpha,\beta}) \quad \begin{cases} -\Delta_p u = m_{\alpha,\beta}(x)|u|^{p-2}u + m_2(x)\widetilde{g}(u) + h(x) & \text{in} \quad \Omega \\ \frac{\partial u}{\partial \nu} = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$\widetilde{g}(s) = g(s) - \beta|s|^{p-2}s,$$

and

$$m_{\alpha,\beta} = \alpha m_1 + \beta m_2.$$

Since $(\alpha, \beta) \in \mathcal{C}_1$, then 1 is the first eigenvalue of p-laplacian operator with weight $m_{\alpha,\beta}$ in Neumann case. In view of theorem 4.4, there exists at least one weak solution $u \in W^{1,p}(\Omega)$ of the problem $(\mathcal{P}_{\alpha,\beta})$ for all $h \in L^\infty(\Omega)$ if the function $\widetilde{g}$ and his potential $\widetilde{G}$ satisfy the two conditions $(\mathbf{g}^\pm)$ and $(\mathbf{G}^\pm)$. Indeed,

$$\limsup_{\pm\infty} \frac{\widetilde{g}(s)}{|s|^{p-2}s} = \limsup_{\pm\infty} \left(\frac{g(s)}{|s|^{p-2}s} - \beta \right) \leq 0$$

and,

$$\liminf_{\pm\infty} \frac{p\widetilde{G}(s)}{|s|^p} = \liminf_{\pm\infty} \left(\frac{pG(s)}{|s|^p} - \beta \right) < 0$$

Consequently, as $(\mathcal{P}_{\alpha,\beta})$ and $(\mathcal{P}_\alpha)$ are equivalent, which gives that the problem $(\mathcal{P}_\alpha)$ has a solution $u \in W^{1,p}(\Omega)$, for every $h \in L^\infty(\Omega)$. □

Acknowledgments

The authors would like to thank reviewers for clear valuable comments and suggestions.

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