Bol. Soc. Paran. Mat. (3s.) **v. 38** 3 (2020): 63–69.
©SPM –ISSN-2175-1188 on line ISSN-00378712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.v38i3.38404

Spectral Mapping Theorem for C_0-Semigroups of Drazin Spectrum

Abdelaziz Tajmouati and Hamid Boua

ABSTRACT: Let $(T(t))_{t\geq 0}$ be a C_0 semigroup of bounded linear operators on a Banach space X and denote its generator by A. A fundamental problem to decide whether the Drazin spectrum of each operator $T(t)$ can be obtained from the Drazin spectrum of A. In particular, one hopes that the Drazin Spectral Mapping Theorem holds, i.e., $e^{t\sigma_D(A)} = \sigma_D(T(t))\backslash\{0\}$ for all $t \geq 0$.

Key Words: Drazin invertibility, Spectrum Drazin, Semigroup of operators.

Contents

1. Introduction

Throughout this work, X denotes a complex Banach space and $\mathcal{B}(X)$ denotes the Banach algebra of all bounded linear operators on X. Let A be a closed operator with domain $D(A)$, we denote by $R(A)$, $N(A)$, $\rho(A)$, $\sigma(A)$, $\sigma_r(T)$ and $\sigma_p(A)$ respectively the range, the kernel, the resolvent set, the spectrum, the residual spectrum and the point spectrum of A. The ascent of A is defined by $a(A) = \min\{p : N(A^p) = N(A^{p+1})\}$, if no such p exists, we let $a(A) = \infty$. Similarly, the descent of A is $d(A) = \min\{q : R(A^q) = R(A^{q+1})\}$, if no such q exists, we let $d(A) = \infty$ (see [7] and [8]). It is well known that if A is bounded, and if both $a(A)$ and $d(A)$ are finite then $a(A) = d(A)$ and therefore we have the decomposition $X = R(A^p) \oplus N(A^p)$ where $p = a(A) = d(A)$. The descend and ascent spectrum are defined by $\sigma_{desc}(A) = \{\lambda \in \mathbb{C} : d(\lambda - A) = \infty\}$ and $\sigma_{asc}(A) = \{\lambda \in \mathbb{C} : a(\lambda - A) = \infty\}$. Recall that A is a Drazin invertible if $p = a(A) = d(A) < \infty$ and $R(A^p)$ is closed. The Drazin spectrum is defined by $\sigma_D(A) = \{\lambda \in \mathbb{C} : \lambda - A \text{ not Drazin inversible }\}$.

A strongly continuous semigroup $(T(t))_{t\geq 0}$ is called eventually norm continuous, if there exists $t_0 \geq 0$ such that the function $t \mapsto T(t)$ is norm continuous from (t_0, ∞) into $\mathcal{B}(X)$. Let $\Delta = \{z \in \mathbb{C} : \alpha_1 < \arg z < \alpha_2\}$ and for $z \in \Delta$ let $T(z)$ be a bounded linear operator. The family $(T(z))_{z\in\Delta}$ is an analytic semigroup in Δ if

(i) $z \mapsto T(z)$ is analytic in Δ.

(ii) $T(0) = I$ and $\lim T(z)x = x$ for every $x \in X$.

2010 *Mathematics Subject Classification:* 47A16, 47D06, 47D03.
Submitted July 29, 2017. Published October 17, 2017

(iii) $T(z_1+z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.

A semigroup $(T(t))_{t\geq 0}$ will be called analytic if it is analytic in some sector Δ containing the nonnegative real axis.

A strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X is called eventually differentiable if there exists $t_0 \geq 0$ such that the orbit maps $\xi_x : t \mapsto T(t)x$ are differentiable on (t_0, ∞) for every $x \in X$. The semigroup is called eventually compact, if there exists $t_0 > 0$ such that $T(t_0)$ is compact.

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X with infinitesimal generator A. We introduce the following operator acting on X and depending on the parameters $\lambda \in \mathbb{C}$ and $t \geq 0$, $B_\lambda(t)x = \int_0^t e^{\lambda(t-s)}T(s)xds,\ x \in X$. It is well known (see [5] and [10]) that $B_\lambda(t)$ is a bounded linear operator on X. Furthermore, for all $n \in \mathbb{N}$, we have $(e^{\lambda t} - T(t))^n x = (\lambda - A)^n B_\lambda^n(t)x$, for all $x \in X$ and $(e^{\lambda t} - T(t))^n x = B_\lambda^n(t)(\lambda - A)^n x$, for all $x \in D(A^n)$. (See [4]).
In [9], Rainer Nagel and Jan Poland showed that, for an eventually norm continuous semigroup $(T(t))_{t\geq 0}$ with generator A one has $e^{t\sigma(A)} = \sigma(T(t))\backslash\{0\}$ for all $t \geq 0$. Rainer Nagel in [5] proved for the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X, we have the identities: $e^{t\sigma_p(A)} = \sigma_p(T(t))\backslash\{0\}$ and $e^{t\sigma_r(A)} = \sigma_r(T(t))\backslash\{0\}$ for all $t \geq 0$. These works push to ask the following question: Does this spectral inclusion hold for the other parts of spectrum?
In this work, we show that this spectral mapping theorem of C_0-semigroups eventually norm continuous holds for Drazin spectrum.

2. Main results

We start by the following lemmas.

Lemma 2.1. *Let $(T(t))_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. For all $\lambda \in \mathbb{C}$ and $t \geq 0$, there exists $F_\lambda(t), G_\lambda(t) \in \mathcal{B}(X)$ such that*

1. *$\forall x \in X$, $F_\lambda(t)x \in D(A)$ and $(\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = tI$,*

2. *The operators $\lambda - A$, $F_\lambda(t)$, $G_\lambda(t)$ and $B_\lambda(t)$ are are mutually commuting.*

Proof:

1. For every $\lambda \in \mathbb{C}$ and $t \geq 0$, let $F_\lambda(t)x = \int_0^t e^{-\lambda s}B_\lambda(s)xds$. $F_\lambda(t)$ is a bounded

linear operator on X. Moreover for every $x \in X$, we have

$$\begin{aligned}\frac{T(h)-I}{h}F_\lambda(t)x =& \frac{T(h)-I}{h}\int_0^t e^{-\lambda s}B_\lambda(s)xds \\ =& \frac{1}{h}\int_0^t\int_0^s e^{-\lambda u}T(u+h)xduds \\ & -\frac{1}{h}\int_0^t\int_0^s e^{-\lambda u}T(u)xduds \\ =& \frac{1}{h}\int_0^t\left(\int_0^s e^{-\lambda u}T(u+h)xdu\right. \\ & \left.-\int_0^s e^{-\lambda u}T(u)xdu\right)ds \\ =& \int_0^t\left(\frac{e^{\lambda h}}{h}\int_h^{h+s} e^{-\lambda u}T(u)xdu\right. \\ & \left.-\frac{1}{h}\int_0^s e^{-\lambda u}T(u)xdu\right)ds \\ =& \int_0^t\left(\frac{e^{\lambda h}-1}{h}\int_h^s e^{-\lambda u}T(u)xdu\right. \\ & +\frac{e^{\lambda h}}{h}\int_s^{h+s} e^{-\lambda u}T(u)xdu \\ & \left.-\frac{1}{h}\int_0^h e^{-\lambda u}T(u)xdu\right)ds.\end{aligned}$$

Therefore

$$\lim_{h\to 0}\frac{T(h)-I}{h}F_\lambda(t)x = \lambda\int_0^t e^{-\lambda s}B_\lambda(s)xds + e^{-\lambda t}\int_0^t e^{-\lambda s}T(s)xds - tx$$

Consequently $F_\lambda(t)x \in D(A)$ and $AF_\lambda(t)x = \lambda F_\lambda(t)x + e^{-\lambda t}B_\lambda(t)x - tx$. Then $(\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = tI$ with $G_\lambda(t) = e^{-\lambda t}I$.

2. For all $t \geq 0$, $F_\lambda(t)$ and $B_\lambda(t)$ commuting. Indeed, for $t, s \geq 0$ we have

$$\begin{aligned}B_\lambda(t)B_\lambda(s)x &= \int_0^t e^{\lambda(t-u)}T(u)B_\lambda(s)xdu \\ &= \int_0^t e^{\lambda(t-u)}T(u)\int_0^s e^{\lambda(s-v)}T(v)xdvdu \\ &= \int_0^t\int_0^s e^{\lambda(t-u)}e^{\lambda(s-v)}T(u)T(v)xdvdu \\ &= \int_0^s e^{\lambda(s-v)}T(v)\int_0^t e^{\lambda(t-u)}T(u)xdudv \\ &= B_\lambda(s)B_\lambda(t)x\end{aligned}$$

Therefore

$$\begin{aligned} F_\lambda(t)B_\lambda(t)x &= \int_0^t e^{-\lambda u}B_\lambda(u)B_\lambda(t)xdu \\ &= \int_0^t e^{-\lambda u}B_\lambda(t)B_\lambda(u)xdu \\ &= B_\lambda(t)\int_0^t e^{-\lambda u}B_\lambda(u)xdu \\ &= B_\lambda(t)F_\lambda(t)x \end{aligned}$$

For all $x \in D(A)$ we have

$$\begin{aligned} F_\lambda(t)(\lambda - A)x &= \int_0^t e^{-\lambda s}B_\lambda(s)(\lambda - A)xds \\ &= \int_0^t e^{-\lambda s}(e^{\lambda s} - T(s))xds \\ &= tx - \int_0^t e^{-\lambda s}T(s)xds \\ &= tx - G_\lambda(t)B_\lambda(t)x \\ &= (\lambda - A)F_\lambda(t)x \end{aligned}$$

□

Lemma 2.2. *Let $(T(t))_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. For all $\lambda \in \mathbb{C}$, $t > 0$ and $n \in \mathbb{N}$, there exists $H_n(t), L_n(t) \in \mathcal{B}(X)$ such that*

1. *$\forall x \in X$, $H_n(t)x \in D(A^n)$ and $(\lambda - A)^n H_n(t) + L_n(t)B_\lambda^n(t) = I$,*
2. *The operators $(\lambda - A)^n$, $H_n(t)$, $L_n(t)$ and $B_\lambda^n(t)$ are mutually commuting.*

Proof: According to lemma 1 there exists tow bounded operators $F_\lambda(t)$ and $G_\lambda(t)$ such that $(\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = I$. For $i \in \{1, ..., n-1\}$ and $x \in X$, we have

$$\begin{aligned} (\lambda - A)^i F_\lambda^n(t)x &= [(\lambda - A)F_\lambda(t)]^i F_\lambda^{n-i}(t)x \\ &= [F_\lambda(t)(\lambda - A)]^i F_\lambda^{n-i}(t)x \in D(A). \end{aligned}$$

Hence $\forall n \in \mathbb{N}^*, F_\lambda^n(t)x \in D(A^n)$. Therefore

$$\begin{aligned} (\lambda - A)^n F_\lambda^n(t) &= [(\lambda - A)F_\lambda(t)]^n \\ &= [I - G_\lambda(t)B_\lambda(t)]^n \\ &= I - L_{1,n}(t)B_\lambda(t) \end{aligned}$$

with $L_{1,n}(t) = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}G_\lambda^k(t)B_\lambda^{k-1}(t)$. Hence $(\lambda - A)^n F_\lambda^n(t) + L_{1,n}(t)B_\lambda(t) = I$

Similarly

$$\begin{aligned} L^n_{1,n}(t)B^n_\lambda(t) &= [I-(\lambda-A)^n F^n_\lambda(t)]^n \\ &= I-(\lambda-A)^n\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(\lambda-A)^{n(k-1)}F^{nk}_\lambda(t) \end{aligned}$$

Let $H_n(t)=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}(\lambda-A)^{n(k-1)}F^{nk}_\lambda(t)$ and $L_n(t)=L^n_{1,n}(t)$, then $(\lambda-A)^n H_n(t)+L_n(t)B^n_\lambda(t)=I$, moreover $(\lambda-A)^n$, $H_n(t)$, $L_n(t)$ and $B^n_\lambda(t)$ are mutually commuting.

□

Lemma 2.3. *Let $(T(t))_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. If $R(e^{\lambda t}-T(t))^p$ is closed, then $R(\lambda-A)^p$ is closed.*

Proof: Suppose that $R(e^{\lambda t}-T(t))^p$ is closed. Let $y_n=(\lambda-A)^p x_n$ be a convergent sequence with limit $y\in X$. From lemma 2, there exists $H_p(t),L_p(t)\in\mathcal{B}(X)$ such that $(\lambda-A)^p H_p(t)+L_p(t)B^p_\lambda(t)=I$, then $x_n=(\lambda-A)^p H_p(t)x_n+L_p(t)B_\lambda(t)^p x_n$ and $y_n=(\lambda-A)^p H_p(t)y_n+(e^{\lambda t}-T(t))^p L_p(t)x_n$. Since $(\lambda-A)^p H_p(t)$ is a linear bounded operator and $R(e^{\lambda t}-T(t))^p$ is closed, then $(e^{\lambda t}-T(t))^p L_p(t)x_n=y_n-(\lambda-A)^p H_p(t)y_n$ tends to $y-(\lambda-A)^p H_p(t)y\in R(e^{\lambda t}-T(t))^p$, therefore there exists $z\in X$ such that $y-(\lambda-A)^p H_p(t)y=(e^{\lambda t}-T(t))^p z$, then $y=(\lambda-A)^p[H_p(t)y+B^p_\lambda(t)z]$, hence $y\in R(\lambda-A)^p$. □

We have the following theorem.

Theorem 2.4. *Let $(T(t))_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. Then*

For all $t\geq 0$, $e^{t\sigma_{desc}(A)}\subseteq\sigma_{desc}(T(t))\backslash\{0\}$ and $e^{t\sigma_{asc}(A)}\subseteq\sigma_{asc}(T(t))\backslash\{0\}$

Proof: If $e^{\lambda t}-T(t)$ has finite descent, then there exists $n\in\mathbb{N}$ such that $R(e^{\lambda t}-T(t))^n=R(e^{\lambda t}-T(t))^{n+1}$, from lemma 3, there exist two operators $H_n(t)$ and $L_n(t)$ such that $(\lambda-A)^n H_n(t)+L_n(t)B^n_\lambda(t)=I$ and $H_n(t)$, $L_n(t)$, $B_\lambda(t)$ and $\lambda-A$ are mutually commuting. Let $y\in R(\lambda-A)^n$ and $x\in D(A^n)$ such that $y=(\lambda-A)^n x$. Therefore

$$\begin{aligned} (\lambda-A)^n x &= (\lambda-A)^n H_n(t)(\lambda-A)^n x+L_n(t)B^n_\lambda(t)(\lambda-A)^n x \\ &= (\lambda-A)^{n+1}H_n(t)(\lambda-A)^{n-1}x+L_n(t)(e^{\lambda t}-T(t))^n x \end{aligned}$$

Moreover, $R(\lambda-A)^n=R(\lambda-A)^{n+1}$, hence $\lambda-A$ has finite descent.
If $e^{\lambda t}-T(t)$ has finite ascent, there exist $n\in\mathbb{N}$ such that $N(e^{\lambda t}-T(t))^n=N(e^{\lambda t}-T(t))^{n+1}$. Let $x\in D(A)^{n+1}$, we have

$$\begin{aligned} (\lambda-A)^n x &= (\lambda-A)^n H_n(t)(\lambda-A)^n x+L_n(t)(e^{\lambda t}-T(t))^n x \\ &= (\lambda-A)^{n-1}H_n(t)(\lambda-A)^{n+1}x+L_n(t)(e^{\lambda t}-T(t))^n x \end{aligned}$$

Moreover, $N(\lambda-A)^n=N(\lambda-A)^{n+1}$, hence $\lambda-A$ has finite ascent. □

Remark 2.5. *Consider the translation group on the space $C_{2\pi}(\mathbb{R})$ of all 2π periodic continuous functions on $\mathbb{R}$ and denote its generator by A (see* [5, Paragraph I.4.15]*).* From [5, Examples 2.6.iv] we have, $\sigma(A) = i\mathbb{Z}$, then $e^{t\sigma(A)}$ is at most countable, therefore $e^{t\sigma_{asc}(A)}$ and $e^{t\sigma_{desc}(A)}$ are also. The spectra of the operators $T(t)$ are always contained in $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ and contain the eigenvalues e^{ikt} for $k \in \mathbb{Z}$. Since $\sigma(T(t))$ is closed, it follows from [5, Theorem IV.3.16] below, that $\sigma(T(t)) = \Gamma$ whenever $t/2\pi \notin \mathbb{Q}$, then $\sigma(T(t))$ is not countable, from [2, Corollary 2.10] and [3, Corollary 1.8], $\sigma_{asc}(T(t))$ and $\sigma_{desc}(T(t))$ are also. Therefore the inclusions of the preceding theorem are strict.

Corollary 2.6. *Let $(T(t))_{t\geq 0}$ a C_0-semigroup on X with infinitesimal generator A. Then*

$$\text{For all } t \geq 0,\ \ e^{t\sigma_D(A)} \subseteq \sigma_D(T(t))\backslash\{0\}$$

Proof: If $e^{\lambda t} - T(t)$ is invertible Drazin, then $e^{\lambda t} - T(t)$ has finite ascent and descnt p, therefore $R(e^{\lambda t} - T(t))^p$ is closed. By lemma 3 and theorem 1, $\lambda - A$ is invertible Drazin.

□

Remark 2.7. *The inclusion of the preceding corollary is strict. Indeed, from remark 1, $e^{t\sigma_D(A)}$ is at most countable, on the other hand $\sigma_D(T(t))$ is not countable.*

Theorem 2.8. *Let $(T(t))_{t\geq 0}$ be an eventually norm-continuous semigroup with generator A on the Banach space X. The spectral mapping theorem*

$$e^{t\sigma_D(A)} = \sigma_D(T(t))\backslash\{0\} \text{ for all } t \geq 0$$

holds.

Proof: Let λ be a complex number such that $\lambda - A$ has finite ascent and descent p such that $R(\lambda - A)^p$ is closed. According to [8, Lemma 3.4] and [8, Lemma 3.5], there is $\delta > 0$ such that, for every $\mu \in \mathbb{C}$ with $0 < |\lambda - \mu| < \delta$, the operator $\mu - A$ is bijective, by [9, Corollary 3.3], for every $\mu \in \mathbb{C}$ with $0 < |\mu - \lambda| < \delta$, $e^{\mu t} - T(t)$ is bijective, from open mapping theorem $e^{\lambda t}$ is isolated in $\sigma(T(t))$. By [1, Theorem 3.81], we have $e^{\lambda t}$ is a pole of the resolvent of $T(t)$. Using [7, Theorem V.10.1], we obtain $e^{\lambda t} - T(t)$ has a finite ascent and descent, moreover $e^{\lambda t} - T(t)$ is Drazin inversible. □

Example 2.9. *On $X := C_0(\Omega)$ take the multiplication operator $M_q f(\lambda) = q(\lambda)f(\lambda)$ for $\lambda \in \Omega$, $f \in X$. From* [5, Proposition I.4.2] we obtain that $\sigma(M) = \overline{q(\Omega)}$ and $\sigma_p(M) = \{\lambda \in \mathbb{C} : \lambda \text{ is isolated in } \Omega\}$. On for some continuous function $q : \Omega \to \mathbb{C}$, if $\sup_{s\in\Omega} Re(q(s)) < \infty$, then $T_q(t)f := e^{tq}f$ defines a strongly continuous semigroup (see [5, Proposition I.4.5]). Suppose that $\Omega = \{\lambda \in \mathbb{C} :\ \mathrm{Re}(\lambda) \leq 1 \text{ and } -1 \leq \mathrm{Im}(\lambda) \leq 1\}$ and for all $\lambda \in \Omega$, $q(\lambda) = \lambda$. Then $\sigma(M) = \Omega$ and $\sigma_p(M) = \emptyset$, by [7, Theorem 5.41-C], we have $\sigma(M) = \sigma_{desc}(M) \cup \sigma_p(M) = \sigma_{desc}(M)$, then $\sigma_D(M) = \Omega$. Furthermore $\overline{q(\Omega)} \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq b\}$ is bounded for every $b \in \mathbb{R}$, from [5, Theorem II.4.32], $(T_q(t))_{t\geq 0}$ is eventually norm-continuous. By theorem 2, for $t > 0$, we have $\sigma_D(T(t)) = \{e^{t\lambda} : \lambda \in \Omega\} \cup \{0\}$.

Corollary 2.10. *The spectral mapping theorem*

$$e^{t\sigma_D(A)} = \sigma_D(T(t))\backslash\{0\} \text{ for all } t \geq 0$$

hold for the following classes of strongly continuous semigroups:

1. *eventually compact semigroups,*
2. *eventually differentiable semigroups,*
3. *analytic semigroups.*

Proof: If a strongly continuous semigroup $(T(t))_{t\geq 0}$ satisfies one of the following conditions:

1. eventually compact semigroups,
2. eventually differentiable semigroups,
3. analytic semigroups.

Then it is an eventually norm-continuous semigroup, from Teorem 2 we have $e^{t\sigma_D(A)} = \sigma_D(T(t))\backslash\{0\}$ for all $t \geq 0$. □

References

1. P.Aiena. *Fredholm and Local Spectral Theory with Applications to Multipliers,* Kluwer.Acad.Press, 2004.
2. O. Bel Hadj Fredj, M. Burgos and M. Oudghiri *Ascent spectrum and essential ascent spectrum,* Studia Math 187 (1) (2008).
3. M. Burgos, A. Kaidi, M. Mbekhta and M. Oudghiri, *The descent spectrum and perturbations,* Journal of Operator Theory 56 (2006), 259-271.
4. A. Elkoutri and M. A. Taoudi, *Spectre singulier pour les générateurs des semigroupes [Singular spectrum for generators of semigroups]* , C. R. Acad. Sci. Paris Sér. I Math. 333 (2001), no. 7, 641-644 (French).
5. K. J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, vol. 194, Springer-Verlag, New York, 2000.
6. K.B. Laursen, M.M. Neumann, *Introduction to Local Spectral Theory,* Clarendon Press, Oxford,(2000).
7. A.E. Taylor, D .C. Lay, *Introduction to Functional Analysis,* Wiley, New York-Chichester-Brisbane 1980.
8. M. Mbekhta, *Ascente, descente et spectre essentiel quasi-Fredholm,* Rend. Circ. Math. Palermo (2) 2(1997), 175-196.
9. R. Nagel, J. Poland *The Critical Spectrum of a Strongly Continuous Semigroup,* Advances in Mathematics 152, 120-133 (2000).
10. A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, vol. 44, Springer-Verlag, New York 1983.

Abdelaziz Tajmouati and Hamid Boua,
Sidi Mohamed Ben Abdellah University,
Faculty of Sciences Dhar El Mahraz,
Laboratory of Mathematical Analysis and Applications,
Fez, Morocco.
E-mail address: abdelaziz.tajmouati@usmba.ac.ma hamid.boua@usmba.ac.ma